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A Discussion on a Pata Type Contraction via Iterate at a Point

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Abstract. In this paper, we introduce the notion of Pata type contraction at a point in the context of a complete metric space. We observe that such contractions possesses unique fixed point without continuity assumption on the given mapping. Thus, is extended the original results of Pata. We also provide an example to illustrate its validity.

1. Introduction and Preliminaries

A century has passed since the appearance of the first metric fixed point result. It was Banach [3] who published the first result in a solely fixed point result in the setting of complete norm spaces in 1922. As is known well, fixed point techniques were used to solve ordinary differential equations by Liouville [1] and also Picard [2] in the late of eighteens century. Banach understood the essence of the method of successive approximation that was used by Picard [2] and he formulated as a sole fixed point result. Since then, many improvements have been made in fixed-point theory, and consequently many results have been reported on this research topic. Among all these results, we underline the renowned results of Sehgal [4] who refined the prominent result of Banach in fixed point theory with an elegant way. Roughly speaking, Sehgal [4] proved that in a complete metric space (X, d), if, for any point x, there is a positive integer k such that k^{th} iteration of continuous self-mapping $\mathcal{T}$ at x fulfills contraction condition, then $\mathcal{T}$ possesses a unique fixed point. Immediately after Sehgal [4], Guseman [5] improved this result by eliminating the continuity condition on the mapping. In other words, despite the Banach's theorem, iterate of a self-mapping at a point forms a contraction where the number of the iteration depends on the given point.

To make it more clear and precise, we give the result of Guseman [5] below. Before that we fix the pairs (X, d) and (X^*, d) to indicate the metric space and complete metric space, respectively.

Theorem 1.1. *[5] Let $\mathcal{T}$ be a self-mapping on (X^*, d). If there $\kappa < 1$ such that for each $u \in X$ there exists a positive integer $m(u)$ so that*

$$d(\mathcal{T}^{m(u)}u, \mathcal{T}^{m(u)}v) \leq \kappa d(u, v)$$

for all $v \in X$, then $\mathcal{T}$ admits a unique fixed point.

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Note that for Theorem 1.1 coincide with Banach's theorem incase $m(u) = 1$, for each $u \in X$. On the other hand, if $m(u) > 1$, for each $u \in X$ then the self-mapping needs not to be continuous that indicate that Guseman's Theorem is more general then Banach's theorem. The following example, due to Bryant [6], indicates the aspect of this arguments:

Example 1.2. *[6] Let $\mathcal{T} : [0, 2] \to [0, 2]$ be defined by*

$$\mathcal{T}(x) = \begin{cases} 0 & \text{if } u \in [0, 1], \\ 1 & \text{if } u \in (1, 2]. \end{cases}$$

Here, $m(u) = 2$, for all $u \in [0, 2]$ and T is not continuous.

Note that Seghal's theorem can not applicable to Example 1.2, since Sehgal [4] presumed the continuity of the considered self-mapping. For more details on the advances of fixed point theory can be found in [7–10] and also [11–14].

Prior to giving more technical details to make it clear the result of the paper, we, first, fix the following notations. The expression $\mathbb{R}_0^+$ stands for the set of all non-negative real numbers, denoted by $\mathbb{R}$. In addition, the letters N and $\mathbb{N}_0$ are preserved for all positive integers and all non-negative integers. Unless otherwise stated throughout this manuscript, all considered sets are non-empty. The expression $\Upsilon =$ represents the class of all auxiliary functions $\psi : [0, 1] \to [0, \infty)$ which are increasing, continuous at zero, and $\psi(0) = 0$. For an arbitrary point u_0 in a (X^*, d), we set a function

$$\|u\| = d(u, u_0), \forall u \in X,$$

that will be called "the zero of X".

On the other hand, V.Pata [15] obtained the following generalization of Banach mapping principle

Theorem 1.3. *[15] Let $\mathcal{T} : X \to X$ and let $\Lambda \geq 0$, $\alpha \geq 1$ and $\beta \in [0, \alpha]$ be fixed constants. If the inequality*

$$d(\mathcal{T}u, \mathcal{T}v) \quad \leq (1 - \varepsilon)d(u, v) + \Lambda(\varepsilon)^{\alpha}\psi(\varepsilon)\left[1 + \|u\| + \|v\|\right]^{\beta}, \tag{1}$$

is satisfied for every $\varepsilon \in [0, 1]$ and all $u, v \in X$, then $\mathcal{T}$ has a fixed point $\sigma \in X$.

2. Main results

Definition 2.1. *A self-mapping $\mathcal{T}$, defined on (X, d), is called Pata type contraction at a point if for every $\varepsilon \in [0, 1]$ and for any $u \in X$, there exists a positive integer $m(u)$such that*

$$d(\mathcal{T}^{m(u)}u, \mathcal{T}^{m(u)}v) \leq (1 - \varepsilon)d(u, v) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)\left[1 + \|u\| + \|v\|\right]^{\beta}. \tag{2}$$

for all $v \in X$, where $\Lambda \geq 0$, $\lambda \geq 1$, $\beta \in [0, \lambda]$ are fixed constants.

Theorem 2.2. *Suppose that a self-mapping $\mathcal{T}$ on (X^*, d) is a Pata type contraction at a point. Then, $\mathcal{T}$ admits a unique fixed point.*

Proof. On account of (2), for each $u \in X^*$, there exists $p = m(u)$ a positive integer, such that for $\varepsilon = 0$ we find

$$d(\mathcal{T}^p u, \mathcal{T}^p v) \leq d(u, v). \tag{3}$$

Choosing an arbitrary point $u_0 \in X$, we can suppose that $\mathcal{T}u_0 \neq u_0$, because on the contrary u_0 is a fixed point of $\mathcal{T}$ that terminates the proof. Starting from this point, we build the sequence $\{u_n\}$ as follows:

$$u_{n+1} = \mathcal{T}^{m_n}u_n \text{ for all } n \in \mathbb{N} \tag{4}$$

(we prefer to use $m_n = m(u_n)$). Assuredly, from

$$u_1 = \mathcal{T}^{m_0} u_0, u_2 = \mathcal{T}^{m_1} u_1 = \mathcal{T}^{m_1+m_0} u_0,$$

we can deduce that $u_n = \mathcal{T}^{m_{n-1}+...+m_1+m_0} u_0$, so we obtain that $\{u_n\}_{n\geq 0}$ is a subsequence of $O(u_0) = \{\mathcal{T}^n u_0 : n = 0, 1, 2, ...\}$, the orbit of $u_0 \in X^*$.

By (2), for $\varepsilon = 1$ we have

$$\begin{aligned} d(u_n, u_{n+1}) &= d(\mathcal{T}^{m_{n-1}} u_{n-1}, \mathcal{T}^{m_n} u_n) = d(\mathcal{T}^{m_{n-1}} u_{n-1}, \mathcal{T}^{m_{n-1}}(\mathcal{T}^{m_n} u_{n-1})) \\ &\leq d(u_{n-1}, \mathcal{T}^{m_n} u_{n-1}) = d(\mathcal{T}^{m_{n-2}} u_{n-2}, \mathcal{T}^{m_{n-2}}(\mathcal{T}^{m_n} u_{n-2})) \\ &\leq d(u_{n-2}, \mathcal{T}^{m_n} u_{n-2}) \\ &\quad ... \\ &\leq d(u_0, \mathcal{T}^{m_n} u_0) \end{aligned} \tag{5}$$

Let also, denote by $\omega(u_0) = \sup\{d(u, v) : u, v \in O(u_0)\}$. On what follows we claim that, $\omega(u_0) < \infty$. Let p be a positive integer, arbitrary but fixed and l be a positive integer, depending on u_0 and p such that

$$c_p = d(u_0, \mathcal{T}^l u_0) = \max\{d(u_0, \mathcal{T}^s u_0) : 0 < s \leq p\}.$$

Let $\Lambda \geq 0$, $\lambda \geq 1$ and $\beta \in [0, \lambda]$ be fixed constants. We can assume that $p > m(u_0)$ and also $l > m(u_0)$. Then, regarding the triangle inequality, by (2) and taking into account (3), we have

$$\begin{aligned} c_p &= d(\mathcal{T}^p(u_0), u_0) \leq d(\mathcal{T}^p u_0, \mathcal{T}^{p+l} u_0) + d(\mathcal{T}^{p+l} u_0, \mathcal{T}^l u_0) + d(\mathcal{T}^l u_0, u_0) \\ &= d(\mathcal{T}^p u_0, \mathcal{T}^p(\mathcal{T}^l u_0)) + d(\mathcal{T}^l(\mathcal{T}^p u_0), \mathcal{T}^l u_0) + d(\mathcal{T}^l u_0, u_0) \\ &\leq 2d(u_0, \mathcal{T}^l u_0) + (1-\varepsilon) d(u_0, \mathcal{T}^p u_0) + \Lambda\varepsilon^\lambda \psi(\varepsilon)\,[1 + \|\mathcal{T}^p u_0\| + \|u_0\|]^\beta \\ &\leq 2d(u_0, \mathcal{T}^l u_0) + (1-\varepsilon) c_p + \Lambda\varepsilon^\lambda \psi(\varepsilon)\left[1 + c_p\right]^\beta \end{aligned}$$

Therefore, we get that

$$\varepsilon c_p \leq A + B\psi(\varepsilon)\varepsilon^\lambda c_p^\lambda.$$

Assuming that c_p is not bounded, we can find a sub-sequence $\left\{c_{p_i}\right\}$ such that $c_{p_i} \to \infty$. Thus, choosing $\varepsilon = \varepsilon_i = \frac{1+A}{c_{p_i}}$, we have

$$\frac{1+A}{c_{p_i}} c_{p_i} \leq A + B\psi(\varepsilon_i)\left(\frac{1+A}{c_{p_i}}\right)^\lambda c_{p_i}^\lambda$$

and then taking into account the properties of ψ

$$1 \leq B\psi(\varepsilon_i)(1+A)^\lambda \to 0$$

we get a contradiction. Because $p > m(u_0)$ were arbitrary chosen, we obtain that $\sup_{p>m(u_0)} \{d(\mathcal{T}^p u_0, u_0)\} < \infty$. Moreover, because from the triangle inequality

$$d(u, v) \leq d(u, u_0) + d(u_0, v),$$

for every $u, v \in O(u_0)$, we deduce that $\omega(u_0) < \infty$. Now, due to the sequence $\{u_n\}$ construction, we have for $n \geq j$,

$$u_n = \mathcal{T}^{m_{n-1}+m_{n-2}+...+m_{j+1}+m_j+...+m_1+m_0} u_0 = \mathcal{T}^{m_{n-1}+...+m_{j+1}+m_j} u_j. \tag{6}$$

Let $u_q, u_r \in O(u_0)$. If we consider a fixed term u_j of the sequence $\{u_n\}$, which precedes u_q, u_r, for some s_1, s_2 we have

$$u_q = \mathcal{T}^{s_1} u_j, \text{ respectively } u_r = \mathcal{T}^{s_2} u_j.$$

Let $\rho_j^q = j^\lambda d(u_q, u_j)$ and $\rho_j^r = j^\lambda d(u_r, u_j)$. By (2), we have

$$\begin{aligned}\rho_j^q = j^\lambda d(u_q, u_j) &\leq j^\lambda d(\mathcal{T}^{s_1} u_j, u_j) = j^\lambda d(\mathcal{T}^{m_{j-1}}(\mathcal{T}^{s_1} u_{j-1}), \mathcal{T}^{m_{j-1}} u_{j-1})\\ &\leq j^\lambda d(\mathcal{T}^{m_{j-1}}(\mathcal{T}^{s_1} u_{j-1}), \mathcal{T}^{m_{j-1}} u_{j-1})\\ &\leq j^\lambda \left[(1-\varepsilon) d(\mathcal{T}^{s_1} u_{j-1}, u_{j-1}) + \Lambda \varepsilon^\lambda \psi(\varepsilon)\left[1 + \left\|\mathcal{T}^{s_1} u_{j-1}\right\| + \left\|u_{j-1}\right\|\right]^\beta\right]\\ &\leq j^\lambda (1-\varepsilon) d(\mathcal{T}^{s_1} u_{j-1}, u_{j-1}) + j^\lambda \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + 2\omega(u_0)]^\beta\end{aligned}$$

Let us denote $\mathcal{K} = \Lambda[1 + 2\omega(u_0)]^\beta$. For each $n \in \mathbb{N}$, we can choose

$$\varepsilon = 1 - \left(\frac{j-1^\lambda}{j}\right) < \frac{\lambda}{j}$$

and then since ψ is increasing

$$\begin{aligned}\rho_j^q &\leq (j-1)^\lambda d(\mathcal{T}^{s_1} u_{j-1}, u_{j-1}) + j^\lambda \mathcal{K} \cdot \left(\tfrac{\lambda}{j}\right)^\lambda \psi\left(\tfrac{\lambda}{j}\right) \leq \rho_{j-1}^q + \mathcal{K} \cdot \lambda^\lambda \psi(\tfrac{\lambda}{j})\\ &\leq \rho_{j-2}^q + \mathcal{K} \cdot \lambda^\lambda \psi\left(\tfrac{\lambda}{j}\right) + \mathcal{K} \cdot \lambda^\lambda \psi(\tfrac{\lambda}{j-1})\\ &\dots\\ &\leq \rho_0^q + \mathcal{K} \cdot \lambda^\lambda \textstyle\sum_{i=1}^{j} \psi\left(\tfrac{\lambda}{i}\right)\\ &= \mathcal{K} \cdot \lambda^\lambda \textstyle\sum_{i=1}^{j} \psi\left(\tfrac{\lambda}{i}\right),\end{aligned} \tag{7}$$

due to $\rho_0^q = 0$. Moreover,

$$d(u_q, u_j) \leq \mathcal{K}\left(\frac{\lambda}{j}\right)^\lambda \sum_{i=1}^{j} \psi\left(\frac{\lambda}{i}\right)$$

Similarly, $\rho_j^r \leq \mathcal{K} \cdot \lambda^\lambda \sum_{i=1}^{j} \psi\left(\frac{\lambda}{i}\right)$ and then

$$d(u_r, u_j) \leq \mathcal{K}\left(\frac{\lambda}{j}\right)^\lambda \sum_{i=1}^{j} \psi\left(\frac{\lambda}{i}\right).$$

By the triangle inequality, we obtain

$$d(u_q, u_r) \leq d(u_q, u_j) + d(u_j, u_r) \leq 2\mathcal{K} \cdot x_j(\lambda),$$

where $x_j(\lambda) = \left(\frac{\lambda}{j}\right)^\lambda \sum_{i=1}^{j} \psi\left(\frac{\lambda}{i}\right) \to 0$. Therefore, $\{u_n\}$ is a Cauchy sequence. Further on, because X^* is complete, we can find $z \in X^*$ such that $u_n \to z$.

Supposing that $\mathcal{T}^{m(z)} z \neq z$, by (2) we have

$$\begin{aligned}d(\mathcal{T}^{m(z)} z, z) &\leq d(\mathcal{T}^{m(z)} z, \mathcal{T}^{m(z)}(\mathcal{T}^p u_0)) + d(\mathcal{T}^{m(z)}(\mathcal{T}^p u_0), z)\\ &\leq (1-\varepsilon) d(z, \mathcal{T}^p u_0) + \Lambda \varepsilon^\lambda \psi(\varepsilon)[1 + \|\mathcal{T}^p u_0\| + \|z\|]^\beta + d(\mathcal{T}^{m(z)}(\mathcal{T}^p u_0), z)\end{aligned}$$

Letting $p \to \infty$ in the above inequality we have

$$d(\mathcal{T}^{m(z)}z, z) \le \Lambda \varepsilon^{\lambda}\psi(\varepsilon)[1 + \|\mathcal{T}^p u_0\| + \|z\|]^{\beta}$$

and taking into account the properties of ψ, when $\varepsilon \to 0$ we get $d(\mathcal{T}^{m(z)}(z), z) \le 0$, which involves

$$d(\mathcal{T}^{m(z)}z, z) = 0.$$

As a consequence, $\mathcal{T}^{m(z)}z = z$. Let us presume now that $\mathcal{T}z \neq z$ and we denote

$$d(z, \mathcal{T}^k z) = \max\{d(z, \mathcal{T}^q z) : 0 < q \le m(z)\}.$$

Thus, we have

$$\begin{aligned} d(z, \mathcal{T}^k z) &\le d(\mathcal{T}^{m(z)}z, \mathcal{T}^k(\mathcal{T}^{m(z)}z)) \\ &\le (1-\varepsilon)d(z, \mathcal{T}^k z) + \Lambda\varepsilon^{\lambda}\psi(\varepsilon)[1 + \|z\| + \left\|\mathcal{T}^k z\right\|]^{\beta} \end{aligned}$$

Taking the limit as $\varepsilon \to 0$ in the above inequality we have $d(z, \mathcal{T}^k z) \le d(z, \mathcal{T}^k z)$, which implies that

$$d(z, \mathcal{T}^k z) = 0.$$

Therefore, z is a fixed point of $\mathcal{T}$.
As a last step, we claim that the fixed point of $\mathcal{T}$ is unique. Suppose now that there are two points $z, w \in X$ such that

$$d(z, \mathcal{T}^{m(z)}z) = z \neq w = d(w, \mathcal{T}^{m(z)}w).$$

Denoting by $\mathcal{C} = [1 + \|z\| + \|w\|]^{\beta}$, for each $\varepsilon \in [0, 1]$, we have

$$d(z, w) = d(\mathcal{T}^{m(z)}z, \mathcal{T}^{m(z)}w) \le (1-\varepsilon)d(z, w) + \mathcal{C}\varepsilon^{\lambda}\psi(\varepsilon).$$

Therefore, taking $\varepsilon = 0$ we have that $d(z, w) = 0$, so that $z = w$. □

Example 2.3. *Let $X = [0, \frac{1}{2}] \cup \{1, 2\}$ be a set endowed with the usual distance $d(u, v) = |u - v|$ and the mapping $\mathcal{T} : X \to X$ be defined as:*

$$\mathcal{T}u = \begin{cases} \frac{u}{2}, & \text{for } u \in [0, \frac{1}{2}] \\ 2, & \text{for } u = 1 \\ \frac{1}{2}, & \text{for } u = 2 \end{cases}$$

Let us note first that for $u = 1$ and $v = 2$ we have $d(\mathcal{T}1, \mathcal{T}2) = d(2, \frac{1}{2}) = \frac{3}{2} > 1 = d(1, 2) = 1$ so that, the inequality (1) does not hold, for $\varepsilon = 0$.

On the other hand, we have $\mathcal{T}^2 u = \begin{cases} \frac{u}{4}, & \text{for } u \in [0, \frac{1}{2}], \\ \frac{1}{2}, & \text{for } u = 1, \\ \frac{1}{4}, & \text{for } u = 2. \end{cases}$

Choosing $\lambda = \beta = 1$, $\Lambda = 3$ and $\psi(\varepsilon) = \varepsilon^2$, we have the following cases:

- $u, v \in [0, \frac{1}{2}]$.

$$\begin{aligned} \alpha(u, v)d(\mathcal{T}^2 u, \mathcal{T}^2 v) &= d(\tfrac{u}{4}, \tfrac{v}{4}) = \tfrac{d(u,v)}{4} \le (1-\varepsilon)d(u, v) + 3\varepsilon^3(1 + d(u, v)) \\ &\le (1-\varepsilon)d(u, v) + 3\varepsilon\psi(\varepsilon)[1 + \|u\| + \|v\|]. \end{aligned}$$

- $u = 1, v \in [0, \frac{1}{2}]$.

$$\begin{aligned}\alpha(1,v)d(\mathcal{T}^2 1, \mathcal{T}^2 v) &= d(\tfrac{1}{2}, \tfrac{v}{4}) = \tfrac{2-v}{4} \leq (1-\varepsilon)d(1,v) + 3\varepsilon^3(1 + d(1,v)) \\ &\leq (1-\varepsilon)d(1,v) + 3\varepsilon\psi(\varepsilon)[1 + \|1\| + \|v\|].\end{aligned}$$

- $u = 2, v \in [0, \frac{1}{2}]$.

$$\begin{aligned}\alpha(2,v)d(\mathcal{T}^2 2, \mathcal{T}^2 v) &= d(\tfrac{1}{4}, \tfrac{v}{4}) = \tfrac{1-v}{4} \leq (1-\varepsilon)d(2,v) + 3\varepsilon^3(1 + d(2,v)) \\ &\leq (1-\varepsilon)d(2,v) + 3\varepsilon\psi(\varepsilon)[1 + \|2\| + \|v\|].\end{aligned}$$

- $u = 1, v = 2$.

$$\begin{aligned}\alpha(1,2)d(\mathcal{T}^2 1, \mathcal{T}^2 2) &= d(\tfrac{1}{2}, \tfrac{1}{4}) = \tfrac{1}{4} \leq (1-\varepsilon)d(1,2) + 3\varepsilon^3(1 + d(2,v)) \\ &\leq (1-\varepsilon)d(1,2) + 3\varepsilon\psi(\varepsilon)[1 + \|1\| + \|2\|].\end{aligned}$$

Consequently, $\mathcal{T}$ is a Pata type contraction at a point and $z = 0$ is the fixed point of $\mathcal{T}$.

3. Conclusion

Note that in Pata's Theorem, a self-mapping is necessarily continuous. Indeed, by letting $\varepsilon = 0$ in the expression (1), we observe that $d(\mathcal{T}u, \mathcal{T}v) \leq d(u,v)$. This yields the continuity of $\mathcal{T}$. In our result, the continuity condition is not necessary anymore. In fact, by letting $\varepsilon = 0$ in (2), we find $d(\mathcal{T}^p u, \mathcal{T}^p v) \leq d(u,v)$ which does not implies the continuity.

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