# Q-FRACTIONAL PROPORTIONAL DERIVATIVES AND Q-LAPLACE TRANSFORMS

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# ABSTRACT

## Q-FRACTIONAL PROPORTIONAL DERIVATIVES AND Q-LAPLACE TRANSFORMS

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**M.Sc. in Mathematics**

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The main goal of this thesis is to develop a new type of q-fractional operators generated from proportional q-differences. To achieve this goal, first the main aspects and tools related to the q and q-fractional calculi are presented. After then, the proportional q-derivative is discussed. The proportional q-fractional differences or derivatives are proposed and the solutions of certain types of q-difference equations embodied by the proportional fractional derivatives are shown in details utilizing the q-Laplace transforms.

**Keywords:** q-proportional fractional derivative, q-fractional calculus, q- Laplace transform.

# ÖZET

## Q-KESİRLİ ORANTISAL TÜREVLER VE Q-LAPLACE DÖNÜŞÜMLERİ

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**Matematik Yüksek Lisans**

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Ağustos 2023, 66 sayfa

Bu çalışmanın temel hedefi q-kesirli orantısal türevleri geliştirmektir. Bu hedefe ulaşmak için q ve q-kesirli analizleriyle ilgili bakış açıları ve araçları gösterilmiştir. Ondan sonra q-orantısal türevler tartışılmıştır. Q-kesirli orantısal türevler gösterildikten sonra bu operatörleri içeren bazı belirli denklemlerin çözümleri q-Laplace dönüşümü kullanılarak, detaylarıyla gösterilmiştir.

**Anahtar Kelimeler:** q-orantısal kesirli türev, q-kesirli kalkülüs, q-Laplace dönüşümü.

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## LIST OF SYMBOLS

**SYMBOLS**

| | |
|---|---|
| $D$ | : Derivative Operator |
| $I$ | : Integral Operator |
| $D_q$ | : q-Derivative Operator |
| $I_q$ | : q-Integral Operator |
| $\varphi, \psi, \Omega$ | : Functions |
| $t$ | : Independent variable |
| $c, d$ | : Constants |
| $e_q$ | : q-analogues of the exponential functions |
| ${}_q\mathcal{L}_s$ | : q-Laplace transformation |
| ${}_qL_s$ | : q-Laplace transformation |
| $E_{\alpha,\theta}^{\gamma}(z)$ | : The generalized version of Mittag-Leffler function |
| $e_{\alpha,\theta}(z;q), E_{\alpha,\theta}(z;q)$ | : The q-analogues of the Mittag-Leffler functions |
| $B_q(y,x)$ | : q-beta function |
| $\Gamma_q(z)$ | : q-gamma function |
| $\|\varphi\|$ | : Norm of $\varphi$ function |
| $\langle\varphi,\psi\rangle$ | : Inner product of $\varphi$ and $\psi$ functions |
| $\Gamma$ | : Gamma Function |
| $*$ | : Convolution |
| $\kappa_0(\beta,t)$ | : Constant of proportional operator |
| $\kappa_1(\beta,t)$ | : Constant of proportional operator |
| ${}^{C}D_q^{\theta}$ | : q-Caputo fractional q-derivative of order $\theta$ |
| $\mathcal{AC}_q[0,b]$ | : The space of all q-absolutely continuous functions defined on $[0,b]$. |

# CHAPTER I

# INTRODUCTION

In the last few decades, there has been a great deal of concern in what so called the fractional calculus which argues about the integrals and derivatives of non-integer order and thus it generalizes the classical calculus [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. This calculus turned out to be a good tool for modeling some real world problems. It was applied in physics by Hilfer [4] and was used in biomedical engineering by Magin [12]. In addition, many fractional mathematical models and their numerical techniques were given by Baleanu et al. [11] and Li et al. [13].

One of the best peculiarity of the fractional calculus is the fact that there are many types of the fractional operators. This enables a researcher to choose the most appropriate operator for the model under consideration. In spite of the diversity of the fractional operators, researchers have not stopped seeking for new fractional operators [14, 15, 16, 17]. There are also types of local fractional derivatives. One of these derivatives is the one called the conformable derivative proposed by Khalil et al. [18, 19, 20]. But the problem in this derivative is that it does not give the function itself when the order is 0. To overpass this problem, Anderson [21] proposed a modified version of the conformable derivative which give the function itself when the order is zero and the classical derivative when the order is one, and called it a proportional derivative. Jarad et al. [22, 23, 24, 25, 26, 27] launched out the nonlocal fractional versions of the proportional derivative.

The quantum calculus is a field of the derivative of the function that can be calculated without the asisstance of the limit process and many mathematicians worked on q-calculus and their applications [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. Moreover, the quantum calculus was applied with impulsive difference equations in [30], introduced with the operator theory in [33] and correlated with q-hypergeometric function in [35].

The fractional versions of the q-derivatives and integrals were discussed in details by [37, 42].

The aim of this master thesis is to introduce a new q-fractional operator called q-fractional proportional operator and to solve related q-fractional proportional differential equations with the help of q-Laplace transforms.

The overview of this thesis is as follows:

The proportional derivative is mentioned in Chapter 2. The concept of q-calculus is reviewed in Chapter 3. The q-proportional derivative and q-Laplace transform are provided in Chapter 4. The q-fractional proportional derivative and some related q-Laplace transforms are defined in Chapter 5. Chapter 6 is devoted to our conclusion.

# CHAPTER II

# PROPORTIONAL DERIVATIVE

The scope of this chapter is to provide the fundamentals and the main properties of the powerful fractional proportional derivative.

## 2.1 DEFINITION OF THE PROPORTIONAL DERIVATIVE

The proportional derivative is related with many fields of science and engineering and was used by many mathematicians [22, 27]. Anderson proposed the proportional derivative as a modification of the conformable derivatives [21]. It can also be defined with different time scales [43]. In fact, the proportional derivative is defined as

$$\mathrm{D}^{\beta}\varphi(t) = \kappa_1(\beta,t)\varphi(t) + \kappa_0(\beta,t)\varphi'(t), \tag{2.1.1}$$

where $\beta \in [0,1]$ and $\kappa_0, \kappa_1 \colon [0,1] \times \mathbb{R} \to [0,\infty)$ are continuous such that for all $t \in \mathbb{R}$,

$$\lim_{\beta\to 0^+} \kappa_1(\beta,t) = 1,$$

$$\lim_{\beta\to 0^+} \kappa_0(\beta,t) = 0,$$

$$\lim_{\beta\to 1^-} \kappa_1(\beta,t) = 0,$$

$$\lim_{\beta\to 1^-} \kappa_0(\beta,t) = 1$$

and $\kappa_1(\beta,t) \neq 0$, $\beta \in [0,1)$, $\kappa_0(\beta,t) \neq 0$, $\beta \in (0,1]$. For a special interest, the case $\kappa_1(\beta,t) = 1-\beta$ and $\kappa_0(\beta,t) = \beta$ will be considered. Therefore, (2.1.1) becomes (see[22]):

$$\mathrm{D}^{\beta}\varphi(t) = (1-\beta)\varphi(t) + \beta\varphi'(t). \tag{2.1.2}$$

## 2.2 SOME PROPERTIES OF PROPORTIONAL DERIVATIVE

Assume that the function $\varphi:\mathbb{R}\to\mathbb{R}$ is continuous and $\kappa_0,\kappa_1:[0,1]\times\mathbb{R}\to[0,\infty)$ are continuous and satisfying the conditions in (2.1.1) . Let $\varphi$ and $\psi$ be differentiable functions on $\mathbb{R}$. Then,

**Property 1.** $D^\beta[c\varphi+d\psi]=cD^\beta\varphi+dD^\beta\psi$ for all $c,d\in\mathbb{R}$;

**Property 2.** $D^\beta c=c\kappa_1(\beta,.)$ for all $c\in\mathbb{R}$;

**Property 3.** $D^\beta(\varphi\psi)=\varphi D^\beta\psi+\psi D^\beta\varphi-\varphi\psi\kappa_1(\beta,.)$;

**Property 4.** $D^\beta(\varphi/\psi)=\frac{\psi D^\beta\varphi-\varphi D^\beta\psi}{\psi^2}+\frac{\varphi}{\psi}\kappa_1(\beta,.)$;

**Property 5.** For $\beta\in(0,1], m\in\mathbb{R}$, we have

$$D_t^\beta[e_n(t,m)]=n(t)e_n(t,m) \tag{2.2.1}$$

and

$$e_n(t,m):=e^{\int_m^t\frac{n(\tau)-\kappa_1(\beta,\tau)}{\kappa_0(\beta,\tau)}d\tau},$$

$$e_0(t,m):=e^{-\int_m^t\left(\frac{\kappa_1(\beta,\tau)}{\kappa_0(\beta,\tau)}\right)d\tau}.$$

**Property 6.** For $\beta\in(0,1]$, we have

$$D^\beta\left[\int_a^t\frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)}dm\right]=\varphi(t). \tag{2.2.2}$$

For a better understanding of this concept , the proofs are given in below.

**Proof of Property 1[21]:**

By direct calculations, one may conclude that

$$\begin{aligned} D^\beta[c\varphi+d\psi]&=\kappa_0[c\varphi+d\psi]'+\kappa_1\varphi\psi\\ &=c\kappa_0\varphi'+\varphi\kappa_1\psi+d\kappa_0\psi'+d\kappa_1\varphi+\kappa_1\varphi\psi-\kappa_1\varphi\psi\\ &=cD^\beta\varphi+dD^\beta\psi\end{aligned}$$

■

**Proof of Property 2[21]:**

By using the definition of the operator, it is obvious that

$$D^\beta c=c\kappa_1(\beta,.),\qquad c\in\mathbb{R}$$

■

**Proof of Property 3[21]:**

Operating $D^\beta$ on $(\varphi\psi)$, one gets

$$\begin{aligned} D^\beta(\varphi\psi)&=\kappa_0(\varphi\psi'+\varphi'\psi)+\kappa_1\varphi\psi\\ &=(\varphi\kappa_0\psi'+\varphi\kappa_1\psi)+(\psi\kappa_0\varphi'+\psi\kappa_1\varphi)-\varphi\psi\kappa_1\\ &=\varphi D^\beta\psi+\psi D^\beta\varphi-\varphi\psi\kappa_1.\end{aligned}$$

■

**Proof of Property 4 [21]:**

Applying the operator $D^{\beta}$ on the ratio $\varphi/\psi$ , one concludes that

$$D^{\beta}[\varphi/\psi] = \frac{\kappa_0(\varphi'\psi - \varphi\psi')}{\psi^2} + \frac{\varphi}{\psi}\kappa_1(\beta,.)$$

$$= \frac{\varphi'\kappa_0\psi - \varphi\kappa_0\psi'}{\psi^2} + \frac{\varphi}{\psi}\kappa_1(\beta,.)$$

$$= \frac{\psi D^{\beta}\varphi - \varphi D^{\beta}\psi}{\psi^2} + \frac{\varphi}{\psi}\kappa_1(\beta,.)$$

■

**Proof of Property 5 [21]:** Starting with

$D_t^{\beta} e_n(t,m) = n(t)e_n(t,m)$, one gets

$$D_t^{\beta} e_n(t,m) = \kappa_0(\beta,t)\left(\frac{n(t) - \kappa_1(\beta,t)}{\kappa_0(\beta,t)}\right)e_n(t,m) + \kappa_1(\beta,t)e_n(t,m)$$

$$= \big(n(t) - \kappa_1(\beta,t)\big)e_n(t,m) + \kappa_1(\beta,t)e_n(t,m)$$

$$= n(t)e_n(t,m) - \kappa_1(\beta,t)e_n(t,m) + \kappa_1(\beta,t)e_n(t,m)$$

$$= n(t)e_n(t,m)\ .$$

so, $D_t^{\beta} e_n(t,m) = n(t)e_n(t,m)$ is obtained. ■

**Proof of Property 6 [21]:** For $\beta \in (0,1]$, one obtains the following

$D^{\beta}\left[\int_a^t \frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)} dm\right] = \varphi(t)$. Therefore,

$$D^{\beta}\left[\int_a^t \frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)} dm\right] =$$

$$= \kappa_0(\beta,t)\frac{d}{dt}\left(\int_a^t \frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)} dm\right) + \kappa_1(\beta,t)\int_a^t \frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)} dm$$

$$= \kappa_0(\beta,t)\left(-\frac{\kappa_1(\beta,t)}{\kappa_0(\beta,t)}\int_a^t \frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)} dm + \frac{\varphi(t)e_0(t,t)}{\kappa_0(\beta,t)}\right)$$

$$+ \kappa_1(\beta,t)\int_a^t \frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)} dm$$

$$= -\kappa_1(\beta,t)\int_a^t \frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)}dm + \varphi(t)e_0(t,t) + \kappa_1(\beta,t)\int_a^t \frac{\varphi(m)e_0(t,m)}{\kappa_0(\beta,m)}dm$$

$$= \varphi(t)e_0(t,t)$$

$$= \varphi(t)$$

■

# CHAPTER III

# Q-CALCULUS

In this chapter, a brief review of q-calculus is presented.

## 3.1 STANDARD EXPRESSIONS OF Q-DERIVATIVE AND Q-INTEGRAL

The q-differential form of a function $\varphi$ is

$$d_q\varphi(t) = \varphi(qt) - \varphi(t). \tag{3.1.1}$$

This is similar to the Nabla-difference operator given by $\nabla\varphi(t) = \varphi(t+1) - \varphi(t)$. Let $\sigma \in \mathbb{R}$ be fixed, and $B$ be subset of $\mathbb{C}$. $B$ is defined to be $\sigma$-geometric if $\sigma t \in B$ whenever $t \in B$, [37]. Also, in case $B$ of $\mathbb{C}$ is a $\sigma$-geometric, it must absorb all geometric sequences in the form $\{t\sigma^m\}_{m=0}^{\infty}$ (see [37]), $t \in B$. If real or complex valued function $\varphi$ is given on a $q$-geometric set $B, |q| \neq 1$ [37], the q-derivative of the function $\varphi(t)$ is shown as [28, 29, 34, 37, 39]

$$D_q\varphi(t) = \frac{\varphi(qt) - \varphi(t)}{qt - t}$$

or

$$D_q\varphi(t) = \frac{\varphi(qt) - \varphi(t)}{(q-1)t}, \quad (0 < q < 1). \tag{3.1.2}$$

The same definition was presented in [40-41]. Particularly, (3.1.2) is called the Jackson q-difference operator, the Euler-Jackson q-difference operator or the Euler-Heine-Jackson q-difference operator. When $0 \in B$ and for $|q| < 1$, the q-derivative at zero is given as [37]

$$D_q\varphi(0) = \lim_{m\to\infty} \frac{\varphi(tq^m) - \varphi(0)}{tq^m}, \qquad t \in B\backslash\{0\}, \tag{3.1.3}$$

where the limit is said to exist and it does not depend on $t$ [37]. When it is assumed that $|q| > 1$, the q-derivative at zero is introduced by (see [37]);

$$D_q\varphi(0) = D_{q^{-1}}\varphi(0).$$

The following are some properties of the q-derivative [28, 29].

**1) (Linearity Property)**

$$D_q\big(c\varphi(t) + d\psi(t)\big) = cD_q\varphi(t) + dD_q\psi(t), \quad c, d \in \mathbb{R} \tag{3.1.4}$$

**Proof:**

$$D_q\big(c\varphi(t) + d\psi(t)\big) = \frac{\big(c\varphi(qt) + d\psi(qt)\big) - \big(c\varphi(t) + d\psi(t)\big)}{qt - t}$$

$$= \frac{c\varphi(qt) + d\psi(qt) - c\varphi(t) - d\psi(t)}{qt - t}$$

$$= c\left(\frac{\varphi(qt) - \varphi(t)}{qt - t}\right) + d\left(\frac{\psi(qt) - \psi(t)}{qt - t}\right)$$

and $(0 < q < 1)$

$$= cD_q\varphi(t) + dD_q\psi(t)$$

■

**2) Q-Derivative of a Product**

$$D_q\big(\varphi(t)\psi(t)\big) = \varphi(qt)D_q\psi(t) + \psi(t)D_q\varphi(t). \tag{3.1.5}$$

**Proof:** (3.1.5) is proved as

$$D_q\big(\varphi(t)\psi(t)\big) = \frac{\varphi(qt)\psi(qt) - \varphi(t)\psi(t)}{qt - t}$$

$$= \frac{\varphi(qt)\psi(qt) - \varphi(qt)\psi(t) + \varphi(qt)\psi(t) - \varphi(t)\psi(t)}{qt - t}$$

$$= \varphi(qt)\left(\frac{\psi(qt) - \psi(t)}{qt - t}\right) + \psi(t)\left(\frac{\varphi(qt) - \varphi(t)}{qt - t}\right)$$

$$= \varphi(qt)D_q\psi(t) + \psi(t)D_q\varphi(t)\ , \quad (0 < q < 1).$$

■

**3) Q-Derivative of a Quotient**

$$D_q\left(\frac{\varphi(t)}{\psi(t)}\right) = \frac{\psi(t)D_q\varphi(t) - \varphi(t)D_q\psi(t)}{\psi(t)\psi(qt)}. \tag{3.1.6}$$

**Proof:** (3.1.6) is demonstrated as;

$$D_q\left(\frac{\varphi(t)}{\psi(t)}\right) = \frac{\frac{\varphi(qt)}{\psi(qt)} - \frac{\varphi(t)}{\psi(t)}}{qt - t}$$

$$= \frac{\varphi(qt)\psi(t) - \varphi(t)\psi(qt)}{\psi(qt)\psi(t)(qt - t)}$$

$$= \frac{\varphi(qt)\psi(t) - \varphi(t)\psi(t) + \varphi(t)\psi(t) - \varphi(t)\psi(qt)}{\psi(qt)\psi(t)(qt-t)}$$

$$= \frac{\psi(t)\left(\frac{\varphi(qt)-\varphi(t)}{qt-t}\right) - \varphi(t)\left(\frac{\psi(qt)-\psi(t)}{qt-t}\right)}{\psi(qt)\psi(t)}$$

$$= \frac{\psi(t)D_q\varphi(t) - \varphi(t)D_q\psi(t)}{\psi(qt)\psi(t)}, \qquad (0 < q < 1).$$

■

The above proofs were handled keeping in mind that the function of $\varphi$ and $\psi$ are continuous. Similar properties can be implemented to q-integral. In fact, the q-integral is introduced (see [30]) as

$\varphi: A_t \to \mathbb{R}$ and $A_t = \{tq^n : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and

$$I_q\varphi(t) = \int_0^t \varphi(s)d_qs = \sum_{m=0}^{\infty} t(1-q)q^m\varphi(tq^m), \tag{3.1.7}$$

when $c, d \in A_t$. So that,

$$\int_c^d \varphi(s)d_qs = I_q\varphi(d) - I_q\varphi(c), \tag{3.1.8}$$

$$\int_c^d \varphi(s)d_qs = (1-q)\sum_{m=0}^{\infty} q^m[d\varphi(dq^m) - c\varphi(cq^m)], \tag{3.1.9}$$

and $(0 < q < 1)$ in (3.1.8) and (3.1.9).

According to (3.1.8) and (3.1.9) (see [30]), the result is obtained.

**4) (Linearity Property)**

$$I_q(\varphi(t) + \psi(t)) = I_q\varphi(t) + I_q\psi(t). \tag{3.1.10}$$

**Proof of Linearity Property:** (3.1.10) is shown as;

$$I_q(\varphi(t) + \psi(t)) = \int_0^t (\varphi(s) + \psi(s))d_qs$$

$$= \int_0^t \varphi(s)d_qs + \int_0^t \psi(s)d_qs$$

$$= \sum_{m=0}^{\infty} t(1-q)q^m\varphi(tq^m) + \sum_{m=0}^{\infty} t(1-q)q^m\psi(tq^m)$$

$$= I_q\varphi(t) + I_q\psi(t).$$

■

**5) (Linearity Property)**

$$I_q(\varphi(t) - \psi(t)) = I_q\varphi(t) - I_q\psi(t). \tag{3.1.11}$$

**Proof of Linearity Property:** From (3.1.11) it is evident that

$$I_q(\varphi(t) - \psi(t)) = \int_0^t (\varphi(s) - \psi(s))d_q s$$

$$= \int_0^t \varphi(s)d_q s - \int_0^t \psi(s)d_q s$$

$$= \sum_{m=0}^{\infty} t(1-q)q^m\varphi(tq^m) - \sum_{m=0}^{\infty} t(1-q)q^m\psi(tq^m)$$

$$= I_q\varphi(t) - I_q\psi(t).$$

■

**Definition 3.1.1:** Let $\varphi$ be a function defined on a q-geometric set $B$. $\varphi$ is q-integrable on $B$ if and only if $\int_0^m \varphi(t)d_q t$ exists for every $m \in B$, (see [37]).

**Definition 3.1.2:** If $\lim_{m\to\infty} \varphi(tq^m) = \varphi(0)$ for every $m \in B$ then $\varphi$ that is described on a q-geometric set $B$, $0 \in B$ is q-regular at zero. In addition to this, when $B$ is also $q^{-1}$-geometric, $\varphi$ is q-reqular at infinity when there exists a constant $c$, (see [37]) such that; $\lim_{m\to\infty} \varphi(tq^{-m}) = c$ for all $t \in B$. Now, $\varphi(0^+)$ and $\varphi(0^-)$ are shown as;

$$\varphi(0^+) = \lim_{\substack{\iota\to\infty \\ n>0}} \varphi(nq^\iota), \qquad \varphi(0^-) = \lim_{\substack{\iota\to\infty \\ n<0}} \varphi(nq^\iota).$$

When $B \subseteq \mathbb{R}$ is q-geometric and $\varphi$ is a q-regular at zero function (see [37]). As a result, q-regularity at zero plays the role of continuity in the classical sense in some settings (see [37]). But, continuity at zero implies q-regularity at zero. However, the converse cannot be true.

## 3.2 MORE PROPERTIES OF THE Q-INTEGRALS

The q-integration by parts is given as, (see [37]),

$$\int_0^b \psi(t)D_q\varphi(t)d_q t = (\varphi\psi)(b) - \lim_{m\to\infty}(\varphi\psi)(bq^m) - \int_0^b D_q\psi(t)\varphi(qt)d_q t, \tag{3.2.1}$$

where $\varphi$ and $\psi$ are q-regular at zero and limit could be changed by $(\varphi\psi)(0)$.

**Theorem 3.2.1:** Let $\varphi$ be a q-regular at zero defined on a q-geometric set B involving zero. Then

$$\bar{\varphi}(z) = \int_a^z \varphi(t) d_q t, \qquad (z \in B)$$

where, $\bar{\varphi}$ is q-regular at zero such that $D_q \bar{\varphi}(z) = \varphi(z)$ for all $z \in B$. In addition, if $c$ and $d$ are two different points in B, then (see [37]);

$$\int_c^d D_q \varphi(t) d_q t = \varphi(d) - \varphi(c). \tag{3.2.2}$$

**Theorem 3.2.2:** [37] Let $\varphi$ be a function defined on $[c, d], 0 \leq c \leq d$. Suppose that there exists a number $\alpha,\ 0 \leq \alpha < 1$ such that $t^{\alpha} \varphi(t)$ is continuous on $[c, d]$. Let,

$$\bar{\varphi}(z) = \int_a^z \varphi(t) d_q t, \quad z \in [c, d],$$

where $a$ is a fixed point in $[c, d]$, (see [37]). Then, $\bar{\varphi}(z)$ is a continuous function in $[c, d]$.

**Proof of Theorem 3.2.2:** The proof is shown as [37],

$\psi(z) = z^{\alpha} \varphi(z)$ for every $z \in [c, d]$. $z_0$ is fix and $z_0 \in [c, d]$. Consider that $z_0 \neq 0$. Later,

$$\bar{\varphi}(z) - \bar{\varphi}(z_0) = (1 - q) \sum_{s=0}^{\infty} z q^s \varphi(z q^s) - (1 - q) \sum_{s=0}^{\infty} z_0 q^s \varphi(z_0 q^s)$$

$$= (1 - q) z^{1-\alpha} \sum_{s=0}^{\infty} z q^{s(1-\alpha)} [\psi(z q^s) - \psi(z_0 q^s)]$$

$$+ z_0^{\alpha} (z^{1-\alpha} - z_0^{1-\alpha}) (1 - q) \sum_{s=0}^{\infty} \psi^s \psi(z_0 q^s) \tag{3.2.3}$$

Because, $\psi(x)$ is continuous on $[c, d], \psi(x)$ is uniformly continuous on $[c, d]$, (see [37]). Therefore, $\delta > 0$ exists for every $\epsilon > 0$ in fact for every $z, w \in [c, d]$ (see [37]),

$$|z - w| < \delta \to |g(z) - g(w)| < \epsilon.$$

So, when $z \in [c, d]$, (see [37]),

$|z - z_0| < \delta$ then $|z q^s - z_0 q^s| < \delta$ for every $s \in \mathbb{N}_0$, and in [37],

$|\psi(z q^s) - \psi(z_0 q^s)| < \epsilon$ for every $s \in \mathbb{N}_0$. Therefore, $\lim_{z \to z_0} \psi(z q^s) = \psi(z_0 q^s)$ uniformly in $s$ and $\lim_{z \to z_0}$ can be applied on the series. Together with limit as $z \to z_0$

on the series could be introduced on (3.2.3) to satisfy $\lim_{z\to z_0} \bar{\varphi}(z) = \bar{\varphi}(z_0)$. Also, consider that $z_0 = 0$. So,

$$\bar{\varphi}(z) - \bar{\varphi}(0) = \int_0^z \varphi(t) d_q t = \int_0^z t^{1-\alpha}(\psi(t) - \psi(0)) + \frac{1-q}{1-q^{2-\alpha}} z^{2-\alpha}\psi(0).$$

As a result,

$$|\bar{\varphi}(z) - \bar{\varphi}(0)| \leq \left(\max_{s\in\mathbb{N}_0} |\psi(zq^s) - \psi(0)| + \psi(0)\right) \frac{1-q}{1-q^{2-\alpha}} z^{2-\alpha}.$$

Since, $\psi$ is continuous at 0 and $0 < \alpha < 1$, $\lim_{z\to 0} \bar{\varphi}(z) = \bar{\varphi}(0)$ is obtained. So, $\bar{\varphi}(z)$ is continuous on $[c, d]$.

■

**Lemma 3.2.1:** In [37], let $u(t, s)$ be a function defined on $[0, b] \times [0, b]$ in fact for every fixed $t$ the functions (see [37]),

$$D_{q,z}^{j} u(t, z) \qquad (j = 0,1,2, \dots, s-1)$$

are q-integrable on $[0, b]$. When for some $z \in (0, b]$ and $s \in \mathbb{N}$ then

$$u(zq^v, zq^j) = 0, \qquad (v = 0,1,2, \dots, j-1;\ j = 1,2, \dots, s) \qquad (3.2.4)$$

then

$$D_{q,z}^{s} \int_0^z u(t, z) d_q t = \int_0^z D_{q,z}^{s} u(t, z) d_q t.$$

**Proof of Lemma 3.2.1:** The m-th order q-derivative, $D_q^m$, of a function $\varphi$ could be given as its values at the points $\{q^j z,\ j = 0,1, \dots, m\}$ through the identity,

$$D_q^m \varphi(z) = (-1)^m (1-q)^{-m} z^{-m} q^{-m(m-1)/2} \sum_{v=0}^{m} (-1)^v \begin{bmatrix} m \\ v \end{bmatrix}_q q^{v(v-1)/2} \varphi(zq^{m-v})$$

for all $z$ in $B\{0\}$. Then,

$$D_{q,z}^{s} \int_0^z u(t, z) d_q t = \sum_{j=0}^{j=s} (-1)^j \begin{bmatrix} s \\ j \end{bmatrix}_q \frac{q^{\frac{j(j+1)}{2} - sj}}{z^s (1-q)^s} \int_0^{zq^j} h(t, zq^j) d_q t, \qquad (3.2.5)$$

and the means of $\varphi$ obtains (3.2.4) implies

$$\int_0^{zq^j} u(t, zq^j) d_q t = \int_0^z u(t, zq^j) d_q t, \quad j = 1,2, \dots s.$$

Therefore,

$$D_{q,z}^{s}\int_{0}^{z} u(t,z)d_qt = \sum_{j=0}^{j=s}(-1)^j \begin{bmatrix} s \\ j \end{bmatrix}_q \frac{q^{\frac{j(j+1)}{2}-sj}}{z^s(1-q)^s}\int_{0}^{z} u(t,zq^j)d_qt$$

$$= \int_{0}^{z}\left(\sum_{j=0}^{j=s}(-1)^j \begin{bmatrix} s \\ j \end{bmatrix}_q \frac{q^{\frac{j(j+1)}{2}-sj}}{z^s(1-q)^s}u(t,zq^j)\right)d_qt$$

$$= \int_{0}^{z} D_{q,z}^{s}u(t,z)d_qt.$$

■

**Lemma 3.2.2: [37]** Assume that $u(t,z)$ is defined on $[b,\infty)\times[b,\infty)$ such that for every fixed $t$ the functions,

$$D_{q,z}^{j}u(t,z), \quad (j = 0,1,\dots,s-1)$$

are q-integrable on $[b,\infty)$. If for some $x$ is defined on $[b,\infty)$ and $s \in \mathbb{N}$

$u(zq^v, zq^j) = 0, \quad (v = 0,1,2,\dots,j-1; j = 1,2,\dots,s),$

then

$$D_{q,z}^{s}\int_{z}^{\infty} u(t,z)d_qt = \int_{z}^{\infty} D_{q,z}^{s}u(t,z)d_qt.$$

**Lemma 3.2.3:** [37] Let $I$ and $J$ be intervals including zero such that $J \subseteq I$. Assume that $\varphi_m, \varphi$ are functions defined on $I$, $m \in \mathbb{N}$, such that for every $t \in I$, $\varphi_m$ tends uniformly to $\varphi$ on $J$; i.e.

$$\lim_{m\to\infty}\varphi_m(t) = \varphi(t), \tag{3.2.6}$$

Then,

$$\lim_{m\to\infty}\int_{0}^{z}\varphi_m(t)d_qt = \int_{0}^{z}\varphi(t)d_qt, \qquad \forall z \in I, \qquad (3.2.7).$$

## 3.3 IMPORTANT SPACES, FUNCTIONS AND NOTATIONS

### 3.3.1 Function Spaces

Let $1 \le p < \infty$, $b > 0$ and $\xi \in \mathbb{R}$. If $L_{q,\xi}^{p}(0,b)$ is the space of all equivalence classes of functions having the property ( [37]);

$$\int_{0}^{b} t^{\xi}|\varphi(t)|^p d_qt < \infty.$$

The norm defined on space $L_{q,\xi}^{p}(0,b)$ [37]

$$\|\varphi\|_{p,\xi,b} = \left(\int_0^b t^{\xi} |\varphi(t)|^p d_q t\right)^{1/p}$$

is a Banach space. In addition to this, when $p = 2$, the space $L^2_{q,\xi}(0,b)$ connected to the inner product

$$\langle\varphi,\psi\rangle = \int_0^b t^{\xi}\varphi(t)\bar{\psi}(t) d_q t, \quad (\varphi,\psi \in L^2_{q,\xi}(0,b)), \tag{3.3.1.1}$$

is a seperable Hilbert space. In this case, an orthonormal basis of $L^2_{q,\xi}(0,b)$ is defined as

$$\varphi_n(x) = \begin{cases} \dfrac{1}{\sqrt{x^{\xi+1}(1-q)}}, & x = bq^n, \quad n \in \mathbb{N}_0, \\ 0, & \text{otherwise} \end{cases} \tag{3.3.1.2}$$

Especially, the case of $\xi = 0$ was shown in [44]. Also, for every $0 < q < 1$ and $0 < a < \infty$, the above mentioned Hilbert spaces are given by

$$L^p(\mathbb{R}_{a,q}) = \left\{\varphi: \int_{-\infty/a}^{\infty/a} |\varphi(t)|^p d_q t < \infty\right\}, \quad (p \geq 1)$$

The notation $\mathbb{R}_q$ and $\widetilde{\mathbb{R}}_q$ should be used to demonstrate $\mathbb{R}_{1,q}$ and $\mathbb{R}_{\sqrt{1-q},q}$ in turn (see [37]), the inner product of $L^2(\mathbb{R}_{a,q})$ is

$$\langle\varphi,\psi\rangle = \int_{-\infty/a}^{\infty/a} \varphi(t)\overline{\psi(t)}\, d_q t, \quad \left(\varphi,\psi \in L^2(\mathbb{R}_{a,q})\right)$$

is a Hilbert space.

**Definition 3.3.1.1:** Let $\xi$ be a real number and $p$ be a positive number, then the space $\ell^p_{q,\xi}[0,b]$ is the space of all functions $\varphi$ identified on $(0,b]$ obtaining

$$\|\varphi\|_{p,\xi} = sup_{y\in(0,b]}\left(\int_0^y t^{\xi}|\varphi(t)|^p d_q t\right)^{1/p} < \infty. \tag{3.3.1.3}$$

The symbols $L^p_q[0,b], \ell^p_q[0,b]$ and $\|\varphi\|_p$ are used to denote $L^p_{q,0}[0,b], \ell^p_{q,0}[0,b]$ and $\|\varphi\|_{p,0}$ (see [37]).

**Proposition 3.3.1.1:** $\left(\ell_{q,\xi}^{p}[0,b],\ \|.\|_{p,\xi}\right)$ is a Banach space.

**Proof of Proposition 3.3.1.1:** It is enough to show that $\left(\ell_{q,\xi}^{p}[0,b],\ \|.\|_{p,\xi}\right)$ is a normed space directly. Consider $(\varphi_n)_n$ is a Cauchy sequence in $\left(\ell_{q,\xi}^{p}[0,b],\ \|.\|_{p,\xi}\right)$. Therefore, $n_0 \in \mathbb{N}$ exists for every $\epsilon > 0$ such that for every $n, m \in \mathbb{N}$.

$$n, m > n_0 \to sup_{y\in(0,b]} \sum_{s=0}^{\infty} (xq^s)^{\xi+1}(1-q)|\varphi_n(xq^s) - \varphi_m(xq^s)|^p < \epsilon \quad (3.3.1.4)$$

so, $x^{\frac{\xi+1}{p}}\varphi_n(x)$ is a uniformly Cauchy sequence on (0,b], (see [37]). Hence, $\varphi$ which exists, identified on (0,b] such that, [37];

$\lim_{n\to\infty} x^{\frac{\xi+1}{p}}\varphi_n(x) = x^{\frac{\xi+1}{p}}\varphi(x)$ is uniformly on $(0, b]$. Also, for the fix $M > 0$ and $n > n_0$, (3.3.1.4) is written as (see [37]);

$$m > n_0 \to \sum_{s=0}^{M} (xq^s)^{\xi+1}(1-q)|\varphi_n(xq^s) - \varphi_m(xq^s)|^p < \epsilon \quad \forall x \in (0,b], \quad (3.3.1.5)$$

Later limit is calculated as $m \to \infty$ on (3.3.1.5) gives for every $M > 0$ and $n > n_0$ (see [37]);

$$\sum_{s=0}^{M} (xq^s)^{\xi+1}(1-q)|\varphi_n(xq^s) - \varphi(xq^s)|^p \le \epsilon, \qquad \forall x \in [0,b].$$

Therefore, when $n \to \infty$ then $\|\varphi_n - \varphi\|_{p,\xi} \to 0$. As a result,

$$\varphi_{n_0+1} - \varphi \in \left(\ell_{q,\xi}^{p}[0,b], \|.\|_{p,\xi}\right)$$

and because $\varphi_{n_0+1} \in \left(\ell_{q,\xi}^{p}[0,b], \|.\|_{p,\xi}\right)$, then similarly $\varphi \in \left(\ell_{q,\xi}^{p}[0,b], \|.\|_{p,\xi}\right)$.

**Definition 3.3.1.2:** [37] Let $H_h(B)$ be the space of all functions determined on $B$ when $\varphi \in H_h(B)$. Then there exists $c > 0$ such that;

$$|\varphi(x) - \varphi(0)| < c|x|^h, \quad \forall x \in B.$$

**Definition 3.3.1.3:** [37] $L_q^2\big((0,b)\times(0,b)\big)$ is defined to be the space for every complex-valued functions $\varphi(x,t)$ defined on $[0,b]\times[0,b]$ such that,

$$\|\varphi(.,.)\|_2 = \left(\int_0^b \int_0^b |\varphi(x,t)|^2 d_q x d_q t\right)^{1/2} < \infty.$$

**Lemma 3.3.1.1:** [37] $L_q^2\big((0,b)\times(0,b)\big)$ related with the inner product;

$$\langle\varphi,\psi\rangle_2=\int_0^b\int_0^b\varphi(x,t)\overline{\psi(x,t)}d_qxd_qt$$

is a seperable Hilbert space.

**Proof of Lemma 3.3.1.1:** It is similar to the proof of lemma 3.3.1.1. It was noted in [44] that $L_q^2\big((0,b)\times(0,b)\big)$ is a Banach space. It is enough now to show separability:

$$\Omega_{ij}(x,t)=\Omega_i(x)\Omega_j(t),\quad (i,j=1,2,\dots)$$

and $\Omega$ is an ortonormal basis of $L_q^2\big((0,b)\times(0,b)\big)$ at anytime $\{\Omega_i(.)\}_{i=1}^{\infty}$ is an orthonormal basis of $L_q^2(0,b)$. Actually,

$$\begin{aligned}\langle\Omega_{jk},\Omega_{mn}\rangle_2&=\int_0^b\int_0^b\Omega_j(x)\Omega_k(t)\overline{\Omega_m(x)\Omega_n(t)}d_qxd_qt\\&=\int_0^b\Omega_j(x)\overline{\Omega_m(x)}d_qx\int_0^b\Omega_k(t)\overline{\Omega_n(t)}d_qt\\&=w_{jm}w_{kn},\end{aligned}$$

showing orthogonality. When $\varphi\in L_q^2\big((0,b)\times(0,b)\big)$ exists $\{\Omega_{ij}\}$ can be proven to be a basis, such that $\langle\varphi,\Omega_{ij}\rangle_2=0$ for every $i,j\in\mathbb{N}_0$, then $\varphi$ is the zero element (see [37]). Mainly,

$$\begin{aligned}0=\langle\varphi,\Omega_{ij}\rangle&=\int_0^b\int_0^b\varphi(x,t)\overline{\Omega_i(x)\Omega_j(t)}\,d_qx\,d_qt\\&=\int_0^b\overline{\Omega_j(t)}\left(\int_0^b\varphi(x,t)\overline{\Omega_i(x)}\,d_qx\right)d_qt\\&=\int_0^b u(t)\overline{\Omega_j(t)}\,d_qt.\end{aligned}$$

Hence,

$$u(t)=\int_0^b\varphi(x,t)\overline{\Omega_i(x)}d_qx$$

is orthogonal to the $\Omega_j$'s implying that $\varphi(bq^n)=0$, for every $n\in\mathbb{N}_0$, (see [37]). Therefore, from the above proof, $\varphi(x,bq^n)$ is orthogonal to each $\Omega_i$, (see [37]). As a result, $\varphi(bq^m,bq^n)=0$, for every $m,n\in\mathbb{N}_0$. ■

**Definition 3.3.1.4:** [37] $C_q^n[b,a]$ is defined as the space of all continuous functions that has continuous q-derivative up to order $n-1$ on interval. Also, $C_q^n[b,a]$ related with the norm function

$$\|\varphi\| = \sum_{s=0}^{n-1} \max_{b\le x\le a} \left|D_q^s \varphi(t)\right|, \qquad (\varphi \in C_q^n[b,a])$$

is a Banach space.

**Lemma 3.3.1.2:** $\left(C_q^n[b,a], \|.\|\right), n \in \mathbb{N}$, is a Banach space.

**Proof of Lemma 3.3.1.2:** In [37], let $(\varphi_m)_m$ be a Cauchy sequence in $C_q^n[b,a]$ and later, for every $\epsilon > 0, n_0 \in \mathbb{N}$ exists such that for every $k,m \in \mathbb{N}$,

$$k,m > n_0 \to \sum_{s=0}^{n-1} \max_{x\in[b,a]} \left|D_q^s \varphi_k(x) - D_q^s \varphi_m(x)\right| < \epsilon$$

Therefore,

$$k,m > n_0 \to \max_{x\in[b,a]} \left|D_q^s \varphi_k(x) - D_q^s \varphi_m(x)\right| < \epsilon.$$

In here, $\left(D_q^s \varphi_m\right)_m$ is a Cauchy sequence in $C[a,b]$ for $s = 0,1,\dots,n-1$. Hence, $\psi_s \in C[b,a]$ exists for every $s \in \{0,1,\dots,n-1\}$. Then,

$$\lim_{s\to\infty} \max_{x\in[b,a]} \left|D_q^s \varphi(x) - \psi_s(x)\right| = 0, \quad s = 0,1,\dots,n-1.$$

then

$\psi_k(x) = D_q^s \psi_0(x), \quad (x \in [b,a]\backslash\{0\}, \qquad s = 0,1,2,\dots,n-1)$ (3.3.1.6).

when $0 \in (b,a)$ then,

$$\lim_{x\to 0} \psi_s(x) = \lim_{x\to 0} D_q^s \psi_0(x) = \lim_{v\to\infty} D_q^s \psi_0(tq^v),$$

for every $t \in (b,a)$ and $t \neq 0$. As a result,

$$\lim_{x\to 0} \psi_s(x) = \lim_{v\to\infty} \frac{D_q^{s-1}\psi_0(tq^v) - D_q^{s-1}\psi_0(tq^{v+1})}{tq^v(1-q)}$$

$$= \frac{1}{1-q} \lim_{v\to\infty} \left[\frac{D_q^{s-1}\psi_0(tq^v) - D_q^{s-1}\psi_0(0)}{tq^v} - q\,\frac{D_q^{s-1}\psi_0(tq^{v+1}) - D_q^{s-1}\psi_0(0)}{tq^{v+1}}\right]$$

$= D_q^s \psi_0(0)$ (3.3.1.7).

Therefore, the identity in (3.3.1.6) is valid for all $x \in [b,a]$ and so $\psi_0 \in C_q^n[b,a]$, (see [37]).

**Definitin 3.3.1.5:** [37]

$$C_\gamma[b,a] = \left\{\psi(x): x^\gamma \psi(x) \in C[b,a], \|\psi\|_{C_\gamma} = \max_{b\le x\le a} |x^\gamma \psi(x)|\right\},$$

where $\gamma \in \mathbb{R}$.

### 3.3.2 Some q-functions

The q-shifted factorial function is presented for $\alpha \in \mathbb{C}$ by,

$$(z;q)_m = \begin{cases} 1, & m = 0 \\ \prod_{i=0}^{m-1} (1 - zq^i), & m \in \mathbb{N} \end{cases},$$

The limit of $(z;q)_m$ as $m \to \infty$ exists and is given by $(z;q)_\infty$. The multiple q-shifted factorial for complex numbers $z_1, \dots, z_k$ is described as

$$(z_1, \dots, z_k; q)_m = \prod_{j=1}^{t} (z_j; q)_m.$$

Let $\alpha$ be a complex number. The following notation is used for the q-binomial coefficients,

$$\begin{bmatrix} \alpha \\ n \end{bmatrix}_q = \begin{cases} 1, & n = 0, \\ \dfrac{(1-q^\alpha)(1-q^{\alpha-1}) \dots (1-q^{\alpha-n+1})}{(q;q)_n}, & n \in \mathbb{N}. \end{cases}$$

If $aq^\alpha \neq q^{-m}$ for all $m \in \mathbb{N}_0$, we define

$$(z;q)_\alpha = \frac{(z;q)_\infty}{(zq^\alpha;q)_\infty}$$

$$(-z;q)_\infty = \sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{(q;q)_m} z^m \quad (z \in \mathbb{C}),$$

and

$$\frac{1}{(z;q)_\infty} = \sum_{m=0}^{\infty} \frac{z^m}{(q;q)_\infty}, \quad (|z| < 1)$$

were proposed by Euler. In [45, 46] these above expressions were mentioned. In [37], the above two expressions connect infinite products to infinite sums. In [47], $(-z;q)_\infty$ was pointed out as $E_q(z)$ and $1/(z;q)_\infty$ was explained by $e_q(z)$. Also, $(-z;q)_\infty$ and $1/(z;q)_\infty$ were defined as $E_q(z)$ and $e_q(z)$ in [45, 48, 49]. Therefore, $E_q(z)$ is an entire function with simple zeros at the points $\{-q^{-m}, \ m \in \mathbb{N}_0\}$, and

$$e_q(z) E_q(-z) \equiv 1, \qquad |z| < 1 \qquad (3.3.2.1)$$

So, the domain of the function $e_q(z)$ are able to be expanded to $\mathbb{C}$ by defining $e_q(z)$, $z \in \mathbb{C}$, to be (see [37]),

$$e_q(z) = \frac{1}{(z;q)_\infty}.$$

and it relates with (3.3.2.1) and it holds in $\mathbb{C}$. Also, the function $e_q(z)$ has simple poles at the points $\{q^{-m}, m \in \mathbb{N}_0\}$, (see [47, 48]). In addition, to this, $e_q(z)$ was shown with series expansions in [48]. In [48], $e_q(z)$ was introduced also, that

$$e_q(z) = \frac{1}{(z;q)_\infty} = \sum_{m=0}^{\infty} \frac{q^{m^2-m}}{(q,z;q)_m} z^m$$

for $z \in \mathbb{C}\backslash\{q^{-s}, s \in \mathbb{N}_0\}$. Now, basic trigonometric functions $sin_q z, cos_q z, Sin_q z$ and $Cos_q z$ are defined by [37, 47].

$$sin_q z = \frac{e_q(iz) - e_q(-iz)}{2i}, \quad |z| < 1,$$

(3.3.2.2)

$$cos_q z = \frac{e_q(iz) + e_q(-iz)}{2}, \quad |z| < 1,$$

and

$$Sin_q z = \frac{E_q(iz) - E_q(-iz)}{2i}, \quad z \in \mathbb{C},$$

(3.3.2.3)

$$Cos_q z = \frac{E_q(iz) + E_q(-iz)}{2}, \quad z \in \mathbb{C},$$

The functions $sin_q z$ and $cos_q z$ could be analytically continued through the identities (see [37]),

$$sin_q z = \frac{Sin_q z}{(-z^2;q^2)_\infty}, \qquad cos_q z = \frac{Cos_q z}{(-z^2;q^2)_\infty}$$

for $z \in \mathbb{C}\backslash\{\pm q^{-m} i; m \in \mathbb{N}_0\}$.

In [37], $sin_q z$ and $cos_q z$ are meromorphic functions with poles at the points $\{\pm q^{-m} i, m \in \mathbb{N}_0\}$. At the same time, q-analogues of hyperbolic functions $sinhz$ and $coshz$ are shown as;

$$sinh_q z = -i sin_q(iz), \qquad cosh_q z = cos_q(iz), \qquad (3.3.2.4)$$

$$Sinh_q z = -i Sin_q(iz), \qquad Cosh_q z = Cos_q(iz). \qquad (3.3.2.5)$$

When $z \in \mathbb{C}\backslash\{0\}, 0 < |q| < 1$ then $\alpha$-function is determined by,

$$\alpha(z;q) = \sum_{m=-\infty}^{\infty} q^{m^2} z^m. \qquad (3.3.2.6)$$

In [37], the following formula was introduced by Jacobi in 1829. And this identity is named by Jacobi's triple product identity (see [48]);

$$\sum_{m=-\infty}^{\infty} q^{m^2} z^m = (q^2;q^2)_\infty(-qz;q^2)_\infty(-qz^{-1};q^2)_\infty \tag{3.3.2.7}$$

where $z \in \mathbb{C}\backslash\{0\}$ and $0 < |q| < 1$ (see [37, 48]). Therefore, $\alpha(z;q)$ has only real simple zeros at the points $\{-q^{2s+1}, s \in \mathbb{Z}\}$, (see [37]). A generalization of (3.3.2.7) is Ramanujan's identity;

$$\sum_{m=-\infty}^{\infty} \frac{(b;q)_m}{(a;q)_m} z^m = \frac{(bz, q/bz, q, a/b; q)_\infty}{(z, a/bz, a, q/b; q)_\infty}, \tag{3.3.2.8}$$

where $|q| < 1$ and $|ab^{-1}| < |z| < 1$, (see [37]).

### 3.3.3 The q-Gamma and q-Beta Functions

The q-Gamma is defined as by [50, 51]

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \qquad (0 < |q| < 1) \tag{3.3.3.1}$$

where $z \in \mathbb{C}\backslash\{-m: m \in \mathbb{N}_0\}$ and the principal values of $q^z$ and $(1-q)^{1-z}$ is used. $\Gamma_q(z)$ is a meromorphic function with poles at $z = -m, m \in \mathbb{N}_0$, [37]. Because, $\Gamma_q(z)$ has no zeros, $1/\Gamma_q(z)$ is an entire function with zeros at $z = -m, m \in \mathbb{N}_0$, [37]). It can also be defined as,

$$\Gamma_q(m) = \frac{(q;q)_{m-1}}{(1-q)^{m-1}},. \quad (m \in \mathbb{N}).$$

$\Gamma_q(y)$ satisfies the following property when $y > 0$.

$$\Gamma_q(y+1) = \frac{1-q^y}{1-q}\Gamma_q(y), \quad \Gamma_q(1) = 1.$$

Now, the q-beta function is proposed as [37]

$$B_q(y,x) = \int_0^1 t^{y-1}(qt;q)_{x-1} d_q t, \qquad (Re(y) > 0;\ Re(x) > 0) \tag{3.3.3.2}$$

The relation between q-beta of q-gamma functions was given in [52] as

$$B_q(y,x) = \frac{\Gamma_q(y)\Gamma_q(x)}{\Gamma_q(y+x)}, \quad (Re(y) > 0, \quad Re(x) > 0). \tag{3.3.3.3}$$

For more details on the properties of the q-gamma and the q-beta functions we refer the reader to [53, 54, 55, 56, 57].

**Lemma 3.3.3.1:** Assume that $\alpha$ and $\theta$ are two complex numbers. Then for all $m \in \mathbb{N}_0$ then,

$$\left(q^{\alpha+\theta};q\right)_m = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q q^{s\theta}(q^{\alpha};q)_s\left(q^{\theta};q\right)_{m-s} \qquad (3.3.3.4)$$

**Proof of Lemma 3.3.3.1:** We have

$$\sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q q^{s\theta}(q^{\alpha};q)_s\left(q^{\theta};q\right)_{m-s}$$

$$= \left(q^{\theta};q\right)_m {}_2\phi_1\left(q^{-m},q^{\alpha};q^{1-\theta-m};q,q\right)$$

$$= \left(q^{\theta};q\right)_m q^{m\alpha}\frac{\left(q^{1-\theta-\alpha-m};q\right)_m}{(q^{1-\theta-m};q)_m} = \left(q^{\alpha+\theta};q\right)_m.$$

**Corollary 3.3.3.1:** Assume that $\alpha$ and $\theta$ are complex numbers, and let $m \in \mathbb{N}_0$, (see [37]). Then,

$$B_q(\alpha,\theta)(q^{m+1};q)_{\alpha+\theta-1} = \sum_{s=0}^{m} q^{s\theta}(1-q)(q^{s+1};q)_{\alpha-1}(q^{m-s+1};q)_{\theta-1} \qquad (3.3.3.5)$$

and

$$q^{-m\theta}(q^{m+1};q)_{\alpha+\theta-1}B_q(\alpha,\theta)$$

$$= \sum_{s=0}^{m} q^{-s\theta}(1-q)(q^{m-s+1};q)_{\alpha-1}(q^{s+1};q)_{\theta-1}. \qquad (3.3.3.6)$$

**Proof of Corollary 3.3.3.1:** This proof is done as (see [37]);

$$\left(q^{\alpha+\theta};q\right)_m = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q q^{s\theta}(q^{\alpha};q)_s\left(q^{\theta};q\right)_{m-s}$$

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \qquad (0<|q|<1)$$

are used and one can easily see that for any $\alpha \in \mathbb{C}, j \in \mathbb{N}_0$;

$$\left(q^{j+1};q\right)_{\alpha-1} = \frac{(q;q)_\infty(q^{\alpha};q)_j}{(q^{\alpha};q)_\infty(q;q)_j} = \Gamma_q(\alpha)(1-q)^{\alpha-1}\frac{(q^{\alpha};q)_j}{(q;q)_j} \qquad (3.3.3.7)$$

Therefore, (3.3.3.4) becomes

$$\sum_{s=0}^{m} q^{s\theta}(1-q)(q^{s+1};q)_{\alpha-1}(q^{m-s+1};q)_{\theta-1}$$

$$= \frac{(1-q)^{\alpha+\theta-1}\Gamma_q(\alpha)\Gamma_q(\theta)}{(q;q)_m}\sum_{s=0}^{m} q^{s\theta}\begin{bmatrix} m \\ s \end{bmatrix}_q (q^{\alpha};q)_s\left(q^{\theta};q\right)_{m-s} \qquad (3.3.3.8)$$

$$= (1-q)^{\alpha+\theta-1}\Gamma_q(\alpha)\Gamma_q(\theta)\frac{\left(q^{\alpha+\theta};q\right)_m}{(q;q)_m}.$$

From (3.3.3.7), one can conclude that

$$\frac{\left(q^{\alpha+\theta};q\right)_m}{(q;q)_m} = \frac{(1-q)^{1-\alpha-\theta}}{\Gamma_q(\alpha+\theta)}(q^{m+1};q)_{\alpha+\theta-1} \tag{3.3.3.9}$$

■

**Lemma 3.3.3.2:** [37] Let $\alpha$ and $\theta$ be two complex numbers with positive real parts, and let $\varphi \in L_q^1(0,a)$ for some $a > 0$. Then,

$$a^{\alpha-1}\int_0^a (qt/a;q)_{\alpha-1}t^{\theta-1}\int_0^t (qv/t;q)_{\theta-1}\varphi(u)d_qud_qt =$$

$$= B_q(\alpha,\theta)a^{\alpha+\theta-1}\int_0^a (qt/a;q)_{\alpha+\theta-1}\varphi(t)d_qt \tag{3.3.3.10}$$

**Proof of Lemma 3.3.3.2:** Assume that $\alpha,\theta > 0$ and $\varphi \in L_q^1(0,a), a > 0$. From the definition of the q-integration, one has

$$a^{\alpha-1}\int_0^a (qt/a;q)_{\alpha-1}t^{\theta-1}\int_0^t (qv/t\,;q)_{\theta-1}\varphi(v)d_qvd_qt$$

$$= a^{\alpha+\theta}(1-q)^2\sum_{m=0}^{\infty} q^{m\theta}(q^{m+1};q)_{\alpha-1}\sum_{n=0}^{\infty} q^{m+n}(q^{m+1};q)_{\theta-1}\varphi(aq^{m+n})$$

$$= a^{\alpha+\theta}(1-q)^2\sum_{m=0}^{\infty} q^{m\theta}(q^{m+1};q)_{\alpha-1}\sum_{s=m}^{\infty} q^{s}(q^{s-m+1};q)_{\theta-1}\varphi(aq^{s})$$

$$= a^{\alpha+\theta-1}\sum_{s=0}^{\infty} aq^s(1-q)\varphi(aq^s)\sum_{m=0}^{s} q^{m\theta}(1-q)(q^{m+1};q)_{\alpha-1}(q^{s-m+1};q)_{\theta-1}.$$

Using (3.3.3.5), one obtains

$$a^{\alpha-1}\int_0^a (qt/a;q)_{\alpha-1}t^{\theta-1}\int_0^t (qv/t\,;q)_{\theta-1}\varphi(v)d_qvd_qt$$

$$= a^{\alpha+\theta-1}B_q(\alpha,\theta)\sum_{s=0}^{\infty} aq^s(1-q)(q^{s+1};q)_{\alpha+\theta-1}\varphi(aq^s)$$

$$= a^{\alpha+\theta-1}B_q(\alpha,\theta)\int_0^a (qt/a;q)_{\alpha+\theta-1}\varphi(t)d_qt.$$

■

### 3.3.4 The q-Mittag Leffler Functions

The classical Mittag-Leffler functions are useful functions in the fractional calculus. In this chapter, the q-analogues of the Mittag-Leffler function are introduced. We are interested in a pair of q-Mittag-Leffler function which might be conceived as a generalization of the q-exponential functions $e_q(z)$ and $E_q(z)$.

First of all, the Mittag-Leffler functions are defined as

$$E_\gamma(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\gamma+1)}, \qquad (\gamma > 0), \tag{3.3.4.1}$$

and

$$E_{\gamma,\theta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\gamma+\theta)} \qquad (\gamma > 0;\ \theta \in \mathbb{C}, z \in \mathbb{C}). \tag{3.3.4.2}$$

(3.3.4.1) is a one-parameter of Mittag-Leffler function, (3.3.4.2) is a two-parameter of Mittag-Leffler function. Especially, the generalized version of Mittag-Leffler function is introduced as

$$E_{\alpha,\theta}^{\gamma}(z) = \sum_{m=0}^{\infty} \frac{z^m(\gamma)_m}{m!\,\Gamma(m\alpha+\theta)}.$$

The q-analogues of the Mittag-Leffler functions have two main q-exponential functions, namely $e_q(z)$ and $E_q(z)$.

$$e_{\alpha,\theta}(z;q) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma_q(m\alpha+\theta)}, \qquad (|z(1-q)^\alpha| < 1),$$

$$E_{\alpha,\theta}(z;q) = \sum_{m=0}^{\infty} \frac{q^{\alpha m(m-1)/2}}{\Gamma_q(m\alpha+\theta)} z^m, \qquad z \in \mathbb{C}, \tag{3.3.4.3}$$

where $\alpha > 0,\ \theta \in \mathbb{C}$. For $\theta = 1$, the functions $e_{\alpha,1}(z;q)$ and $E_{\alpha,1}(z;q)$ shows classes of q-exponential functions of one parameter, (see [37]). Other one parameter class of q-exponential functions is shown by [49, 58, 59],

$$E_q^{(\alpha)}(z) = \sum_{s=0}^{\infty} \frac{q^{as^2/2}}{(q;q)_s} z^s, \qquad (\alpha \in \mathbb{C}).$$

In [60], other pair of q-Mittag-Leffler functions were defined $e_{q;\alpha,\theta}(z;c)$ and $E_{q;\alpha,\theta}(z;c)$ such that

$$e_{q;\alpha,\theta}(z;c) = \sum_{m=0}^{\infty} \frac{(c/z;q)_{\alpha m+\theta-1}}{(q;q)_{\alpha m+\theta-1}} z^{\alpha m+\theta-1}, \quad (|z| > |c|),$$

$$E_{q;\alpha,\theta}(z;c) = \sum_{m=0}^{\infty} \frac{q^{\binom{\alpha m+\theta-1}{2}}(c/z;q)_{\alpha m+\theta-1}}{(q;q)_{\alpha m+\theta-1}(-c;q)_{\alpha m+\theta-1}}, \quad (z \in \mathbb{C}),$$

where $\{q,z,c,\alpha\} \subset \mathbb{C}$, $Re(\alpha), Re(\theta) > 0$ and $|q| < 1$. $e_{q;\alpha,\theta}(z;c)$ was called the small q-Mittag-Leffler function and $E_{q;\alpha,\theta}(z;c)$ was called as the big q-Mittag-Leffler function. Clearly, [37],

$$e_{q;\alpha,\theta}(z;0) = (1-q)^{-\theta} z^{\theta-1} e_{\alpha,\theta}(z^{\alpha}(1-q)^{-\alpha};q).$$

### 3.3.5 q-Analogues of the Laplace Transform

q-analogues of the Laplace transform was given in [61]. In fact, there are q-versions of the Laplace transform

$${}_qL_s\{\varphi(t)\} = \Omega(s) = \frac{1}{1-q}\int_0^{s^{-1}} E_q(-qst)\varphi(t)d_qt, \qquad (3.3.5.1)$$

and

$${}_q\mathcal{L}_s\{\varphi(t)\} = \mathbf{\Omega(s)} = \frac{1}{1-q}\int_0^{\infty} e_q(-st)\varphi(t)d_qt, \qquad (3.3.5.2)$$

where $Re(s) > 0$, (see [37]).

**Properties of the ${}_qL_s$ Transform:**

The most significant features of the ${}_qL_s$ transform needed in the sequel are summarized here. By using the definition of below the q-integration;

$$\int_0^t \varphi(\tau)d_q\tau = (1-q)\sum_{m=0}^{\infty} tq^m\varphi(tq^m), \qquad (t \in A), \qquad (3.3.5.3)$$

the q-Laplace transform of (3.3.5.1) can be written as

$${}_qL_s\{\varphi(t)\} = \frac{(q;q)_{\infty}}{s}\sum_{j=0}^{\infty}\frac{q^j}{(q;q)_j}\varphi(s^{-1}q^j).$$

The convolution of two functions $\bar{\varphi}$ and $\bar{\psi}$ is defined [61] as

$$(\bar{\varphi} * \bar{\psi})(t) = \frac{t}{1-q}\int_0^1 \bar{\varphi}(t\tau)\bar{\psi}(t-tq\tau)d_q\tau \qquad (3.3.5.4)$$

where $\bar{\psi}[t-t_i]$, for (see [37]);

$$\bar{\psi}(t) = \sum_{m=0}^{\infty} a_m t^m,$$

is shown as,

$$\bar{\psi}[t-t_i] = \sum_{m=0}^{\infty} a_m [t-t_i]_m$$

with

$$[t-t_i]_m = t^m \left(\frac{t_i}{t}; q\right)_m$$

By using the definition of q-integration, then $(\bar{\varphi} * \bar{\psi})$ satisfies;

$$(\bar{\varphi} * \bar{\psi})(t) = \frac{1}{1-q} \int_0^t \bar{\varphi}(\tau)\, \bar{\psi}(t-q\tau) d_q\tau, \qquad (3.3.5.5)$$

and thus $\bar{\psi}(t-q\tau) = \varepsilon^{-q\tau}\bar{\psi}(t)$ is defined in this manner. In [49], the convolution of two functions $\bar{\varphi}, \bar{\psi}$ is shown as

$$(\bar{\varphi} * \bar{\psi}) = \frac{1}{1-q} \int_0^t \bar{\varphi}(\tau)\varepsilon^{-q\tau}\bar{\psi}(t) d_q\tau, \qquad (3.3.5.6).$$

In [37], it is remarked by Hahn (see [61]) that the convolution theorem

$${}_qL_s\{\bar{\varphi} * \bar{\psi}\} = {}_qL_s\, \bar{\varphi}\ {}_qL_s\, \bar{\psi}, \qquad (3.3.5.7)$$

valid only for ${}_qL_s$ transform and does not hold for the ${}_qL_s$ transform, (see [37]). Now, there are some basic properties of the ${}_qL_s$ transform are mentioned. Initially, let

$${}_qL_s\{\varphi(t)\} = \Omega(s).$$

**Property 1:**

$${}_qL_s\{\varphi(bt)\} = (1/b)\Omega(s/b), \quad (b \neq 0).$$

**Property 2:**

$${}_qL_s\{D_q^m \varphi(t)\} = \left(\frac{s}{1-q}\right)^m \Omega(s) - \sum_{k=1}^{m} D_q^{m-k}\varphi(0)\frac{s^{k-1}}{(1-q)^k}, \quad (m \in \mathbb{N}). \qquad (3.3.5.8)$$

When the case $m = 1$ then, (see [37]),

$${}_qL_s\{D_q\varphi(t)\} = \left(\frac{s}{1-q}\right)\Omega(s) - \frac{\varphi(0)}{1-q} = \frac{s\Omega(s) - \varphi(0)}{1-q}.$$

**Property 3:**

$${}_qL_s\left\{\frac{\varphi(t)}{t}\right\} = \frac{1}{1-q}\int_{qs}^{\infty} \Omega(\tau) d_q\tau$$

and ${}_qL_s\left\{\frac{\varphi(t)}{t}\right\}$ exists.

**Property 4:**

$$ {}_qL_s\left\{\int_t^\infty \frac{\varphi(\tau)}{\tau} d_q\tau\right\} = \frac{1}{s}\int_0^{qs} \Omega(\tau) d_q\tau, $$

provided that [37]

i) the two integral exists,

ii) $\Omega(s)$ exists for all $s$,

iii) $q^j \sum_{r=1}^{\infty}\left|\varphi\left(q^j/s\right)\right| = O\left(|h|^j\right)$, for all fixed $j$ and $|h| < 1$.

**Property 5:**

If

$$ I_q^m\varphi(t) = \int_0^t \int_0^{t_{m-1}} \int_0^{t_{m-2}} \dots \int_0^{t_1} \varphi(\tau) d_q\tau d_q t_1 d_q t_2 \dots d_q t_{m-1}, \tag{3.3.5.9} $$

then

$$ {}_qL_s\{I_q^m\varphi(t)\} = \left(\frac{1-q}{s}\right)^m \Omega(s). $$

**Property 6:**

If ${}_qL_s\{\varphi_k(t)\} = \Omega_k(s)$ then

$$ {}_qL_s\left\{\sum_{k=0}^{m} \varphi_k(t)\right\} = \sum_{k=0}^{m} \Omega_k(s) \tag{3.3.5.10} $$

holds if the following conditions hold

a) $m$ is finite

b) $m$ is infinite and

i) $\sum_{k=0}^{\infty}\left|\varphi_k(tq^j)\right|$ is convergent for every $t_0 q^j$, $t_0$ being fixed,

ii) $q^j \sum_{k=0}^{\infty}\left|\varphi_k(tq^j)\right| = O\left(h^j\right)$, where $j$ is greater than some fixed $J$ and $h$ is a fixed quantity, $|h|$ being less than unity.

**Property 7:**

Set

$$ \bar{\varphi}(t) = \int_0^t \frac{\varphi(\tau)}{\tau} d_q\tau $$

and consider that $\lim_{t\to 0}\frac{\varphi(t)}{t} = 0$. Then,

$$ {}_qL_s\{\bar{\varphi}(t)\} = \frac{1}{s}\int_{qs}^{\infty} \Omega(s) d_q s. $$

The inversion formula of the ${}_qL_s$ transform is given below and it was proposed in [61].

**Definition of Hahn's inversion formula:** If ${}_qL_s\{\varphi(t)\} = \Omega(s)$ then,

$$\varphi(t) = \int_c \Omega(s) e_q(st) ds, \qquad (3.3.5.11).$$

where the path of integration C encircles the origin and could be deformed into a loop, parallel to the imaginary axis. Especially, if $\Omega(s)$ is analytic then according to the above, the inversion can be written as

$$\varphi(t) = \frac{1}{t}\sum_{i=0}^{\infty}(-1)^i q^{i(i-1)/2}\frac{\Omega\left(t^{-1}q^{-i}\right)}{(q;q)_i}.$$

**Definition of the ${}_q\mathcal{L}_s$ Transform:** Now, even though there is no q-analogue of the convolution theorem exiting for the ${}_q\mathcal{L}_s$ transform, the ${}_q\mathcal{L}_s$ transform has an advantage over the ${}_qL_s$ transform. This advantage is the recognition of q-analogue of the Goldstein theorem. By using the definition of below q-integration,

$$\int_0^t \varphi(\tau) d_q\tau = (1-q)\sum_{m=0}^{\infty} tq^m \varphi(tq^m), \qquad (t \in A)$$

then the q-Laplace transform of (3.3.5.2) is,

$${}_q\mathcal{L}_s\varphi(t) = \frac{1}{(-s;q)_\infty}\sum_{j=-\infty}^{\infty} q^j(-s;q)_j\varphi\left(q^j\right). \qquad (3.3.5.12)$$

Now, there are many basic properties of the ${}_q\mathcal{L}_s$ transform are introduced. Particularly, Abdi introduced the following properties of ${}_q\mathcal{L}_s$ transform [62].

**Property 8:** If ${}_q\mathcal{L}_s\{\varphi(t)\} = \Omega(s)$ and $b \in \mathbb{R}_{q,+}$ then,

$${}_q\mathcal{L}_s\{\varphi(bt)\} = (1/b)\Omega(s/b).$$

**Property 9:**

$${}_q\mathcal{L}_s\left\{D_q^m\varphi(t)\right\} = \left(\frac{s}{1-q}\right)^m \Omega(s) - \sum_{k=1}^{m} D_q^{m-k}\varphi(0)\frac{s^{k-1}}{q^{k-1}(1-q)^k}, \qquad (3.3.5.13)$$

and $m \in \mathbb{N}$ in (3.3.5.13).

**Property 10:** In [37],

$$ {}_q\mathcal{L}_s\{\varphi(t)/t\} = \frac{1}{1-q}\int_{qs}^{\infty} \Omega(\tau) d_q\tau, \tag{3.3.5.14}$$

obtained that ${}_q\mathcal{L}_s\{\varphi(t)/t\}$ exists.

**Property 11:**

$$ {}_q\mathcal{L}_s\left\{\int_t^{\infty} \frac{\varphi(\tau)}{\tau} d_q\tau\right\} = \frac{1}{s}\int_0^{qs} \Omega(\tau) d_q\tau, \tag{3.3.5.15}$$

Provided that

i) the two integrals exist,

ii) $\Omega(s)$ exists for every s,

iii) $q^j \sum_{r=1}^{\infty}\left|\varphi\left(q^j/s\right)\right| = O\left(|h|^j\right)$, for all fixed $j$ and $|h| < 1$.

**Property 12:**

$$ {}_q\mathcal{L}_s\{I_q^m \varphi(t)\} = \left(\frac{1-q}{s}\right)^m \Omega(\mathrm{s}), \tag{3.3.5.16}$$

where $I_q^m$ is defined in (3.3.5.7).

**Property 13:** If ${}_q\mathcal{L}_s\{\varphi_k(t)\} = \Omega_k(s)$, then

$$ {}_q\mathcal{L}_s\left\{\sum_{k=0}^{m} \varphi_k(t)\right\} = \sum_{k=0}^{m} \Omega_k(s), \tag{3.3.5.17}$$

once  at least one of the following conditions hold,

a) $m$ is finite

b) $m$ is infinite and

i) $\sum_{k=0}^{\infty}\left|\varphi_k\left(tq^j\right)\right|$ is convergent for every to $t_0 q^j$, $t_0$ being fixed,

ii) $q^j \sum_{k=0}^{\infty}\left|\varphi_k\left(tq^j\right)\right| = O\left(h^j\right)$, where $j$ is gretaer than some fixed $J$ and $h$ is a fixed quantity, $|h|$ being less than unity.

**Property 14:** If $\lim_{t\to 0}\frac{\varphi(t)}{t} = 0$ then,

$$ {}_q\mathcal{L}_s\left\{\int_0^{t} \frac{\varphi(t)}{t} d_q t\right\} = \frac{1}{s}\int_{qs}^{\infty} \Omega(\tau) d_q\tau. \tag{3.3.5.18}$$

**Property 15:** If ${}_q\mathcal{L}_s\{\varphi(t)\} = \Omega(s)$, then

$$ {}_q\mathcal{L}_s\{t^m \varphi(t)\} = (q-1)^m D_{q,s}^m \Omega(s), \tag{3.3.5.19}$$

$$ {}_q\mathcal{L}_s\left\{t^m D_q^n \varphi(t)\right\} = (-q^{-1})^n (q-1)^{m-n} D_{q,s}^m\left(s^n \Omega(s/q^n)\right) \text{ for } n \leq m, \tag{3.3.5.20}$$

and

$${}_q\mathcal{L}_s\{t^m D_q^n \varphi(t)\} = (q-1)^m D_q^n \left[\frac{s^n}{(q(1-q))^n}\Omega(s/q^n) - \sum_{k=1}^{n} \frac{s^{k-1} D_q^{n-k}\varphi(0)}{q^{k-1}(1-q)^k}\right], \quad (3.3.5.21)$$

for every $n > m$.

## 3.4 THE Q-FRACTIONAL CALCULUS

### 3.4.1 Classical Fractional Calculus

In this part, we present the continuous case of the fractional calculus version. $\varphi \in \mathcal{AC}^{(m)}[a,b]$ if $\varphi$ has continous derivatives up to order $m-1$ on $[a,b]$ with $\varphi^{(m-1)} \in \mathcal{AC}[a,b]$.

**Lemma 3.4.1.1:** Let $\mathcal{AC}^{(m)}[a,b]$ exists the functions $\varphi$ giving in the form,

$$\varphi(t) = \sum_{k=0}^{m-1} c_k (t-a)^k + \frac{1}{(m-1)!}\int_a^t (t-\tau)^{m-1}\Omega(\tau)d\tau,$$

where $\Omega \in L_1(a,b)$ and the $c_k$'s are arbitrary constants (see [37]). In addition to this, $\Omega(t) = \varphi^{(m)}(t)$, and $c_t = \frac{\varphi^{(t)}(a)}{t!}$, $t = 0,1,\dots,m-1$.

The first fractional Riemann-Liouville integral operator is related to Abel's integral equation,

$$\frac{1}{\Gamma(\gamma)}\int_a^t (t-\tau)^{\gamma-1}\Omega(\tau)d\tau = \varphi(t),\ \ t > a, \quad \gamma > 0, \quad \varphi \in L_1(a,b) \qquad (3.4.1.1)$$

**Theorem 3.4.1.1** (3.4.1.1) with $0 < \gamma < 1$ has a unique solution in $L_1(a,b)$ if and only if $\varphi_{1-\gamma}$ is written as

$$\varphi_{1-\gamma}(t) = \frac{1}{\Gamma(1-\gamma)}\int_a^t (t-\tau)^{-\gamma}\varphi(\tau)d\tau$$

and $\varphi_{1-\gamma}$ is absolutely continuous on $[a,b]$ such that $\varphi_{1-\gamma}(a) = 0$. Therefore, $\Omega$ can be shown directly as,

$$\Omega(t) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dt}\int_a^t (t-\tau)^{-\gamma}\varphi(\tau)d\tau = \frac{d}{dt}\varphi_{1-\gamma}(t), \qquad (3.4.1.2)$$

when $\varphi \in \mathcal{AC}[a,b]$, then $\varphi_{1-\gamma} \in \mathcal{AC}[a,b]$ and also (3.4.1.2) becomes

$$\Omega(t) = \frac{1}{\Gamma(1-\gamma)}\left[\frac{\varphi(a)}{(t-a)^\gamma} + \int_a^t \frac{\varphi'(s)}{(t-s)^\gamma}ds\right].$$

Similarly, $n$-th primitive of a function $\varphi \in L_1(a,b)$ can be used with Cauchy formulae (see [37]),

$$\int_a^t \int_a^{t_{m-1}} \dots \int_a^{t_1} \varphi(\tau) d\tau dt_1 \dots d_{t_{m-1}} = \frac{1}{(m-1)!} \int_a^t (t-\tau)^{m-1} \varphi(\tau) d\tau \quad (3.4.1.3)$$

$$\int_t^b \int_{t_{m-1}}^b \dots \int_{t_1}^b \varphi(\tau) d\tau dt_1 \dots d_{t_{m-1}} = \frac{1}{(m-1)!} \int_t^b (\tau-t)^{m-1} \varphi(\tau) d\tau \quad (3.4.1.4)$$

and $m \in \mathbb{N}$. The right hand sides of (3.4.1.3) and (3.4.1.4) hold for non integer values of $m$, the Riemann-Liouville fractional integral operator can be given when $\gamma \in \mathbb{R}^+$and $\varphi \in L_1(a,b)$;

$$I_{a+}^{\gamma} \varphi(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-\tau)^{\gamma-1} \varphi(\tau) d\tau$$

$$(3.4.1.5)$$

$$I_{b-}^{\gamma} \varphi(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (\tau-t)^{\gamma-1} \varphi(\tau) d\tau$$

and $t \in (a,b)$.

When $\varphi \in L_1(a,b)$, then not only $I_{a+}^{\gamma}\varphi$ but also $I_{b-}^{\gamma}\varphi$ exist and they are $L_1(a,b)$ functions. In addition, if $\varphi \in L_1(a,b)$, the below expression is obtained

$$\lim_{\gamma \to 0+} I_{a+}^{\gamma} \varphi(t) = \lim_{\gamma \to 0+} I_{b-}^{\gamma} \varphi(t) = \varphi(t), \quad (3.4.1.6)$$

For $\varphi \in L_1(a,b)$, the left and right sided Riemann-Liouville fractional derivatives of order $\gamma, \gamma \in \mathbb{R}^+$, are defined formally by

$$D_{a+}^{\gamma} \varphi(t) = D^p I_{a+}^{p-\gamma} \varphi(t) = \frac{1}{\Gamma(p-\gamma)} \frac{d^p}{dt^p} \int_a^t (t-\tau)^{p-\gamma-1} \varphi(\tau) d\tau. \quad (3.4.1.7)$$

Now, in this part, some properties of Riemann-Liouville fractional calculus are indicated which their q-analogues are derived. Especially, consider the case of the left-sided Riemann-Liouville fractional operator because, its basic analogue will be studied. Let $\gamma, \theta \in \mathbb{R}^+$ and if $\varphi \in L_1(a,b)$ then the semigroup property

$$I_{a+}^{\gamma} I_{a+}^{\theta} \varphi(t) = I_{a+}^{\theta} I_{a+}^{\gamma} \varphi(t) = I_{a+}^{\gamma+\theta} \varphi(t), \quad (3.4.1.8)$$

holds for almost every $t \in [a,b]$. When $\varphi(t)$ satisfies the conditions $\varphi \in L_1(a,b)$ and $I_{a+}^{p-\gamma}\varphi \in \mathcal{AC}^{(p)}[a,b], p = \lceil \gamma \rceil$ then,

$$\begin{cases} D_{a+}^{\gamma-j} \varphi \in L_1(a,b), & j = 0,1,\dots,p, \\ D_{a+}^{\gamma-j} \varphi \in \mathcal{AC}^{(j)}[a,b], & j = 1,2,\dots,p-1. \end{cases} \quad (3.4.1.9)$$

In addition,

$$D_{a+}^{\gamma} I_{a+}^{\gamma} \varphi(t) = \varphi(t) \tag{3.4.1.10}$$

$$D_{a+}^{\gamma} I_{a+}^{\theta} \varphi(t) = I_{a+}^{\theta-\gamma} \varphi(t) \qquad , \theta \geq \gamma \geq 0, \tag{3.4.1.11}$$

$$D_{a+}^{\gamma} I_{a+}^{\theta} \varphi(t) = D_{a+}^{\gamma-\theta} \varphi(t) \qquad , \gamma > \theta \geq 0, \tag{3.4.1.12}$$

Also,

$$I_{a+}^{\gamma} D_{a+}^{\gamma} \varphi(t) = \varphi(t) - \sum_{j=1}^{p} D_{a+}^{\gamma-j} \varphi(a^{+}) \frac{(t-a)^{\gamma-j}}{\Gamma(1+\gamma-j)}, \tag{3.4.1.13}$$

When $\varphi \in L_1(a,b)$ and $D_{a+}^{-(m-\theta)} \varphi \in \mathcal{AC}^{(m)}[a,b], m = \lceil \theta \rceil$, then the expression

$$I_{a+}^{\gamma} D_{a+}^{\theta} \varphi(t) = D_{a+}^{\theta-\gamma} \varphi(t) - \sum_{j=1}^{p} D_{a+}^{\theta-j} \varphi(a^{+}) \frac{(t-a)^{\gamma-j}}{\Gamma(1+\gamma-j)}, \tag{3.4.1.14}$$

holds almost everywhere in $(a,b)$ for any $\gamma > 0$. Also, when $0 \leq p-1 \leq \gamma < p$, $\gamma + \theta < p$ and $D_{a+}^{-(m-\gamma)} \varphi \in \mathcal{AC}^{(m)}[a,b]$ then,

$$D_{a+}^{\gamma} D_{a+}^{\theta} \varphi(t) = D_{a+}^{\gamma+\theta} \varphi(t) - \sum_{j=1}^{m} D_{a+}^{\theta-j} \varphi(a^{+}) \frac{(t-a)^{-\gamma-j}}{\Gamma(1-\gamma-j)} \tag{3.4.1.15}$$

holds almost everywhere in $(a,b)$. Finally,

$I_{a+}^{\gamma} D_{a+}^{\gamma} \varphi(t) = \varphi(t)$ and $I_{a+}^{\gamma} D_{a+}^{\theta} \varphi(t) = D_{a+}^{\theta-\gamma} \varphi(t)$, where $\varphi \in I_{a+}^{\gamma}(L_1)$ and $\varphi \in I_{a+}^{\theta}(L_1)$. Many fractional operators were defined in [1]. Especially, the Caputo-fractional operator is introduced as,

$$ {}_{a}^{C}D_{t}^{\gamma} \varphi(t) = \frac{1}{\Gamma(m-\gamma)} \int_{a}^{t} (t-\tau)^{m-\gamma-1} \varphi^{(m)}(\tau) d\tau$$

and $m-1 < \gamma \leq m, m \in \mathbb{N}$. From [37], the Riemann-Liouville integral and the Caputo fractional derivative can be related by,

$$ {}_{a}^{C}D_{t}^{\gamma} \varphi(t) = I_{a+}^{m-\gamma} \varphi^{(m)}(t).$$

If the function $\varphi(t)$ has $m+1$ continuous derivatives in $[a,b]$, then [63],

$\lim_{\gamma \to m} {}_{a}^{C}D_{t}^{\gamma} \varphi(t) = \varphi^{(m)}(t)$ for $t \in [a,b]$.

While constant $c$ is put instead of the function $\varphi$ when the Caputo-fractional derivative is taken then,

$$D_{a+}^{\gamma}c = D^m I_{a+}^{m-\gamma}c$$
$$= \frac{c}{\Gamma(m-\gamma+1)} D^m (t-a)^{m-\gamma}$$
$$= \frac{c}{\Gamma(1-\gamma)} (t-a)^{-\gamma} ,$$

where $m = \lceil \gamma \rceil$.

### 3.4.2 q-fractional Riemann-Liouville Operator

The q-fractional Riemann-Liouville integral is introduced [37] as

$$I_q^{\theta}\varphi(t) = \frac{t^{\theta-1}}{\Gamma_q(\theta)} \int_0^t (q\tau/t;q)_{\theta-1}\varphi(\tau)d_q\tau$$

And the q-fractional Riemann-Liouville derivative is given below

$$D_q^{\theta}\varphi(t) = D_q^{[\theta]} I_q^{\theta-\lceil\theta\rceil}\varphi(t), \qquad (\theta \geq 0). \qquad (3.4.2.1)$$

**Properties of the q-fractional operators**

**Lemma 3.4.2.1** [37] If $\varphi \in L_q^1[0,b]$ then (3.4.2.1) holds

**Proof:** Initially, the proof is done by [37]

$$I_q^{\gamma}\varphi(t) = t^{\gamma}(1-q)^{\gamma} \sum_{m=0}^{\infty} q^m \frac{(q^{\gamma};q)_m}{(q;q)_m} \varphi(tq^m) . \qquad (3.4.2.2)$$

Therefore,

$$I_q^{\gamma}\left(I_q^{\theta}\varphi(t)\right) = t^{\gamma+\theta}(1-q)^{\gamma+\theta} \sum_{k=0}^{\infty} q^{k(1+\theta)} \frac{(q^{\gamma};q)_k}{(q;q)_k} \sum_{n=0}^{\infty} q^n \frac{\left(q^{\theta};q\right)_n}{(q;q)_n} \varphi(tq^{m+n}) .$$

If the substitution $m = k + n$ is used, then the following expression is obtained.

$$I_q^{\gamma}\left(I_q^{\theta}\varphi(t)\right)$$
$$= t^{\gamma+\theta}(1-q)^{\gamma+\theta} \sum_{k=0}^{\infty} q^{k(1+\theta)} \frac{(q^{\gamma};q)_k}{(q;q)_k} \sum_{m=k}^{\infty} q^{m-k} \frac{\left(q^{\theta};q\right)_{m-k}}{(q;q)_{m-k}} \varphi(tq^m). \qquad (3.4.2.3)$$

Because $\varphi \in L_q^1[0,b]$, the sums in (3.4.2.3) is absolutely convergent. Thus, the order of summations can be interchanged to satisfy, namely

$$I_q^{\gamma}\left(I_q^{\theta}\varphi(t)\right) = t^{\gamma+\theta}(1-q)^{\gamma+\theta} \sum_{m=0}^{\infty} q^m \varphi(tq^m) \sum_{k=0}^{m} q^{k\theta} \frac{(q^{\gamma};q)_k}{(q;q)_k} \frac{\left(q^{\theta};q\right)_{m-k}}{(q;q)_{m-k}} .$$

It is easy to see that, [37];

$$\frac{\left(q^{\theta};q\right)_{m-k}}{(q;q)_{m-k}} = \frac{(q^{-m};q)_k}{(q^{1-m-\theta};q)_k} q^{(1-\theta)k}$$

So it is concluded that,

$$I_q^{\gamma}\left(I_q^{\theta}\varphi(t)\right)$$

$$= t^{\gamma+\theta}(1-q)^{\gamma+\theta}\sum_{m=0}^{\infty} q^m \varphi(tq^m)\sum_{k=0}^{\infty} q^{k\theta}\frac{(q^{\gamma};q)_k}{(q;q)_k}\frac{(q^{-m};q)_k}{(q^{1-m-\theta};q)_k}q^{(1-\theta)k}.$$

As a result,

$$I_q^{\gamma}\left(I_q^{\theta}\varphi(t)\right)$$

$$= t^{\gamma+\theta}(1-q)^{\gamma+\theta}\sum_{m=0}^{\infty} q^m \varphi(tq^m)\frac{\left(q^{\theta};q\right)_m}{(q;q)_m}\ {}_2\phi_1\left(q^{-m},q^{\gamma};q^{1-m-\theta};q,q\right). \quad (3.4.2.4)$$

Hence,

$${}_2\phi_1\left(q^{-m},q^{\gamma};q^{1-m-\theta};q,q\right) = \frac{\left(q^{1-m-\theta-\gamma};q\right)_m}{\left(q^{1-m-\theta};q\right)_m}q^{m\gamma} = \frac{\left(q^{\gamma+\theta};q\right)_m}{\left(q^{\theta};q\right)_m}. \quad (3.4.2.5)$$

**Lemma 3.4.2.2:** When $\varphi \in \mathcal{L}_q^1[0,b]$ then

$$D_q^{\gamma}I_q^{\gamma}\varphi(t) = \varphi(t), \qquad (\gamma > 0; t \in (0,b]). \quad (3.4.2.6)$$

**Proof:** If $\gamma = m$, $m \in \mathbb{N}$, then $D_q^m I_q^m \varphi(t) = \varphi(t)$. [37] If $\gamma$ is a nonpositive integer such that, $m-1 < \gamma < m,\ m \in \mathbb{N}$, the semigroup property can be applied to satisfy $D_q^{\gamma}I_q^{\gamma}\varphi(t) = D_q^m I_q^{m-\gamma}I_q^{\gamma}\varphi(t) = D_q^m I_q^m\varphi(t) = \varphi(t)$, for every $t \in (0,b]$.

■

**Lemma 3.4.2.3:** If $\gamma \in \mathbb{R}^+$ and $m := \lceil\gamma\rceil$. Also when $\varphi \in L_q^1[0,b]$, such that $I_q^{m-\gamma}\varphi \in \mathcal{AC}_q^{(m)}[0,b]$, then,

$$I_q^{\gamma}D_q^{\gamma}\varphi(t) = \varphi(t) - \sum_{p=1}^{m} D_q^{\gamma-p}\varphi(0^+)\frac{t^{\gamma-p}}{\Gamma_q(\gamma-p+1)}, \qquad t \in (0,b]. \quad (3.4.2.7)$$

**Proof:** We start with,

$$h(\tau,t) = t^{\gamma}(q\tau/t;q)_{\gamma}D_q^{\gamma}\varphi(\tau), \quad t \in (0,b], \quad (0 < \tau \le t).$$

So, it is said that $h(t,gt) = 0$ for all $t \in (0,b]$ (see [37]). Besides,

$$I_q^{\gamma}D_q^{\gamma}\varphi(t) = \frac{t^{\gamma-1}}{\Gamma_q(\gamma)}\int_0^t (q\tau/t\,;q)_{\gamma-1}D_q^{\gamma}\varphi(\tau)d_q\tau$$

$$= D_{q,t}\left(\frac{t^{\gamma}}{\Gamma_q(\gamma+1)}\int_0^t (q\tau/t;q)_{\gamma}D_q^{\gamma}\varphi(\tau)d_q\tau\right)$$

$$= D_{q,t}\left(\frac{t^{\gamma}}{\Gamma_q(\gamma+1)}\int_0^t (q\tau/t\,;q)_{\gamma}D_q^m I_q^{m-\gamma}\varphi(\tau)d_q\tau\right) \quad (3.4.2.8)$$

and the q-integration is used n times on the last q-integral of (3.4.2.8), then,

$$\int_0^t \psi(\tau)D_q\varphi(\tau)d_q\tau = (\varphi\psi)(t) - \lim_{m\to\infty}(\varphi\psi)(tq^m) - \int_0^t D_q\psi(\tau)\varphi(q\tau)d_q\tau$$

Later,

$$\frac{t^\gamma}{\Gamma_q(\gamma+1)}\int_0^t (q\tau/t;q)_\gamma D_q^\gamma\varphi(\tau)d_q\tau$$

is shown in details in the next page. This calculation are introduced as,

$$\frac{t^\gamma}{\Gamma_q(\gamma+1)}\int_0^t (q\tau/t;q)_\gamma D_q^\gamma\varphi(\tau)d_q\tau$$

$$= -\sum_{p=1}^{m} D_{q,t}^{\gamma-p}\varphi(0^+)\frac{t^{\gamma-p+1}}{\Gamma_q(\gamma-p+2)}$$

$$+\frac{t^{\gamma-m}}{\Gamma_q(\gamma-m+1)}\int_0^t (q\tau/t;q)_{\gamma-m} I_q^{m-\gamma}\varphi(\tau)d_q\tau$$

$$= -\sum_{p=1}^{m} D_{q,t}^{\gamma-p}\varphi(0^+)\frac{t^{\gamma-p+1}}{\Gamma_q(\gamma-p+2)} + I_q^{\gamma-m+1}I_q^{m-\gamma}\varphi(t). \qquad (3.4.2.9)$$

After (3.4.2.1) and (3.4.2.2) are used, then it is satisfied that

$$\frac{t^\gamma}{\Gamma_q(\gamma+1)}\int_0^t (q\tau/t;q)_\gamma D_q^\gamma\varphi(\tau)d_q\tau$$

$$= I_q\varphi(t) - \sum_{p=1}^{m} D_q^{\gamma-p}\varphi(0^+)\frac{t^{\gamma-p+1}}{\Gamma_q(\gamma-p+2)}. \qquad (3.4.2.10)$$

According to (3.4.2.7), $I_q^\gamma D_q^\gamma\varphi(t) = \varphi(t)$, $(t \in (0,b])$ holds if and only if

$$D_q^{\gamma-p}\varphi(0^+) = 0, \qquad (p = 1,2,\ldots,m).$$

**Lemma 3.4.2.4:** If $\varphi \in L_q^1[0,b]$, then

$$D_q^\gamma I_q^\theta\varphi(t) = I_q^{\theta-\gamma}\varphi(t), \qquad (\theta \ge \gamma \ge 0;\ t \in (0,b]), \qquad (3.4.2.11)$$

In addition to this, if $D_q^{\gamma-\theta}\varphi(t)$ exists in $(0,b]$ then,

$$D_q^\gamma I_q^\theta\varphi(t) = D_q^{\gamma-\theta}\varphi(t), \qquad (\gamma > \theta \ge 0). \qquad (3.4.2.12)$$

**Proof:** First of all, we consider that $\theta \ge \gamma$. Later, $\theta = \gamma + (\theta-\gamma)$ and from (3.3.1) the proof is introduced by (see [37])

$$D_q^\gamma I_q^\theta\varphi(t) = D_q^\gamma I_q^\gamma I_q^{\theta-\gamma}\varphi(t) = I_q^{\theta-\gamma}\varphi(t),$$

and if $\theta < \gamma, n = \lceil\gamma\rceil$ and $m = \lceil\gamma - \theta\rceil$

The following is obtained.

$$D_q^\gamma I_q^\theta \varphi(t) = D_q^n I_q^{n-\gamma} I_q^\theta \varphi(t) = D_q^n I_q^{n-m} I_q^{m-\gamma+\theta} \varphi(t)$$

$$= D_q^m I_q^{m-\gamma+\theta} \varphi(t) = D_q^{\gamma-\theta} \varphi(t). \qquad \blacksquare$$

**Lemma 3.4.2.5:** When $\varphi \in \mathcal{L}_q^1[0,b]$, such that $I_q^{m-\theta}\varphi \in \mathcal{AC}_q^{(m)}[0,b]$, where $\theta > 0$ and $m = \lceil\theta\rceil$, (see [37]). Later, for every, $\gamma \geq 0$ then

$$I_q^\gamma D_q^\theta \varphi(t) = D_q^{-\gamma+\theta} \varphi(t) - \sum_{p=1}^{m} D_q^{\theta-p} \varphi(0^+) \frac{t^{\gamma-p}}{\Gamma_q(\gamma - p + 1)}, \quad t \in (0,b]. \quad (3.4.2.13)$$

**Proof:** It is known that

$$\frac{1}{\Gamma_q(z)}$$

has zeros at the negative integers and (3.4.2.13) exists for every $\theta > 0$, when $\gamma = 0$, [37]. So, we consider that $\gamma > 0$. Especially, we assume below two different cases.

**First Case:** If $\gamma \geq \theta$, then this condition is satisfied,

$$I_q^\gamma D_q^\theta \varphi(t) = I_q^{\gamma-\theta}\left(I_q^\theta D_q^\theta\right)\varphi(t)$$

$$= I_q^{\gamma-\theta}\left(\varphi(t) - \sum_{p=1}^{m} D_q^{\theta-p} \varphi(0^+) \frac{t^{\theta-p}}{\Gamma_q(\theta - p + 1)}\right)$$

$$= I_q^{\gamma-\theta}\varphi(t) - \sum_{p=1}^{m} D_q^{\theta-p} \varphi(0^+) \frac{t^{\gamma-p}}{\Gamma_q(\gamma - p + 1)}$$

for all $t \in (0,b]$, (see [37]).

**Second case:** If $\theta \geq \gamma$, then this condition is obtained,

$$I_q^\gamma D_q^\theta \varphi(t) = D_q^{\theta-\gamma}\left(I_q^\theta D_q^\theta \varphi(t)\right)$$

$$= D_q^{\theta-\gamma}\left(\varphi(t) - \sum_{p=1}^{m} D_q^{\theta-p} \varphi(0^+) \frac{t^{\theta-p}}{\Gamma_q(\theta - p + 1)}\right),$$

$$= D_q^{\theta-\gamma}\varphi(t) - \sum_{p=1}^{m} D_q^{\theta-p} \varphi(0^+) \frac{t^{\gamma-p}}{\Gamma_q(\gamma - p + 1)}$$

for every $t \in (0,b]$ (see [37]).

### 3.4.3 q-fractional Caputo Operator

**Definition 3.4.3.1:** For $\gamma > 0$, the q-fractional Caputo operator is given by (see [37]);

$$ {}^{C}D_q^{\gamma}\varphi(t) = I_q^{m-\gamma} D_q^{m}\varphi(t), \qquad m := \lceil \gamma \rceil, \tag{3.4.3.1}$$

when $\gamma$ is nonnegative integer then the operator is introduced by

$$ {}^{C}D_q^{m}\varphi(t) = I_q^{0} D_q^{m}\varphi(t) = D_q^{m}\varphi(t).$$

**Theorem 3.4.3.2:** Let $\gamma > 0$ and $m = \lceil \gamma \rceil$. When $\varphi \in \mathcal{AC}_q^{(m)}[0,b]$, then ${}^{C}D_q^{\gamma}\varphi(t) \in \mathcal{L}_q^1[0,b]$.

**Proof:** when $\varphi \in \mathcal{AC}_q^{(m)}[0,b]$, then it is clear that $D_q^{(m)}\varphi \in \mathcal{L}_q^1[0,b]$. Thus, we conclude

$$ {}^{C}D_q^{\gamma}\varphi(t) = I_q^{m-\gamma} D_q^{m}\varphi(t) \in \mathcal{L}_q^1[0,b].$$

■

**Theorem 3.4.3.3:** Let $\gamma$ and $\theta$ be positive numbers and if $m = \lceil \gamma \rceil$ and $n = \lceil \theta \rceil$ (see [37]), then

**1.** When $\varphi \in \mathcal{AC}_q^{(n)}[0,b]$, then [37];

$$ I_q^{\gamma}\, {}^{C}D_q^{\theta}\varphi(t) = \begin{cases} I_q^{\theta-\gamma}\varphi(t) - \sum_{p=0}^{n-1} \frac{D_q^{p}\varphi(0^+)}{\Gamma_q(\theta-\gamma+p+1)} t^{\theta-\gamma+p}, & \theta \ge \gamma, \\ D_q^{\theta-\gamma}\varphi(t) - \sum_{p=0}^{m-1} \frac{D_q^{p}\varphi(0^+)}{\Gamma_q(\theta-\gamma+p+1)} t^{\theta-\gamma+p}, & \theta < \gamma, \end{cases} \tag{3.4.3.2}$$

for every $t \in (0,b]$.

**2.** When $\varphi \in \mathcal{L}_q^1[0,b]$ such that $I_q^{\gamma}\varphi \in \mathcal{AC}_q^{(n)}[0,b]$, then [37];

$$ {}^{C}D_q^{\theta} I_q^{\gamma}\varphi(t) = \begin{cases} I_q^{\gamma-\theta}\varphi(t), & \gamma \ge n \ge \theta, \\ D_q^{\theta-\gamma}\varphi(t) - \sum_{p=0}^{\lceil n-\gamma \rceil} \frac{D_q^{n-\gamma-p}\varphi(0^+)}{\Gamma_q(n-\theta-p+1)} t^{n-\theta-p}, & \theta > \gamma, \end{cases} \tag{3.4.3.3}$$

for every $t \in (0,\mathrm{b}]$.

**Proof:** Initially we show (3.4.2) by (see [37]);

$$ I_q^{\gamma}\, {}^{C}D_q^{\theta}\varphi(t) = I_q^{\gamma} I_q^{n-\theta} D_q^{n}\varphi(t) = I_q^{n+\gamma-\theta} D_q^{n}\varphi(t).$$

Then (3.4.2) follows for $\gamma \ge n \ge \theta$ by using,

$D_q^{\gamma} I_q^{\theta}\varphi(t) = I_q^{\theta-\gamma}\varphi(t)$ and $\varphi \in \mathcal{L}_q^1[0,a]$ and $\theta > \gamma$. Later, if we use

$$ D_q^{\gamma} I_q^{\theta}\varphi(t) = D_q^{\gamma-\theta}\varphi(t), \qquad (\gamma > \theta \ge 0),$$

where $\gamma$ and $\theta$ are interchanged by $n + \gamma - \theta$ and $n$. Taking into account that

$$^{C}D_q^{\theta} I_q^{\gamma} \varphi(t) = I_q^{n-\theta} D_q^n I_q^{\gamma} \varphi(t),$$

We reach at,

$$D_q^n I_q^{\gamma} \varphi(t) = \begin{cases} I_q^{\gamma-n}, & \gamma \geq n \geq \theta, \\ D_q^{n-\gamma}, & \gamma < n. \end{cases}$$

Therefore, when $\gamma \geq n \geq \theta$ from the semigroup property (see [37]);

$$^{C}D_q^{\theta} I_q^{\gamma} \varphi(t) = I_q^{n-\theta} I_q^{\gamma-n} \varphi(t) = I_q^{\gamma-\theta} \varphi(t).$$

if $\gamma < n$, then

$$^{C}D_q^{\theta} I_q^{\gamma} \varphi(t) = I_q^{n-\theta} D_q^{n-\gamma} \varphi(t).$$

■

### 3.4.4 q-Laplace transform of fractional q-Integrals and q-Derivatives

In this part, q-Laplace transform are shown for the Riemann-Liouville fractional q-integral and q-derivatives.

**Theorem 3.4.4.1:** In [37], if $\bar{\varphi} \in \ell_q^1[0, b]$ and $\mathbf{\Omega}(s) = {}_qL_s\{\bar{\varphi}(t)\}$ then,

$${}_qL_s\{I_q^{\theta} \bar{\varphi}(t)\} = \frac{(1-q)^{\theta}}{s^{\theta}} \mathbf{\Omega}(s), \qquad \forall \theta > 0, \tag{3.4.4.1}$$

when $m - 1 < \theta \leq m$ and $I_q^{m-\theta} \bar{\varphi}(t) \in \mathcal{AC}_q^{(m)}[0, b]$ then,

$${}_qL_s\{D_q^{\theta} \bar{\varphi}(t)\} = \frac{s^{\theta}}{(1-q)^{\theta}} \mathbf{\Omega}(s) - \sum_{k=1}^{m} D_q^{\theta-k} \bar{\varphi}(0^+) \frac{s^{k-1}}{(1-q)^k}, \tag{3.4.4.2}$$

**Proof:**

$$I_q^{\theta} \bar{\varphi}(t) = (1-q) \frac{t^{\theta-1}}{\Gamma_q(\theta)} * {}_q\bar{\varphi}(t)$$

$$= (1-q)\big(\Omega_{\theta-1}(t) * {}_q\bar{\varphi}(t)\big),$$

(3.4.4.1) is satisfied from,

$$\Omega_{\theta}(t) = \frac{t^{\theta}}{\Gamma_q(\theta+1)}, \quad (\theta > -1)$$

then,

$${}_qL_s\{\Omega_{\theta}(t)\} = \frac{(1-q)^{\theta}}{s^{\theta+1}}, \quad Re(s) > 0,$$

$$(\bar{\varphi} * \bar{\psi}) = \frac{1}{1-q} \int_0^t \bar{\varphi}(\tau) \varepsilon^{-q\tau} \bar{\psi}(\tau) d_q\tau$$

and

$${}_qL_s\{\bar{\varphi} * \bar{\psi}\} = {}_qL_s\bar{\varphi} \; {}_qL_s\bar{\psi}.$$

Now, (3.4.4.2) was proven from,

$$D_q^\theta \varphi(t) = \Omega(t) = D_q^k I_q^{k-\theta} \varphi(t), \quad (k = \lceil \theta \rceil),$$

$$_qL_s\{D_q^\theta \varphi(t)\} = \left(\frac{s}{1-q}\right)^\theta \Omega(s) - \sum_{k=1}^{m} D_q^{m-k} \varphi(0) \frac{s^{k-1}}{(1-q)^k}, \quad (m \in \mathbb{N}),$$

and

$$_qL_s\{I_q^\theta \bar{\varphi}(t)\} = \frac{(1-q)^\theta}{s^\theta} \mathbf{\Omega}(s), \quad (\theta > 0).$$

We obtain,

$$_qL_s\{D_q^\theta \bar{\varphi}(t)\} = {}_qL_s\{D_q^k I_q^{k-\theta} \bar{\varphi}(t)\}$$

$$= \frac{s^k}{(1-q)^k} \, {}_qL_s\{I_q^{k-\theta} \bar{\varphi}(t)\} - \sum_{k=1}^{m} D_q^{m-k} I_q^{m-\theta} \bar{\varphi}(0^+) \frac{s^{k-1}}{(1-q)^k}$$

$$= \frac{s^\theta}{(1-q)^\theta} \mathbf{\Omega}(s) - \sum_{k=1}^{m} D_q^{\theta-k} \bar{\varphi}(0^+) \frac{s^{k-1}}{(1-q)^k}, \quad (3.4.4.3)$$

■

**Theorem 3.4.4.2:** [37] If $_qL_s\{\varphi(t)\} = \mathbf{\Omega}(s)$ then the q-Laplace transform of the Caputo fractional q-derivative is introduced by,

$$_qL_s\{{}^C D_q^\theta \varphi(t)\} = \frac{s^\theta}{(1-q)^\theta} \left( \mathbf{\Omega}(s) - \sum_{k=0}^{m-1} D_q^k \varphi(0^+) \frac{(1-q)^k}{s^{k+1}} \right)$$

**Proof:** Because, ${}^C D_q^\theta \varphi(t) = (1-q) D_q^m \varphi(t) * \Omega_{m-\theta-1}(t)$ then by,

$$\Omega_\theta(t) = \frac{t^\theta}{\Gamma_q(\theta+1)}, \quad (\theta > -1)$$

then,

$$_qL_s\{\Omega_\theta(t)\} = \frac{(1-q)^\theta}{s^{\theta+1}}, \quad Re(s) > 0$$

$$_qL_s\{D_q^\theta \varphi(t)\} = \left(\frac{s}{1-q}\right)^\theta \Omega(s) - \sum_{k=1}^{m} D_q^{m-k} \varphi(0) \frac{s^{k-1}}{(1-q)^k}, \quad (m \in \mathbb{N})$$

then

$$_qL_s\{{}^C D_q^\theta \varphi(t)\} = \frac{(1-q)^{m-\theta}}{s^{m-\theta}} \, {}_qL_s\{D_q^m \varphi(t)\}$$

$$= \frac{(1-q)^{m-\theta}}{s^{m-\theta}} \left( \left(\frac{s}{1-q}\right)^m \mathbf{\Omega}(s) - \sum_{k=1}^{m} D_q^{m-k} \varphi(0^+) \frac{s^{k-1}}{(1-q)^k}, \right)$$

$$= \left( \frac{s^{\theta}}{(1-q)^{\theta}} \mathbf{\Omega}(s) - \sum_{k=0}^{m-1} D_q^k \varphi(0^+) \frac{s^{-k+\theta-1}}{(1-q)^{-k+\theta}} \right)$$

■

**Lemma 3.4.4.1:** [37] If ${}_qL_s\{\varphi(t)\} = \mathbf{\Omega}(s)$ then q-Laplace transform of the Riemann-Liouville sequential q-derivative of order $k\theta, 0 < \theta < 1$, is defined by,

$${}_qL_s\{\mathfrak{D}_q^{k\theta}\varphi(t)\} = r^{k\theta}\Omega(s) - \frac{1}{1-q}\sum_{k=0}^{m-1} r^{\theta(m-1-k)} I_q^{1-\theta} D_q^{\theta p}\varphi(0^+).$$

and $r = \frac{s}{1-q}$. When $\mathfrak{D}_q^{k\theta}$ is given in;

$$\mathfrak{D}_q^{\theta} y = D_q^{\theta} y, \qquad \mathfrak{D}_q^{k\theta} y = D_q^{\theta}\mathfrak{D}_q^{(k-1)\theta} y, \quad (k \in \mathbb{N}).$$

**Lemma 3.4.4.2:** [37] If ${}_qL_s\{\varphi(t)\} = \mathbf{\Omega}(s)$, then the q-Laplace transform of the Caputo sequential q-derivative of order $k\theta, 0 < \theta < 1$, is introduced as,

$${}_qL_s\{{}^{C}\mathfrak{D}_q^{k\theta}\varphi(t)\} = r^{k\theta}\Omega(s) - \frac{1}{1-q}\sum_{k=0}^{m-1} r^{\theta(m-1-k)}\, {}^{C}\mathfrak{D}_q^{\theta p}\varphi(0^+).$$

**Lemma 3.4.4.3:** In [37], let $\theta, \gamma, a \in \mathbb{R}^+$ and $n \in \mathbb{N}$. So the expression is

$${}_qL_s\left\{t^{n\theta+\gamma-1} e_{\theta,\gamma}^{(n)}\left(\pm a t^{\theta}; q\right)\right\} = \frac{r^{\theta-\gamma}}{1-q}\frac{n!}{(r^{\theta} \mp a)^{n+1}}, \qquad |r|^{\theta} > a, \qquad (3.4.4.4)$$

is valid in the disc $\{t \in \mathbb{C}: a|t(1-q)|^{\theta} < 1\}$.

**Proof:** From (3.3.4.3) then we satisfy,

$$e_{\theta,\gamma}^{(n)}(v;q) = \sum_{k=0}^{\infty} \frac{k(k-1)\dots(k-n+1)}{\Gamma_q(\theta k+\gamma)} v^{k-n}, \ \ |v| < (1-q)^{-\theta},$$

So, for $|a t^{\theta}| < (1-q)^{-\theta}$ we satisfy,

$$t^{n\theta+\gamma-1} e_{\theta,\gamma}^{(n)}\left(\pm a t^{\theta}; q\right) = \sum_{k=0}^{\infty} \frac{k(k-1)\dots(k-n+1)}{\Gamma_q(\theta k+\gamma)} (\pm a)^{k-n} t^{\theta k+\gamma-1},$$

Hence,

$${}_qL_s\left\{t^{n\theta+\gamma-1} e_{\theta,\gamma}^{(n)}\left(\pm a t^{\theta}; q\right)\right\} = \sum_{k=0}^{\infty} k(k-1)\dots(k-n+1)(\pm a)^{k-n} \frac{(1-q)^{\theta k+\gamma-1}}{s^{\theta k+\gamma}}$$

$$= \frac{r^{-n\theta-\gamma}}{1-q}\sum_{k=0}^{\infty} k(k-1)\dots(k-n+1)\left(\pm a r^{-\theta}\right)^{k-n}$$

$$= \frac{r^{-n\theta-\gamma}}{1-q}\frac{d^n}{d\xi^n}\sum_{k=0}^{\infty}\xi^k, \quad \xi = \pm a r^{-\theta}.$$

By using $a^{-1}|r|^{\theta} > 1$ then we satisfy, $|\xi| = |ar^{-\theta}| < 1$. Because,

$$\frac{r^{-n\theta-\gamma}}{1-q}\frac{d^n}{d\xi^n}\sum_{k=0}^{\infty}\xi^k = \frac{r^{-n\theta-\gamma}}{1-q}\frac{d^n}{d\xi^n}\frac{1}{1-\xi}$$

$$= \frac{r^{-n\theta-\gamma}}{1-q}\frac{n!}{(1-\xi)^{n+1}}$$

$$= \frac{r^{-n\theta-\gamma}}{1-q}\frac{n!}{(1\mp ar^{-\theta})^{n+1}}$$

$$= \frac{r^{\theta-\gamma}}{1-q}\frac{n!}{(r^{\theta}\mp a)^{n+1}}. \qquad \blacksquare$$

**Lemma 3.4.4.4:** Now, consider $\Omega_{\theta,l}(t,\mu_i)$ is the function introduced for $l = 1,2,\dots,\sigma_i$, $i = 1,\dots,k$ and $\mu_i \in \mathbb{C}$ by,

$$\Omega_{\theta,l}(t,\mu_i) = t^{l\theta+\theta-1}\sum_{k=0}^{\infty}(k+l)(k+l-1)\dots(k+1)\frac{(\mu_i t^{\theta})^k}{\Gamma_q(k\theta+l\theta+\theta)},$$

$i = 1,\dots.,K,\ \ l = 1,\dots,\sigma_i$ and $|\mu_i||t(1-q)|^{\theta} < 1$. Then,

$$_qL_s\{\Omega_{\theta,l}(t,\mu_i)\} = \frac{l!}{1-q}\left(r^{\theta}-\mu_i\right)^{-l-1},$$

holds for $|r|^{\theta} > |\mu_i|$ when $r = \frac{s}{1-q}$, [42].

**Proof:** By using the properties of the q-Laplace transform (see [37]), the following is obtained

$$_qL_s\{\Omega_{\theta,l}(t,\mu_i)\} = {}_qL_s\left\{\sum_{k=0}^{\infty}(k+l)(k+l-1)\dots(k+1)\frac{(\mu_i)^k t^{k\theta+l\theta+\theta-1}}{\Gamma_q(k\theta+l\theta+\theta)}\right\}$$

$$= \frac{r^{-l\theta-\theta}}{1-q}\sum_{k=0}^{\infty}(k+l)(k+l-1)\dots(k+1)\left(\mu_i r^{-\theta}\right)^k.$$

So, for $|\mu_i| < r^{\theta}$, then

$$_qL_s\{\Omega_{\theta,l}(t,\mu_i)\} = \frac{r^{-l\theta-\theta}}{1-q}\frac{d^l}{dz^l}\sum_{k=0}^{\infty}z^k|_{z=\mu_i r^{-\theta},}$$

$$= \frac{r^{-l\theta-\theta}}{1-q}\frac{d^l}{dz^l}\frac{1}{1-z}|_{z=\mu_i r^{-\theta},}$$

$$= \frac{l!}{(1-q)(r^{\theta}-\mu_i)^{l+1}}.$$

$\blacksquare$

# CHAPTER IV

# Q-PROPORTIONAL DERIVATIVE AND APPLICATION OF Q-LAPLACE TRANSFORM

In this chapter, we focus on the definition of the q-proportional derivative.

## 4.1 DEFINITION OF THE Q-PROPORTIONAL DERIVATIVE

In this part, q-proportional derivative will be introduced. The discrete version of proportional derivative was applied with coronavirus model by [64]. Also, it was applied to other fields of engineering models in [65]. Similarly, the q-proportional derivative can be related with control systems, dynamical systems and mathematical biology. Also, q-Laplace transforms were showed to solve both q-proportional derivative and q-fractional proportional derivative in this thesis.

Initially, q-proportional derivative is defined as;

$$\mathbf{D_q}\varphi(t) = \kappa_1(\beta, t)\varphi(t) + \kappa_0(\beta, t)D_q\varphi(t) \tag{4.1.1}$$

and $\beta \in [0,1]$. With this information, it is concluded that $\kappa_0, \kappa_1: [0,1] \times \mathbb{R} \to [0, \infty)$ be continuous such that for every $t \in \mathbb{R}$, [21, 22].

$$\lim_{\beta \to 0^+} \kappa_1(\beta, t) = 1,$$

$$\lim_{\beta \to 0^+} \kappa_0(\beta, t) = 0,$$

$$\lim_{\beta \to 1^-} \kappa_1(\beta, t) = 0,$$

$$\lim_{\beta \to 1^-} \kappa_0(\beta, t) = 1,$$

and $\kappa_1(\beta, t) \neq 0, \beta \in [0,1), \kappa_0(\beta, t) \neq 0, \beta \in (0,1]$. Specifically, the case is assumed when $\kappa_1(\beta, t) = 1 - \beta$ and $\kappa_0(\beta, t) = \beta$. Then, (4.1.1) becomes

$$\mathbf{D_q}\varphi(t) = (1 - \beta)\varphi(t) + \beta D_q\varphi(t). \tag{4.1.2}$$

Notably, $\lim_{\beta \to 0^+} \mathbf{D}_q\varphi(t) = \varphi(t)$ and $\lim_{\beta \to 1^-} \mathbf{D}_q\varphi(t) = D_q\varphi(t)$.

Also, ($0 < q < 1$ and $0 < \beta < 1$ in (4.1.2). Now, $\mathbf{D}_q^2\varphi(t), \mathbf{D}_q^3\varphi(t), \dots, \mathbf{D}_q^m\varphi(t)$ can be modified in (4.1.2) but $m$ is higher order of q-derivative. The second, third and higher order terms are given as;

$$\mathbf{D}_q\big(\mathbf{D}_q\varphi(t)\big) = (1-\beta)D_q\varphi(t) + \beta D_q^2\varphi(t)$$

then

$$\mathbf{D}_q^2\varphi(t) = (1-\beta)\big[(1-\beta)\varphi(t) + \beta D_q\varphi(t)\big] + \beta\big[(1-\beta)D_q\varphi(t) + \beta D_q^2\varphi(t)\big],$$

so we say,

$$\mathbf{D}_q^2\varphi(t) = (1-\beta)^2\varphi(t) + \beta(1-\beta)D_q\varphi(t) + \beta(1-\beta)D_q\varphi(t) + \beta^2 D_q^2\varphi(t),$$

$$\mathbf{D}_q^2\varphi(t) = (1-\beta)^2\varphi(t) + 2\beta(1-\beta)D_q\varphi(t) + \beta^2 D_q^2\varphi(t). \qquad (4.1.3)$$

Specifically, the second order q-derivative in (4.1.3) is identified with summation formula. (4.1.3) and (4.1.4) are similar. Therefore, (4.1.3) can be written as;

$$\mathbf{D}_q^2\varphi(t) = \sum_{k=0}^{2}\binom{2}{k}(1-\beta)^{2-k}\beta^k D_q^{(k)}\varphi(t) \qquad (4.1.4)$$

and ($0 < q, \beta < 1$).

The form of $\mathbf{D}_q^3\varphi(t)$ is given as;

$\mathbf{D}_q^3\varphi(t) = (1-\beta)^2 D_q\varphi(t) + 2\beta(1-\beta)D_q^2\varphi(t) + \beta^2 D_q^3\varphi(t)$, or

$$\mathbf{D}_q^3\varphi(t) = (1-\beta)^2 D_q\varphi(t) + 2\beta(1-\beta)D_q\left(D_q\varphi(t)\right) + \beta^2 D_q\big(D_q^2\varphi(t)\big)$$

then

$$\mathbf{D}_q^3\varphi(t) = (1-\beta)^2\big[(1-\beta)\varphi(t) + \beta D_q\varphi(t)\big] +$$
$$+2\beta(1-\beta)\big[(1-\beta)D_q\varphi(t) + \beta D_q^2\varphi(t)\big] + \beta^2\big[(1-\beta)D_q^2\varphi(t) + \beta D_q^3\varphi(t)\big]$$

and later,

$$\mathbf{D}_q^3\varphi(t) = (1-\beta)^3\varphi(t) + \beta(1-\beta)^2 D_q\varphi(t) + 2\beta(1-\beta)^2 D_q\varphi(t) +$$
$$+2\beta^2(1-\beta)D_q^2\varphi(t) + \beta^2(1-\beta)D_q^2\varphi(t) + \beta^3 D_q^3\varphi(t).$$

Finally, the new form of $\mathbf{D}_q^3\varphi(t)$ is found in (4.1.5).

$$\mathbf{D}_q^3\varphi(t) = (1-\beta)^3\varphi(t) + 3\beta(1-\beta)^2 D_q\varphi(t)$$
$$+ 3\beta^2(1-\beta)D_q^2\varphi(t) + \beta^3 D_q^3\varphi(t). \qquad (4.1.5)$$

Also, as mentioned above, (4.1.5) could be denoted as;

$$\mathbf{D}_q^3\varphi(t) = \sum_{k=0}^{3}\binom{3}{k}(1-\beta)^{3-k}\beta^k D_q^k\varphi(t). \qquad (4.1.6)$$

Similarly, fourth-order q-derivative can be defined with summation formula as below;

$$\mathbf{D}_q^4\varphi(t) = \sum_{k=0}^{4}\binom{4}{k}(1-\beta)^{4-k}\beta^k D_q^k\varphi(t) \tag{4.1.7}$$

and higher order q-derivative is written as below;

$$\mathbf{D}_q^m\varphi(t) = \sum_{k=0}^{m}\binom{m}{k}(1-\beta)^{m-k}\beta^k D_q^k\varphi(t). \tag{4.1.8}$$

## 4.2 SOLUTION OF Q-PROPORTIONAL DIFFERENTIAL EQUATIONS WITH Q-LAPLACE TRANSFORM

This section is about solution of q-proportional derivative and q-Laplace transform. Previously, q-Laplace transformation is introduced by [66]. Generally, by using the Laplace transformation, we can solve easily many differential equations. Similarly, by using the q-Laplace transformation we can solve q-proportional differential equations. Besides, q-Laplace transform is shown for the considered function and higher order derivatives of the function as;

$${}_q\mathcal{L}_s\{\varphi(bt)\} = (1/b)\Omega(s/b), \quad (b \neq 0)$$

so

$${}_q\mathcal{L}_s\{\varphi(t)\} = \Omega(s) \tag{4.2.1}$$

and

$${}_q\mathcal{L}_s\{D_q^m\varphi(t)\} = \left(\frac{s}{1-q}\right)^m\Omega(s) - \sum_{k=1}^{m} D_q^{m-k}\varphi(0)\frac{s^{k-1}}{(1-q)^k}, \qquad (m \in \mathbb{N}). \tag{4.2.2}$$

By using (4.2.2) for $m = 1$ we satisfy;

$${}_q\mathcal{L}_s\{D_q\varphi(t)\} = \left(\frac{s}{1-q}\right)\Omega(s) - \frac{\varphi(0)}{1-q}, \qquad (q \neq 1). \tag{4.2.3}$$

Similarly, we get q-Laplace transformation of q-derivative and it says $(0 < q < 1)$ in (4.2.3).

Now, we define the q-proportional derivative in order to apply q-Laplace transform and solve it easily. By using (4.1.2), we investigate the following q-differential equation

$$\mathbf{D}_q\varphi(t) \coloneqq (1-\beta)\varphi(t) + \beta D_q\varphi(t) = \psi(t). \tag{4.2.4}$$

**Proof of (4.2.4):** When q-Laplace transformation is used with (4.2.4) then, the proof is shown as,

$$ {}_q\mathcal{L}_s\{\psi(t)\} = {}_q\mathcal{L}_s\{(1-\beta)\varphi(t) + \beta D_q\varphi(t)\}, $$

or

${}_q\mathcal{L}_s\{\psi(t)\} = (1-\beta)\ {}_q\mathcal{L}_s\{\varphi(t)\} + \beta\ {}_q\mathcal{L}_s\{D_q\varphi(t)\}.$

$\Psi(s)$ is obtained as,

$$ \Psi(s) = (1-\beta)\Omega(s) + \beta\left[\frac{s}{1-q}\Omega(s) - \frac{\varphi(0)}{1-q}\right]. $$

After some standard calculations this new expressions are done

$\Psi(s) = (1-\beta)\Omega(s) + \frac{\beta s}{1-q}\Omega(s) - \frac{\beta\varphi(0)}{1-q}.$

As a result, it is reported that

$$ \Psi(s) = \left[\frac{(1-\beta)(1-q)+\beta s}{1-q}\right]\Omega(s) - \frac{\beta\varphi(0)}{1-q}, $$

$\Psi(s) + \frac{\beta\varphi(0)}{1-q} = \left[\frac{(1-\beta)(1-q)+\beta s}{1-q}\right]\Omega(s)$, respectively.

Finally, the following is obtained, namely

$$ \frac{(1-q)\Psi(s)+\beta\varphi(0)}{1-q} = \left[\frac{(1-\beta)(1-q)+\beta s}{1-q}\right]\Omega(s), $$

for $(0 < q < 1)$.

Then, it is concluded

$$ \begin{aligned} \Omega(s) &= \frac{(1-q)\Psi(s)+\beta\varphi(0)}{(1-\beta)(1-q)+\beta s} \\ &= \frac{(1-q)\Psi(s)}{(1-\beta)(1-q)+\beta s} + \frac{\beta\varphi(0)}{(1-\beta)(1-q)+\beta s} \\ &= \frac{(1-q)\Psi(s)}{\beta\left(s+\frac{(1-\beta)(1-q)}{\beta}\right)} + \frac{\beta\varphi(0)}{\beta\left(s+\frac{(1-\beta)(1-q)}{\beta}\right)}. \end{aligned} $$

Now the inverse q-Laplace transform is applied to find the expression of $\varphi(t)$, namely ${}_q\mathcal{L}_s^{-1}\{\Omega(s)\} = \varphi(t)$. After some calculations it is the followings is reported

$$ \begin{aligned} {}_q\mathcal{L}_s^{-1}\{\Omega(s)\} &= {}_q\mathcal{L}_s^{-1}\left\{\frac{(1-q)\Psi(s)}{\beta\left(s+\frac{(1-\beta)(1-q)}{\beta}\right)}\right\} \\ &\quad + {}_q\mathcal{L}_s^{-1}\left\{\frac{\beta\varphi(0)}{\beta\left(s+\frac{(1-\beta)(1-q)}{\beta}\right)}\right\}. \end{aligned} $$

Later, $\alpha = \frac{(1-\beta)(1-q)}{\beta}$, $(\beta > 1)$, $(0 < q < 1)$ and $(\alpha < 0)$ are assumed to show step by step easily. So, $\varphi$ is reported as,

$$\varphi(t) = \left(\frac{1-q}{\beta}\right) {}_q\mathcal{L}_s^{-1}\left\{\frac{\Psi(s)}{s-\alpha}\right\} + {}_q\mathcal{L}_s^{-1}\left\{\frac{\varphi(0)}{s-\alpha}\right\}$$

and

$\varphi(t) = \frac{(1-q)}{\beta}\left(\psi(t) * e_q(\alpha t)\right) + \varphi(0)e_q(\alpha t)$, respectively.

We recall that the convolution was introduced and inverse q-Laplace transformation were mentioned in [37]-[67]. So, the following is obtained

$$\varphi(t) = \frac{1-q}{\beta}\int_0^t \psi(\tau-\alpha)d_q\tau + \varphi(0)e_q(\alpha t). \tag{4.2.5}$$

# CHAPTER V

# Q-FRACTIONAL PROPORTIONAL DERIVATIVE

The original contributions of this thesis are reported below.

## 5.1 DEFINITION OF Q-FRACTIONAL PROPORTIONAL DERIVATIVE

In this section, we mention q-fractional proportional derivative where the Caputo q-fractional operator will be used, namely

$$^{CP}\mathbf{D}_q^{\theta}\varphi(t) = (1-\beta)\varphi(t) + \beta\, {}^{C}D_q^{\theta}\varphi(t), \qquad (0<\theta<1).$$

Similarly, the q-Laplace transform was identified for Caputo version in [66] and q-Laplace transform of q-fractional Caputo derivative is given as

$${}_q\mathcal{L}_s\{{}^{C}D_q^{\theta}\varphi(t)\} = \frac{s^{\theta}}{(1-q)^{\theta}}\, {}_q\mathcal{L}_s\{\varphi(t)\} - \frac{\varphi(0)s^{\theta-1}}{(1-q)^{\theta}}, \qquad (0<\theta<1)$$

and

$${}_q\mathcal{L}_s\{\varphi(t)\} = \Omega(s).$$

Then, from the above expression, (5.1.1) was obtained as,

$${}_q\mathcal{L}_s\{{}^{C}D_q^{\theta}\varphi(t)\} = \frac{s^{\theta}}{(1-q)^{\theta}}\Omega(s) - \frac{\varphi(0)s^{\theta-1}}{(1-q)^{\theta}}, \qquad (0<\theta<1), \quad (5.1.1)$$

In the next part, we investigate the solution of the following differential equation, as an example

$$^{CP}\mathbf{D}_q^{\theta}\varphi(t) = (1-\beta)\varphi(t) + \beta\, {}^{C}D_q^{\theta}\varphi(t) = \psi(t), \qquad (0<\theta<1). \qquad (5.1.2)$$

The detailed steps of finding the solution of (5.1.2) is given below:

We start with

$${}_q\mathcal{L}_s\{\psi(t)\} = {}_q\mathcal{L}_s\{(1-\beta)\varphi(t) + \beta\, {}^{C}D_q^{\theta}\varphi(t)\},$$

Thus, we conclude that

$$\Psi(s) = (1-\beta)\, {}_q\mathcal{L}_s\{\varphi(t)\} + \beta\, {}_q\mathcal{L}_s\{{}^{C}D_q^{\theta}\varphi(t)\}.$$

After some sample derivations we report that

$$\Psi(s) = (1-\beta)\Omega(s) + \beta\left[\frac{s^\theta}{(1-q)^\theta}\Omega(s) - \frac{\varphi(0)s^{\theta-1}}{(1-q)^\theta}\right]$$

or

$$\Psi(s) = \Omega(s)\left[(1-\beta) + \frac{\beta s^\theta}{(1-q)^\theta}\right] - \frac{\beta\varphi(0)s^{\theta-1}}{(1-q)^\theta}.$$

As a result, we conclude that

$$\Psi(s) + \frac{\beta\varphi(0)s^{\theta-1}}{(1-q)^\theta} = \Omega(s)\left[(1-\beta) + \frac{\beta s^\theta}{(1-q)^\theta}\right].$$

Thus, we report

$$\frac{\Psi(s)(1-q)^\theta + \beta\varphi(0)s^{\theta-1}}{(1-q)^\theta} = \Omega(s)\left[\frac{(1-\beta)(1-q)^\theta + \beta s^\theta}{(1-q)^\theta}\right].$$

From the above expression we get

$$\Omega(s) = \frac{\Psi(s)(1-q)^\theta + \beta\varphi(0)s^{\theta-1}}{(1-q)^\theta}\frac{(1-q)^\theta}{(1-\beta)(1-q)^\theta + \beta s^\theta}$$

or

$$\Omega(s) = \frac{\Psi(s)(1-q)^\theta + \beta\varphi(0)s^{\theta-1}}{(1-\beta)(1-q)^\theta + \beta s^\theta},$$

respectively. Rearranging the terms we conclude that

$$\Omega(s) = \frac{\Psi(s)(1-q)^\theta}{(1-\beta)(1-q)^\theta + \beta s^\theta} + \frac{\beta\varphi(0)s^{\theta-1}}{(1-\beta)(1-q)^\theta + \beta s^\theta}$$

or

$$\Omega(s) = \frac{\Psi(s)(1-q)^\theta}{\beta\left(s^\theta + \frac{(1-\beta)(1-q)^\theta}{\beta}\right)} + \frac{\beta\varphi(0)s^{\theta-1}}{\beta\left(s^\theta + \frac{(1-\beta)(1-q)^\theta}{\beta}\right)}.$$

Now, we consider that $\alpha = \frac{(1-\beta)(1-q)^\theta}{\beta}$ and $(0 < \theta < 1,\ 0 < q < 1)$, then $\alpha > 0$. So, we conclude

$$\Omega(s) = \frac{\Psi(s)(1-q)^\theta}{\beta(s^\theta + \alpha)} + \frac{\varphi(0)s^{\theta-1}}{(s^\theta + \alpha)}. \quad (5.1.3)$$

The inverse q-Laplace transform was used for (5.1.3), namely

$$_q\mathcal{L}_s^{-1}\{\Omega(s)\} = {}_q\mathcal{L}_s^{-1}\left\{\frac{\Psi(s)(1-q)^\theta}{\beta(s^\theta + \alpha)} + \frac{\varphi(0)s^{\theta-1}}{(s^\theta + \alpha)}\right\}$$

$$= \frac{(1-q)^\theta}{\beta}\ {}_q\mathcal{L}_s^{-1}\left\{\frac{\Psi(s)}{s^\theta + \alpha}\right\} + \varphi(0)\ {}_q\mathcal{L}_s^{-1}\left\{\frac{s^{\theta-1}}{s^\theta + \alpha}\right\}.$$

Convolution is used in here in order to introduce the inverse q-Laplace transform. Particularly, convolution was reported in [37]. By using the q-Laplace transformation, we obtained q-convolution

$$\varphi(t) = \frac{(1-q)^{\theta}}{\beta}\left[\ {}_q\mathcal{L}_s^{-1}\left\{\ {}_q\mathcal{L}_s\{\psi(t)\} *\ {}_q\mathcal{L}_s\left\{(1-q)t^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha t^{\theta};q)\right\}\right\}\right] + \varphi(0)e_{\theta,1}^{(0)}(-\alpha t^{\theta};q)(1-q).$$

Therefore we conclude

$$\varphi(t) = \frac{(1-q)^{\theta}}{\beta}\left[\ {}_q\mathcal{L}_s^{-1}\left(\ {}_q\mathcal{L}_s\left\{\psi(t) * (1-q)t^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha t^{\theta};q)\right\}\right)\right] + \varphi(0)e_{\theta,1}^{(0)}\left(-\alpha t^{\theta};q\right)(1-q).$$

The q-convolution formula was obtained from above steps of solutions

$$\varphi(t) = \frac{(1-q)^{\theta}}{\beta}\left(\psi(t) * (1-q)t^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha t^{\theta};q)\right) + \varphi(0)e_{\theta,1}^{(0)}\left(-\alpha t^{\theta};q\right)(1-q)\,. \tag{5.1.4}$$

Finally, q-convolution formula of (5.1.4) is written as;

$$\varphi(t) = \left(\frac{(1-q)^{\theta}}{\beta}\right)\left(\frac{1}{1-q}\right)\int_0^t \psi(\tau)\varepsilon^{-q\tau}(1-q)\tau^{\theta-1}e_{\theta,\theta}^{(0)}\left(-\alpha\tau^{\theta};q\right)d_q\tau + \varphi(0)e_{\theta,1}^{(0)}\left(-\alpha t^{\theta};q\right)(1-q),$$

or

$$\varphi(t) = \left(\frac{(1-q)^{\theta}}{\beta}\right)\int_0^t \psi(\tau)\varepsilon^{-q\tau}\tau^{\theta-1}e_{\theta,\theta}^{(0)}\left(-\alpha\tau^{\theta};q\right)d_q\tau + \varphi(0)e_{\theta,1}^{(0)}\left(-\alpha t^{\theta};q\right)(1-q). \tag{5.1.5}$$

Hence, the solution of a q-fractional proportional differential equation in Caputo sense given by equation (5.1.2) is derived, as an illustrative example.

## CHAPTER VI

## CONCLUSION

In this thesis, the definitions and some properties of q-derivative and q-integral were reviewed. After that, the q-analogue of the  proportional derivative is mentioned. Next, the fractional counterpart of the q-proportional derivatives are discussed. Thereafter, for the first time, the q-analogue of the proportional derivative was introduced. More precisely, we defined the following operator:

$$^{CP}\mathbf{D}_q^{\theta}\varphi(t) = (1-\beta)\varphi(t) + \beta\, {}^{C}D_q^{\theta}\varphi(t), \qquad (0 < \theta < 1).$$

The solutions of some q-differential equations in the frame of q-proportional fractional derivatives were found with the assistance of the q-Laplace transforms.

It is worth mentioning that since the operators discussed in this thesis, any qualitative results such as Gronwall inequality and Lyapunov inequality may be obtained for related q-difference operators. On the top of this, optimal control and biological problems can be reformulated in the framework of such operators.

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## APPENDICES

**APPENDIX 1:** Our used formulas of q-Laplace transforms and inverse q-Laplace transforms are given below:

1. $\mathcal{L}_q\{\varphi(t)\} = \Omega(s)$

2. $\mathcal{L}_q\{D_q\varphi(t)\} = \left(\frac{s}{1-q}\right)\Omega(s) - \frac{\varphi(0)}{1-q},\ (q \neq 1, 0 < q < 1)$

3. $\mathcal{L}_q\{\varphi(bt)\} = (1/b)\Omega(s/b),\quad (b \neq 0)$

4. $\mathcal{L}_q\{D_q^m\varphi(t)\} = \left(\frac{s}{1-q}\right)^m \Omega(s) - \sum_{k=1}^{m} D_q^{m-k}\varphi(0)\frac{s^{k-1}}{(1-q)^k},\qquad (m \in \mathbb{N})$

5. $\mathcal{L}_q^{-1}\left\{\frac{\varphi(0)}{s-\alpha}\right\} = \varphi(0)e_q(\alpha t)$

6. $\mathcal{L}_q^{-1}\left\{\frac{\Psi(s)}{s-\alpha}\right\} = \psi(t) * e_q(\alpha t) = \int_0^t \psi(t-\alpha)d_q\tau$

7. $\mathcal{L}_q\{{}^{C}D_q^{\theta}\varphi(t)\} = \dfrac{s^\theta}{(1-q)^\theta}\Omega(s) - \dfrac{\varphi(0)s^{\theta-1}}{(1-q)^\theta},\quad (0 < \theta < 1,\qquad 0 < q < 1)$

8. $\mathcal{L}_q^{-1}\left\{\frac{s^{\theta-1}}{s^\theta+\alpha}\right\} = \varphi(0)e_{\theta,1}^{(0)}(-\alpha t^\theta; q)(1-q),\quad (0 < \theta < 1)$

9. $\mathcal{L}_q^{-1}\left\{\frac{\Psi(s)}{s^\theta+\alpha}\right\} = \psi(t) * (1-q)t^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha t^\theta; q)$

$$= \frac{1}{1-q}\int_0^t \psi(\tau)\varepsilon^{-q\tau}(1-q)\tau^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha\tau^\theta; q)d_q\tau$$

$$= \int_0^t \psi(\tau)\varepsilon^{-q\tau}\tau^{\theta-1}e_{\theta,\theta}^{(0)}(-\alpha\tau^\theta; q)d_q\tau.$$