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Research article

A variety of dynamic α-conformable Steffensen-type inequality on a time scale measure space

Ahmed A. El-Deeb[1,*], **Osama Moaaz**[2], **Dumitru Baleanu**[3,4,5,*] **and Sameh S. Askar**[6]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City (11884), Cairo, Egypt
[2] Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt
[3] Institute of Space Science, Magurele-Bucharest, Romania
[4] Department of Mathematics, Cankaya University, Ankara 06530, Turkey
[5] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
[6] Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

* **Correspondence:** Email: ahmedeldeeb@zhar.edu.eg; dumitru@cankaya.edu.tr.

Abstract: The main objective of this work is to establish several new alpha-conformable of Steffensen-type inequalities on time scales. Our results will be proved by using time scales calculus technique. We get several well-known inequalities due to Steffensen, if we take $\alpha = 1$. Some cases we get continuous inequalities when $\mathbb{T} = \mathbb{R}$ and discrete inequalities when $\mathbb{T} = \mathbb{Z}$.

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1. Introduction

Riemann-Liouville fractional integral given by

$$I^{\alpha}_{a+}\xi(\wp) = \frac{1}{\Gamma(\alpha)} \int_a^{\chi} (\chi - \wp)^{\alpha-1} \xi(\wp) dt.$$

Many different concepts of fractional derivative maybe found in [9–11]. In [12] studied a conformable derivative:

$$\wp_{\alpha} f(\wp) = \lim_{\epsilon \to 0} \frac{f(\wp + \epsilon \wp^{1-\alpha}) - f(\wp)}{\epsilon}.$$

The time scale conformable derivatives was introduced by Benkhettou et al. [17].

Further, in recent years, numerous mathematicians claimed that non-integer order derivatives and integrals are well suited to describing the properties of many actual materials, such as polymers. Fractional derivatives are a wonderful tool for describing memory and learning. a variety of materials and procedures inherited properties is one of the most significant benefits of fractional ownership. For more concepts and definition on time scales see [13–19, 33–35].

Continuous version of Steffensen's inequality [7] is written as: For $0 \leq g(\wp) \leq 1$ on $\in [a, b]$. Then

$$\int_{b-\lambda}^{b} f(\wp)dt \leq \int_{a}^{b} f(\wp)g(\wp)dt \leq \int_{a}^{a+\lambda} f(\wp)dt, \tag{1.1}$$

where $\lambda = \int_a^b g(\wp)dt$.

Supposing f is nondecreasing gets the reverse of (1.1).

Also, the discrete inequality of Steffensen [6] is: For $\lambda_2 \leq \sum_{\ell=1}^{n} g(\ell) \leq \lambda_1$. Then

$$\sum_{\ell=n-\lambda_2+1}^{n} f(\ell) \leq \sum_{\ell=1}^{n} f(\ell)g(\ell) \leq \sum_{\ell=1}^{\lambda_1} f(\ell). \tag{1.2}$$

Recently, a large number of dynamic inequalities on time scales have been studied by a small number of writers who were inspired by a few applications (see [1–4,8,28–32,36,37,40–42,44,48–53]).

In [5] Jakšetić et al. proved that, if $\hat{\mu}([c, d]) = \int_{[a,b]} g(\wp)d\hat{\mu}(\wp)$, where $[c, d] \subseteq [a, b]$. Then

$$\int_{[a,b]} f(\wp)g(\wp)d\hat{\mu}(\wp) \leq \int_{[c,d]} f(\wp)g(\wp)d\hat{\mu}(\wp) + \int_{[a,c]} (f(\wp) - f(d))g(\wp)d\hat{\mu}(\wp),$$

and

$$\int_{[c,d]} f(\wp)d\hat{\mu}(\wp) - \int_{[d,b]} (f(c) - f(\wp))g(\wp)d\hat{\mu}(\wp) \leq \int_{[a,b]} f(\wp)g(\wp)d\hat{\mu}(\wp).$$

Anderson, in [3], studied the inequality:

$$\int_{b-\lambda}^{b} \phi(\wp)\nabla\wp \leq \int_{a}^{b} \phi(\wp)\psi(\wp)\nabla\wp \leq \int_{a}^{a+\lambda} \phi(\wp)\nabla\wp, \tag{1.3}$$

In [47] the authors have proved, for

$$\int_{m}^{m+\lambda_1} \zeta(\wp)d\wp = \int_{m}^{k} \zeta(\wp)g(\wp)d\wp,$$

and

$$\int_{n-\lambda_2}^{n} \zeta(\wp)d\wp = \int_{k}^{n} \zeta(\wp)g(\wp)d\wp.$$

If there exists a constant A such that $r(\wp)/\zeta(\wp) - At$ is monotonic on the intervals $[m, k]$, $[k, n]$, and

$$\int_{m}^{n} tq(\wp)g(\wp)d\wp = \int_{m}^{m+\lambda_1} tq(\wp)d\wp + \int_{n-\lambda_2}^{n} tq(\wp)d\wp,$$

then

$$\int_m^n r(\wp)g(\wp)d\wp \le \int_m^{m+\lambda_1} r(\wp)d\wp + \int_{n-\lambda_2}^n r(\wp)d\wp.$$

In particularly, Anderson [3] proved

$$\int_{n-\lambda}^n r(\wp)\nabla\wp \le \int_m^n r(\wp)g(\wp)\nabla\wp \le \int_m^{m+\lambda} r(\wp)\nabla\wp.$$

where $m, n \in \mathbb{T}_\kappa$ with $m < n$, r, $g : [m,n]_\mathbb{T} \to \mathbb{R}$ are ∇-integrable functions such that r is of one sign and nonincreasing and $0 \le g(\wp) \le 1$ on $[m,n]_\mathbb{T}$ and $\lambda = \int_m^n g(\wp)\nabla\wp$, $n-\lambda, m+\lambda \in \mathbb{T}$.

We prove the next two needed results:

Theorem 1.1. Assume $q > 0$ with $0 \le g(\wp) \le \zeta(\wp)$ $\forall \wp \in [m,n]_\mathbb{T}$ and λ is given from $\int_m^n g(\wp)\Delta_\alpha\wp = \int_m^{m+\lambda} \zeta(\wp)\Delta_\alpha\wp$, then

$$\int_m^n r(\wp)g(\wp)\Delta_\alpha\wp \le \int_m^{m+\lambda} r(\wp)\zeta(\wp)\Delta_\alpha\wp. \tag{1.4}$$

Also, provided with $0 \le g(\wp) \le \zeta(\wp)$ and $\int_{n-\lambda}^n \zeta(\wp)\Delta_\alpha\wp = \int_m^n g(\wp)\Delta_\alpha\wp$, we have

$$\int_{n-\lambda}^n r(\wp)\zeta(\wp)\Delta_\alpha\wp \le \int_m^n r(\wp)g(\wp)\Delta_\alpha\wp. \tag{1.5}$$

We get the reverse inequalities of (1.4) and (1.5) when assuming r/ζ is nondecreasing.

Theorem 1.2. Assume ψ is integrable on time scales interval $[m,n]$, with $\zeta(\wp)-\psi(\wp) \ge g(\wp) \ge \psi(\wp) \ge 0$ $\forall \wp \in [m,n]_\mathbb{T}$ and $\int_m^{m+\lambda} \zeta(\wp)\Delta_\alpha\wp = \int_m^n g(\wp)\Delta_\alpha\wp = \int_{n-\lambda}^n \zeta(\wp)\Delta_\alpha\wp$ and g, r and ζ are Δ_α-integrable functions, $\zeta(\wp) \ge g(\wp) \ge 0$, we have

$$\begin{aligned}
&\int_{n-\lambda}^n r(\wp)\zeta(\wp)\Delta_\alpha\wp + \int_m^n \Big|(r(\wp) - r(n-\lambda))\psi(\wp)\Big|\Delta_\alpha\wp \\
&\le \int_m^n r(\wp)g(\wp)\Delta_\alpha\wp \\
&\le \int_m^{m+\lambda} r(\wp)\zeta(\wp)\Delta_\alpha\wp - \int_m^n \Big|(r(\wp) - r(m+\lambda))\psi(\wp)\Big|\Delta_\alpha\wp,
\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}
\int_{n-\lambda}^n r(\wp)\zeta(\wp)\Delta_\alpha\wp &\le \int_{n-\lambda}^n \Big[r(\wp)\zeta(\wp) - (r(\wp) - r(n-\lambda))\Big][\zeta(\wp) - g(\wp)]\Delta_\alpha\wp \\
&\le \int_m^n r(\wp)g(\wp)\Delta_\alpha\wp \\
&\le \int_m^{m+\lambda} \Big[r(\wp)\zeta(\wp) - (r(\wp) - r(m+\lambda))\Big][\zeta(\wp) - g(\wp)]\Delta_\alpha\wp \\
&\le \int_m^{m+\lambda} r(\wp)\zeta(\wp)\Delta_\alpha\wp.
\end{aligned} \tag{1.7}$$

Proof. The proof techniques of Theorems 1.6 and 1.7 are like to that in [4] and is removed. □

Several authors proved conformable Hardy's inequality [20, 21], conformable Hermite-Hadamard's inequality [22–24], conformable inequality of Opial's [26, 27] and conformable inequality of Steffensen's [25]. In [45] Anderson proved the followong results:

Theorem 1.3. [45] Suppose $\alpha \in (0,1]$ and $r_1, r_2 \in \mathbb{R}$ such that $0 \leq r_1 \leq r_2$. Suppose $\prod : [r_1, r_2] \to [0,\infty)$ and $\Gamma : [r_1, r_2] \to [0,1]$ are α-fractional integrable functions on $[r_1, r_2]$ with Π is decreasing, we get

$$\int_{r_2-\aleph}^{r_2} \Pi(\zeta) d_\alpha \zeta \leq \int_{r_1}^{r_2} \Pi(\zeta)\Gamma(\zeta) d_\alpha \zeta \leq \int_{r_1}^{r_1+\aleph} \Pi(\zeta) d_\alpha \zeta,$$

where $\aleph = \frac{\alpha(r_2-r_1)}{r_2^\alpha - r_1^\alpha} \int_{r_1}^{r_2} \Gamma(\zeta) d_\alpha \zeta \in [0, r_2 - r_1]$.

In [46] the authors gave an extension for Theorem 1.8:

Theorem 1.4. Assume $\alpha \in (0,1]$ and $r_1, r_2 \in \mathbb{R}$ such that $0 \leq r_1 \leq r_2$. Suppose $\prod, \Gamma, \Sigma : [r_1, r_2] \to [0,\infty)$ are integrable on $[r_1, r_2]$ with the decreasing function Π and $0 \leq \Gamma \leq \Sigma$, we get

$$\int_{r_2-\aleph}^{r_2} \Sigma(\zeta)\Pi(\zeta) d_\alpha \zeta \leq \int_{r_1}^{r_2} \Pi(\zeta)\Gamma(\zeta) d_\alpha \zeta \leq \int_{r_1}^{r_1+\aleph} \Sigma(\zeta)\Pi(\zeta) d_\alpha \zeta,$$

where $\aleph = \frac{(r_2-r_1)}{\int_{r_1}^{r_2} \Sigma(\zeta) d_\alpha \zeta} \int_{r_1}^{r_2} \Gamma(\zeta) d_\alpha \zeta \in [0, r_2 - r_1]$.

In this paper, we prove and explore several novel speculations of the Steffensen inequality obtained in [47] through the conformable integral containing time scale concept. We furthermore recover certain known results as special cases of our results.

2. Main results

Lemma 2.1. Assume $\zeta > 0$ is rd-continuous function on $[m,n] \cap \mathbb{T}$, g, r be rd-continuous on $[m,n] \cap \mathbb{T}$ such that r/ζ nonincreasing function and $0 \leq g(\wp) \leq 1$ $\forall \wp \in [m,n] \cap \mathbb{T}$. Then

(Λ_1)

$$\int_m^n r(\wp) g(\wp) \Delta_\alpha \wp \leq \int_m^{m+\lambda} r(\wp) \Delta_\alpha \wp, \tag{2.1}$$

where λ is given by

$$\int_m^n \zeta(\wp) g(\wp) \Delta_\alpha \wp = \int_m^{m+\lambda} \zeta(\wp) \Delta_\alpha \wp.$$

(Λ_2)

$$\int_{n-\lambda}^n r(\wp) \Delta_\alpha \wp \leq \int_m^n r(\wp) g(\wp) \Delta_\alpha \wp, \tag{2.2}$$

such that

$$\int_{n-\lambda}^n \zeta(\wp) \Delta_\alpha \wp = \int_m^n \zeta(\wp) g(\wp) \Delta_\alpha \wp.$$

(2.1) and (2.2) are reversed when r/ζ is nondecreasing.

Proof. Putting $g(\wp) \mapsto \zeta(\wp) g(\wp)$ and $r(\wp) \mapsto r(\wp)/\zeta(\wp)$ in (1.4), (1.5) to get (Λ_1) and (Λ_2) simultaneously. □

Lemma 2.2. Under the same hypotheses of Lemma 2.1. with ψ be integrable functions on $[m,n]\cap\mathbb{T}$ and $0\le\psi(\wp)\le g(\wp)\le 1-\psi(\wp)$ for all $\wp\in[m,n]_{\mathbb{T}}$. Then

$$\begin{aligned}&\int_{n-\lambda}^{n} r(\wp)\Delta_\alpha\wp+\int_m^n\left|\left(\frac{r(\wp)}{\zeta(\wp)}-\frac{r(n-\lambda)}{\zeta(n-\lambda)}\right)\zeta(\wp)\psi(\wp)\right|\Delta_\alpha\wp\\ &\le\int_m^n r(\wp)g(\wp)\Delta_\alpha\wp\\ &\le\int_m^{m+\lambda} r(\wp)\Delta_\alpha\wp-\int_m^n\left|\left(\frac{r(\wp)}{\zeta(\wp)}-\frac{r(m+\lambda)}{\zeta(m+\lambda)}\right)\zeta(\wp)\psi(\wp)\right|\Delta_\alpha\wp,\end{aligned}$$

where λ is obtained from

$$\int_m^{m+\lambda} h(\wp)\Delta_\alpha\wp=\int_m^n\zeta(\wp)g(\wp)\Delta_\alpha\wp=\int_{n-\lambda}^n\zeta(\wp)\Delta_\alpha\wp.$$

Proof. Putting $g(\wp)\mapsto\zeta(\wp)g(\wp)$, $r(\wp)\mapsto r(\wp)/h(\wp)$ and $\psi(\wp)\mapsto\zeta(\wp)\psi(\wp)$ in (1.6) . □

Lemma 2.3. Under the same conditions of Lemma 2.1. Then

$$\begin{aligned}\int_{n-\lambda}^{n} r(\wp)\Delta_\alpha\wp &\le \int_{n-\lambda}^{n}\left(r(\wp)-\left[\frac{r(\wp)}{\zeta(\wp)}-\frac{r(n-\lambda)}{\zeta(n-\lambda)}\right]\zeta(\wp)[1-g(\wp)]\right)\Delta_\alpha\wp\\ &\le\int_m^n r(\wp)g(\wp)\Delta_\alpha\wp\\ &\le\int_m^{m+\lambda}\left(r(\wp)-\left[\frac{r(\wp)}{\zeta(\wp)}-\frac{r(a+\lambda)}{\zeta(m+\lambda)}\right]\zeta(\wp)[1-g(\wp)]\right)\Delta_\alpha\wp\\ &\le\int_m^{m+\lambda} r(\wp)\Delta_\alpha\wp,\end{aligned}$$

where λ is obtained from

$$\int_m^{m+\lambda}\zeta(\wp)\Delta_\alpha\wp=\int_m^n g(\wp)\Delta_\alpha\wp=\int_{n-\lambda}^n\zeta(\wp)\Delta_\alpha\wp.$$

Proof. Taking $g(\wp)\mapsto\zeta(\wp)g(\wp)$ and $r(\wp)\mapsto r(\wp)/\zeta(\wp)$ in (1.7). □

Theorem 2.1. Under the same conditions of Lemma 2.3 such that $k\in(m,n)$ and λ_1, λ_2 are given from (Λ_3)

$$\int_m^{m+\lambda_1}\zeta(\wp)\Delta_\alpha\wp=\int_m^k\zeta(\wp)g(\wp)\Delta_\alpha\wp,$$

$$\int_{n-\lambda_2}^n\zeta(\wp)\Delta_\alpha\wp=\int_k^n\zeta(\wp)g(\wp)\Delta_\alpha\wp.$$

If $r^\sigma/\zeta\in\mathbb{AH}_1^k[m,n]$ and

$$\int_m^n\phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp=\int_m^{m+\lambda_1}\phi(\wp)\zeta(\wp)\Delta_\alpha\wp+\int_{n-\lambda_2}^n\phi(\wp)\zeta(\wp)\Delta_\alpha\wp, \tag{2.3}$$

then

$$\int_m^n r^\sigma(\wp)g(\wp)\Delta_\alpha\wp \le \int_m^{m+\lambda_1} r^\sigma(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^n r^\sigma(\wp)\Delta_\alpha\wp. \tag{2.4}$$

(2.4) is reversed if $r^\sigma/\zeta \in \mathbb{AH}_2^k[m,n]$ and (2.3).

(Λ_4)

$$\int_{k-\lambda_1}^k \zeta(\wp)\Delta_\alpha\wp = \int_m^k \zeta(\wp)g(\wp)\Delta_\alpha\wp,$$

$$\int_k^{k+\lambda_2} \zeta(\wp)\Delta_\alpha\wp = \int_k^n \zeta(\wp)g(\wp)\Delta_\alpha\wp.$$

If $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$ and

$$\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp = \int_{k-\lambda_1}^{k+\lambda_2} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp, \tag{2.5}$$

then

$$\int_m^n r^\sigma(\wp)g(\wp)\Delta_\alpha\wp \ge \int_{k-\lambda_1}^{k+\lambda_2} r^\sigma(\wp)\Delta_\alpha\wp. \tag{2.6}$$

If $r^\sigma/\zeta \in \mathbb{AH}_2^k[m,n]$ and (2.5) satisfied, then we reverse (2.6).

(Λ_5) If λ_1, λ_2 be the same as in (Λ_3) and $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$ so that

$$\begin{aligned}&\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp\\ &= \int_m^{m+\lambda_1}\Big(\phi(\wp)\zeta(\wp) - [\phi(\wp) - m - \lambda_1]\zeta(\wp)[1-g(\wp)]\Big)\Delta_\alpha\wp\\ &\quad + \int_{n-\lambda_2}^n\Big(\phi(\wp)\zeta(\wp) - [\phi(\wp) - n + \lambda_2]\zeta(\wp)[1-g(\wp)]\Big)\Delta_\alpha\wp,\end{aligned} \tag{2.7}$$

then

$$\begin{aligned}&\int_m^n r^\sigma(\wp)g(\wp)\Delta_\alpha\wp\\ &\le \int_m^{m+\lambda_1}\left(r^\sigma(\wp) - \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(m+\lambda_1)}{\zeta(m+\lambda_1)}\right|\zeta(\wp)[1-g(\wp)]\right)\Delta_\alpha\wp\\ &\quad + \int_{n-\lambda_2}^n\left(r^\sigma(\wp) - \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(n-\lambda_2)}{\zeta(n-\lambda_2)}\right|\zeta(\wp)[1-g(\wp)]\right)\Delta_\alpha\wp.\end{aligned} \tag{2.8}$$

If $r^\sigma/\zeta \in \mathbb{AH}_2^k[m,n]$ and (2.7) satisfied, the inequality in (2.8) is reversed.

(Λ_6) If λ_1, λ_2 be defined as in (Λ_4) and $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$ and

$$\begin{aligned}&\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp\\ &= \int_{k-\lambda_1}^k\Big(\phi(\wp)\zeta(\wp) - [\phi(\wp) - k + \lambda_1]\zeta(\wp)[1-g(\wp)]\Big)\Delta_\alpha\wp\end{aligned}$$

$$= \int_{m}^{m+\lambda_1} \Big(\phi(\wp)\zeta(\wp) - [\phi(\wp) - k + \lambda_2]\zeta(\wp)[1 - g(\wp)]\Big)\Delta_\alpha\wp, \tag{2.9}$$

then

$$\begin{aligned}
&\int_{m}^{n} r^\sigma(\wp)g(\wp)\Delta_\alpha\wp \\
&\geq \int_{k-\lambda_1}^{k} \Big(r^\sigma(\wp) - \Big[\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(k-\lambda_1)}{\zeta(k-\lambda_1)}\Big]\zeta(\wp)[1-g(\wp)]\Big)\Delta_\alpha\wp \\
&\quad + \int_{k}^{k+\lambda_2} \Big(r^\sigma(\wp) - \Big[\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(k+\lambda_2)}{\zeta(k+\lambda_2)}\Big]\zeta(\wp)[1-g(\wp)]\Big)\Delta_\alpha\wp. \qquad (2.10)
\end{aligned}$$

If $r^\sigma/\zeta \in \mathbb{AH}_2^k[m,n]$ and (2.9) satisfied, we reverse (2.10).

Proof. (Λ_3) Consider $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$, and $R_1(\ell) = r^\sigma(\ell) - A\phi(\ell)\zeta(\ell)$, since A is given in Definition 2.1. Since $R_1/\zeta : [m,k] \cap \mathbb{T} \to \mathbb{R}$, using Lemma 2.1($\Lambda_1$), we deduce

$$\begin{aligned}
0 &\leq \int_{m}^{m+\lambda_1} R_1(\wp)\Delta_\alpha\wp - \int_{m}^{k} R_1(\wp)g(\wp)\Delta_\alpha\wp \\
&= \int_{m}^{m+\lambda_1} r^\sigma(\wp)\Delta_\alpha\wp - \int_{m}^{k} r^\sigma(\wp)g(\wp)\Delta_\alpha\wp \\
&\quad -A\Big(\int_{m}^{m+\lambda_1} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp - \int_{m}^{k} \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp\Big). \qquad (2.11)
\end{aligned}$$

As $R_1/\zeta : [k,n] \cap \mathbb{T} \to \mathbb{R}$ is nondecreasing, using Lemma 2.1(Λ_2), we obtain

$$\begin{aligned}
0 &\geq \int_{k}^{n} R_1(\wp)g(\wp)\Delta_\alpha\wp - \int_{n-\lambda_2}^{n} R_1(\wp)\Delta_\alpha\wp \\
&= \int_{k}^{n} r^\sigma(\wp)g(\wp)\Delta_\alpha\wp - \int_{n-\lambda_2}^{n} r^\sigma(\wp)\Delta_\alpha\wp \\
&\quad -A\Big(\int_{k}^{n} \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp - \int_{n-\lambda_2}^{n} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp\Big). \qquad (2.12)
\end{aligned}$$

(2.11) and (2.12) imply that

$$\begin{aligned}
&\int_{m}^{m+\lambda_1} r^\sigma(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^{n} r^\sigma(\wp)\Delta_\alpha\wp - \int_{m}^{n} r^\sigma(\wp)g(\wp)\Delta_\alpha\wp \\
&\geq A\Big(\int_{m}^{m+\lambda_1} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^{n} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp - \int_{m}^{n} \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp\Big)
\end{aligned}$$

Hence, if (2.3) is hold, then (2.4) holds. For $r^\sigma/\zeta \in \mathbb{AH}_2^k[m,n]$, we get the some steps.

(Λ_4) Let $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$, also $R_1(x) = r^\sigma(x) - A\phi(x)\zeta(x)$, where A as in Definition 2.1. R_1/ζ : $[m,k] \cap \mathbb{T} \to \mathbb{R}$ is nonincreasing, so from Lemma 2.1(Λ_1) we obtain

$$0 \leq \int_{m}^{k} r^\sigma(\wp)g(\wp)\Delta_\alpha\wp - \int_{k-\lambda_1}^{k} r^\sigma(\wp)\Delta_\alpha\wp$$

$$-A\Big(\int_m^k \phi(\wp)h(\wp)g(\wp)\Delta_\alpha\wp - \int_{c-\lambda_1}^k \phi(\wp)\zeta(\wp)\Delta_\alpha\wp\Big). \tag{2.13}$$

Using Lemma 2.1(Λ_1) we have

$$\begin{aligned} 0 \;\; \geq \;\; & \int_k^{k+\lambda_2} r^\sigma(\wp)\Delta_\alpha\wp - \int_k^n r^\sigma(\wp)g(\wp)\Delta_\alpha\wp \\ & -A\Big(\int_k^{k+\lambda_2} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp - \int_k^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp\Big). \end{aligned} \tag{2.14}$$

Thus, from (2.13), (2.14), we get

$$\begin{aligned} & \int_m^n r^\sigma(\wp)g(\wp)\Delta_\alpha\wp - \int_{k-\lambda_1}^{k+\lambda_2} r^\sigma(\wp)\Delta_\alpha\wp \\ & \geq A\Big(\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp - \int_{k-\lambda_1}^{k+\lambda_2} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp\Big) \end{aligned}$$

Therefore, if $\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp = \int_{k-\lambda_1}^{k+\lambda_2} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp$ is satisfied, then (2.8) holds. Follow the same steps for $r^\sigma/\zeta \in \mathbb{AH}_2^k[m,n]$.

Using Lemma 2.3 and repeat the steps of Theorem 2.1(Λ_3) and Theorem 2.1(Λ_4) in the proof of (Λ_5) and (Λ_6) respectively. □

Corollary 2.1. The inequalities (2.4), (2.6), (2.8) and (2.10) of Theorem 2.1 letting $\mathbb{T} = \mathbb{R}$ takes

$$(i) \qquad \int_m^n f^\sigma(\wp)g(\wp)d_\alpha\wp \leq \int_m^{m+\lambda_1} r^\sigma(\wp)d_\alpha\wp + \int_{n-\lambda_2}^n r^\sigma(\wp)d_\alpha\wp. \tag{2.15}$$

$$(ii) \qquad \int_m^n r^\sigma(\wp)g(\wp)d_\alpha\wp \geq \int_{k-\lambda_1}^{k+\lambda_2} r^\sigma(\wp)d_\alpha\wp. \tag{2.16}$$

$$\begin{aligned} (iii) \qquad & \int_m^n r^\sigma(\wp)g(\wp)d_\alpha\wp \\ & \leq \int_m^{m+\lambda_1} \Big(r^\sigma(\wp) - \Big[\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(m+\lambda_1)}{\zeta(m+\lambda_1)}\Big]\zeta(\wp)[1-g(\wp)]\Big)d_\alpha\wp \\ & \quad + \int_{n-\lambda_2}^n \Big(r^\sigma(\wp) - \Big[\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(n-\lambda_2)}{\zeta(n-\lambda_2)}\Big]\zeta(\wp)[1-g(\wp)]\Big)d_\alpha\wp. \end{aligned} \tag{2.17}$$

$$\begin{aligned} (iv) \qquad & \int_m^n r^\sigma(\wp)g(\wp)d_\alpha\wp \\ & \geq \int_{k-\lambda_1}^k \Big(r^\sigma(\wp) - \Big[\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(k-\lambda_1)}{\zeta(k-\lambda_1)}\Big]\zeta(\wp)[1-g(\wp)]\Big)d_\alpha\wp \\ & \quad + \int_k^{k+\lambda_2} \Big(r^\sigma(\wp) - \Big[\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(k+\lambda_2)}{\zeta(k+\lambda_2)}\Big]\zeta(\wp)[1-g(\wp)]\Big)d_\alpha\wp. \end{aligned} \tag{2.18}$$

Corollary 2.2. We get [47, Theorems 8, 10, 21 and 22], if we put $\alpha = 1$ and $\phi(\wp) = \wp$ in Corollary 2.1 $[(i), (ii), (iii), (iv)]$ simultaneously.
Corollary 2.3. In Corollary 2.1 taking $\mathbb{T} = \mathbb{Z}$, the results (2.15)–(2.18) will be equivalent to

$$(i) \qquad \sum_{\wp=m}^{n-1} r(\wp+1)g(\wp)\wp^{\alpha-1} \leq \sum_{\wp=m}^{m+\lambda_1-1} r(\wp+1) + \sum_{\wp=n-\lambda_2}^{n-1} r(\wp+1)\wp^{\alpha-1}.$$

$$(ii) \qquad \sum_{\wp=m}^{n-1} r(\wp+1)g(\wp)\wp^{\alpha-1} \geq \sum_{\wp=k-\lambda_1}^{k+\lambda_2-1} r(\wp+1)\wp^{\alpha-1}.$$

$$\begin{aligned}(iii) \qquad & \sum_{\wp=m}^{n-1} r(\wp+1)g(\wp)\wp^{\alpha-1} \\ & \leq \sum_{\wp=m}^{m+\lambda_1-1} \Big(r(\wp+1) - \Big[\frac{r(\wp+1)}{\zeta(\wp)} - \frac{r(a+\lambda_1+1)}{\zeta(m+\lambda_1)}\Big]\zeta(\wp)[1-g(\wp)]\Big] \Big)\wp^{\alpha-1} \\ & \quad + \sum_{\wp=n-\lambda_2}^{n-1} \Big(r(\wp+1) - \Big[\frac{r(\wp+1)}{\zeta(\wp)} - \frac{r(n-\lambda_2+1)}{\zeta(n-\lambda_2)}\Big]\zeta(\wp)[1-g(\wp)]\Big] \Big)\wp^{\alpha-1}.\end{aligned}$$

$$\begin{aligned}(iv) \qquad & \sum_{\wp=m}^{n-1} r(\wp+1)g(\wp))\wp^{\alpha-1} \\ & \geq \sum_{\wp=k-\lambda_1}^{k-1} \Big(r(\wp+1) - \Big[\frac{r(\wp+1)}{\zeta(\wp)} - \frac{r(k-\lambda_1+1)}{\zeta(k-\lambda_1)}\Big]\zeta(\wp)[1-g(\wp)]\Big]\Big))\wp^{\alpha-1} \\ & \quad + \sum_{\wp=k}^{k+\lambda_2-1} \Big(r(\wp+1) - \Big[\frac{r(\wp+1)}{\zeta(\wp)} - \frac{r(k+\lambda_2+1)}{\zeta(k+\lambda_2)}\Big]\zeta(\wp)[1-g(\wp)]\Big]\Big))\wp^{\alpha-1}.\end{aligned}$$

Theorem 2.2. Under the assumptions in Lemma 2.1 with $0 \leq g(\wp) \leq \zeta(\wp)$ and λ_1, λ_2 be defined as (Λ_7)

$$\int_m^{m+\lambda_1} \zeta(\wp)\Delta_\alpha\wp = \int_m^k g(\wp)\Delta_\alpha\wp,$$

$$\int_{n-\lambda_2}^{n} \zeta(\wp)\Delta_\alpha\wp = \int_k^n g(\wp)\Delta_\alpha\wp.$$

If $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$ and

$$\int_m^n \phi(\wp)g(\wp)\Delta_\alpha\wp = \int_m^{m+\lambda_1} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^n \phi(\wp)\zeta(\wp)\Delta_\alpha\wp, \tag{2.19}$$

then

$$\int_m^n r^\sigma(\wp)g(\wp)\Delta_\alpha\wp \leq \int_m^{m+\lambda_1} r^\sigma(\wp)\zeta(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^n r^\sigma(\wp)\zeta(\wp)\Delta_\alpha\wp. \tag{2.20}$$

(Λ_8)

$$\int_{k-\lambda_1}^{k} \zeta(\wp)\Delta_\alpha \wp = \int_{m}^{k} g(\wp)\Delta_\alpha \wp,$$

$$\int_{k}^{k+\lambda_2} \zeta(\wp)\Delta_\alpha \wp = \int_{k}^{n} g(\wp)\Delta_\alpha \wp.$$

If $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$ and

$$\int_{m}^{n} \phi(\wp)g(\wp)\Delta_\alpha \wp = \int_{k-\lambda_1}^{k+\lambda_2} \phi(\wp)\zeta(\wp)\Delta_\alpha \wp, \tag{2.21}$$

then

$$\int_{m}^{n} r^\sigma(\wp)g(\wp)\Delta_\alpha \wp \geq \int_{k-\lambda_1}^{k+\lambda_2} r^\sigma(\wp)\zeta(\wp)\Delta_\alpha \wp. \tag{2.22}$$

If $r^\sigma/\zeta \in \mathbb{AH}_2^k[m,n]$ and (2.19), (2.21) satisfied, we get the reverse of (2.20) and (2.22).

Proof. By using Theorem 2.1 $[(\Lambda_3),(\Lambda_4)]$ and by putting $g \mapsto g/h$ and $f \mapsto fh$, we get the proof of (Λ_7) and (Λ_8). □

Corollary 2.4. In Theorem 2.2 $[(\Lambda_7),(\Lambda_8)]$, assuming $\mathbb{T} = \mathbb{R}$, the following results obtains:

$$(i) \qquad \int_{m}^{n} r^\sigma(\wp)g(\wp)d_\alpha \wp \leq \int_{m}^{m+\lambda_1} r^\sigma(\wp)\zeta(\wp)d_\alpha \wp + \int_{n-\lambda_2}^{n} r^\sigma(\wp)\zeta(\wp)d_\alpha \wp. \tag{2.23}$$

$$(ii) \qquad \int_{m}^{n} r^\sigma(\wp)g(\wp)d_\alpha \wp \geq \int_{k-\lambda_1}^{k+\lambda_2} r^\sigma(\wp)\zeta(\wp)d_\alpha \wp. \tag{2.24}$$

Corollary 2.5. In Corollary 2.4 $[(i),(ii)]$, when we put $\alpha = 1$ and $\phi(\wp) = \wp$ then [47, Theorems 16 and 17] gotten.

Corollary 2.6. In (2.23) and (2.24) letting $\mathbb{T} = \mathbb{Z}$, gets

$$(i) \qquad \sum_{\wp=m}^{n-1} r(\wp+1)g(\wp)\wp^{\alpha-1} \leq \sum_{\wp=m}^{m+\lambda_1-1} r(\wp+1)h(\wp) + \sum_{\wp=n-\lambda_2}^{n-1} r(\wp+1)h(\wp)\wp^{\alpha-1}.$$

$$(ii) \qquad \sum_{\wp=m}^{n-1} r(\wp+1)g(\wp)\wp^{\alpha-1} \geq \sum_{\wp=k-\lambda_1}^{k+\lambda_2-1} r(\wp+1)\zeta(\wp)\wp^{\alpha-1}.$$

Theorem 2.3. Using the same conditions in Lemma 2.3. Letting $w : [m,n]\cap\mathbb{T} \to \mathbb{R}$ be integrable with $0 \leq g(\wp) \leq w(\wp)\ \forall \wp \in [m,n]\cap\mathbb{T}$ and

$$(\Lambda_9) \qquad \int_{m}^{m+\lambda_1} w(\wp)\zeta(\wp)\Delta_\alpha \wp = \int_{m}^{k} \zeta(\wp)g(\wp)\Delta_\alpha \wp,$$

$$\int_{n-\lambda_2}^{n} w(\wp)\zeta(\wp)\Delta_\alpha \wp = \int_{k}^{n} \zeta(\wp)g(\wp)\Delta_\alpha \wp.$$

If $r^{\sigma}/\zeta \in \mathbb{AH}_1^k[m,n]$ and

$$\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp = \int_m^{m+\lambda_1} \phi(\wp)w(\wp)\zeta(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^n \phi(\wp)w(\wp)\zeta(\wp)\Delta_\alpha\wp, \quad (2.25)$$

then

$$\int_m^n r^{\sigma}(\wp)g(\wp)\Delta_\alpha\wp \le \int_m^{m+\lambda_1} r^{\sigma}(\wp)w(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^n r^{\sigma}(\wp)w(\wp)\Delta_\alpha\wp. \quad (2.26)$$

$$(\Lambda_{10}) \qquad \int_{k-\lambda_1}^k w(\wp)\zeta(\wp)\Delta_\alpha\wp = \int_m^k \zeta(\wp)g(\wp)\Delta_\alpha\wp,$$

$$\int_k^{k+\lambda_2} w(\wp)\zeta(\wp)\Delta_\alpha\wp = \int_k^n \zeta(\wp)g(\wp)\Delta_\alpha\wp.$$

If $r^{\sigma}/\zeta \in \mathbb{AH}_1^k[m,n]$ and

$$\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp = \int_{k-\lambda_1}^{k+\lambda_2} \phi(\wp)w(\wp)\zeta(\wp)\Delta_\alpha\wp, \quad (2.27)$$

$$\int_m^n r^{\sigma}(\wp)g(\wp)\Delta_\alpha\wp \ge \int_{k-\lambda_1}^{k+\lambda_2} r^{\sigma}(\wp)w(\wp)\Delta_\alpha\wp. \quad (2.28)$$

The inequalities in (2.26) and (2.28) are reversible if $r^{\sigma}/\zeta \in \mathbb{AH}_2^c[a,b]$ and (2.25), (2.27) hold.

Proof. In Theorem 2.1 $[(\Lambda_3),(\Lambda_4)]$, ζ changes wq, g changes g/w and r changes rw. □

Corollary 2.7. In (2.26) and (2.28). Letting $\mathbb{T}=\mathbb{R}$, we have

$$(i) \qquad \int_m^n r^{\sigma}(\wp)g(\wp)d_\alpha\wp \le \int_m^{m+\lambda_1} r^{\sigma}(\wp)w(\wp)d_\alpha\wp + \int_{n-\lambda_2}^n r^{\sigma}(\wp)w(\wp)d_\alpha\wp. \quad (2.29)$$

$$(ii) \qquad \int_m^n r^{\sigma}(\wp)g(\wp)d_\alpha\wp \ge \int_{k-\lambda_1}^{k+\lambda_2} r^{\sigma}(\wp)w(\wp)d_\alpha\wp. \quad (2.30)$$

Corollary 2.8. In Corollary 2.7 $[(i),(ii)]$, letting $\alpha=1$ and $\phi(\wp)=\wp$ we get [47, Theorems 18 and 19].
Corollary 2.9. In (2.29) and (2.30), crossing $\mathbb{T}=\mathbb{Z}$, gets

$$(i) \qquad \sum_{\wp=m}^{n-1} r(\wp+1)g(\wp)\wp^{\alpha-1} \le \sum_{\wp=m}^{m+\lambda_1-1} r(\wp+1)w(\wp) + \sum_{\wp=n-\lambda_2}^{n-1} r(\wp+1)w(\wp)\wp^{\alpha-1}.$$

$$(ii) \qquad \sum_{\wp=m}^{n-1} r(\wp+1)g(\wp)\wp^{\alpha-1} \ge \sum_{\wp=k-\lambda_1}^{k+\lambda_2-1} r(\wp+1)w(\wp)\wp^{\alpha-1}.$$

Theorem 2.4. Using the same conditions in Lemma 2.1, and Theorem 2.1 $[(\Lambda_3),(\Lambda_4)]$ with $\psi:[m,n]\cap\mathbb{T}\to\mathbb{R}$ be a integrable: $0\le\psi(\wp)\le g(\wp)\le 1-\psi(\wp)$.

(Λ_{11}) If $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$ and

$$\begin{aligned}&\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp\\ &= \int_m^{m+\lambda_1} \phi(\wp)\zeta(\wp)\Delta_\alpha\wp - \int_m^k \left|\phi(\wp) - m - \lambda_1\right|\zeta(\wp)\psi(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^n \phi(\wp)\zeta(\wp)\Delta_\alpha\wp\\ &\quad + \int_k^n \left|\phi(\wp) - n + \lambda_2\right|\zeta(\wp)\psi(\wp)\Delta_\alpha\wp, \end{aligned} \tag{2.31}$$

then

$$\begin{aligned}&\int_m^n r^\sigma(\wp)g(\wp)\Delta_\alpha\wp\\ &\le \int_m^{m+\lambda_1} r^\sigma(\wp)\Delta_\alpha\wp - \int_m^k \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(m+\lambda_1)}{\zeta(m+\lambda_1)}\right|\zeta(\wp)\psi(\wp)\Delta_\alpha\wp + \int_{n-\lambda_2}^n r^\sigma(\wp)\Delta_\alpha\wp\\ &\quad + \int_k^n \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(n-\lambda_2)}{\zeta(n-\lambda_2)}\right|\zeta(\wp)\psi(\wp)\Delta_\alpha\wp. \end{aligned} \tag{2.32}$$

(Λ_{12}) If $r^\sigma/\zeta \in \mathbb{AH}_1^k[m,n]$ and

$$\begin{aligned}&\int_m^n \phi(\wp)\zeta(\wp)g(\wp)\Delta_\alpha\wp\\ &= \int_{k-\lambda_1}^k \phi(\wp)\zeta(\wp)\Delta_\alpha\wp - \int_m^k \left|\phi(\wp) - k + \lambda_1\right|\zeta(\wp)\psi(\wp)\Delta_\alpha\wp\\ &\quad + \int_k^n \left|\phi(\wp) - k - \lambda_1\right|\zeta(\wp)\psi(\wp)\Delta_\alpha\wp, \end{aligned} \tag{2.33}$$

then

$$\begin{aligned}&\int_m^n r^\sigma(\wp)g(\wp)\Delta_\alpha\wp\\ &\ge \int_{k-\lambda_1}^{k+\lambda_2} r^\sigma(\wp)\Delta_\alpha\wp + \int_m^k \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(k-\lambda_1)}{\zeta(k-\lambda_1)}\right|\zeta(\wp)\psi(\wp)\Delta_\alpha\wp\\ &\quad - \int_k^n \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(k+\lambda_2)}{\zeta(k+\lambda_2)}\right|\zeta(\wp)\psi(\wp)\Delta_\alpha\wp. \end{aligned} \tag{2.34}$$

If $r^\sigma/\zeta \in \mathbb{AH}_2^k[m,n]$ and (2.31) and (2.33) satisfied, we get the reverse of (2.32) and (2.34).

Proof. The same steps of Theorem 2.1 $[(\Lambda_3), (\Lambda_4)]$ with Lemma 2.1, $R_1/\zeta : [m,k] \cap \mathbb{T} \to \mathbb{R}$ nonincreasing, $R_1/\zeta : [k,n] \cap \mathbb{T} \to \mathbb{R}$ nondecreasing. □

Corollary 2.10. In Theorem 2.4 $[(\Lambda_{11}), (\Lambda_{12})]$, letting $\mathbb{T} = \mathbb{R}$ we get:

(i) $$\int_m^n r^\sigma(\wp)g(\wp)d_\alpha\wp$$

$$\leq \int_m^{m+\lambda_1} r^\sigma(\wp) d_\alpha\wp - \int_m^k \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(m+\lambda_1)}{\zeta(m+\lambda_1)}\right| \zeta(\wp)\psi(\wp) d_\alpha\wp + \int_{n-\lambda_2}^n r^\sigma(\wp) d_\alpha\wp$$
$$+ \int_k^n \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(n-\lambda_2)}{\zeta(n-\lambda_2)}\right| \zeta(\wp)\psi(\wp) d_\alpha\wp. \tag{2.35}$$

(ii)

$$\int_m^n r^\sigma(\wp) g(\wp) d_\alpha\wp$$
$$\geq \int_{k-\lambda_1}^{k+\lambda_2} r^\sigma(\wp) d_\alpha\wp + \int_m^k \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(k-\lambda_1)}{\zeta(k-\lambda_1)}\right| \zeta(\wp)\psi(\wp) d_\alpha\wp$$
$$- \int_k^n \left|\frac{r^\sigma(\wp)}{\zeta(\wp)} - \frac{r^\sigma(k+\lambda_2)}{\zeta(k+\lambda_2)}\right| \zeta(\wp)\psi(\wp) d_\alpha\wp. \tag{2.36}$$

Corollary 2.11. In (2.35) and (2.36), we put $\alpha = 1$, with $\phi(\wp) = \wp$ we get [47, Theorems 23 and 24].
Corollary 2.12. Our results (2.35) and (2.36), by using $\mathbb{T} = \mathbb{Z}$ gets

(i)

$$\sum_{\wp=m}^{n-1} r(\wp+1) g(\wp) \wp^{\alpha-1}$$
$$\leq \sum_{\wp=m}^{m+\lambda_1-1} r(\wp+1)\wp^{\alpha-1} - \sum_{\wp=m}^{k-1} \left|\frac{r(\wp+1)}{\zeta(\wp)} - \frac{r(m+\lambda_1+1)}{\zeta(m+\lambda_1)}\right| \zeta(\wp)\psi(\wp)\hat{\nabla}\wp$$
$$+ \sum_{\wp=n-\lambda_2}^{n-1} r(\wp+1)\wp^{\alpha-1} + \sum_{\wp=k}^{n-1} \left|\frac{r(\wp+1)}{\zeta(\wp)} - \frac{r(n-\lambda_2+1)}{\zeta(n-\lambda_2)}\right| \zeta(\wp)\psi(\wp)\wp^{\alpha-1}.$$

(ii)

$$\sum_{\wp=m}^{n-1} r(\wp+1) g(\wp) \wp^{\alpha-1}$$
$$\geq \sum_{\wp=k-\lambda_1}^{k+\lambda_2-1} r(\wp+1)\wp^{\alpha-1} + \sum_{\wp=m}^{k-1} \left|\frac{r(\wp+1)}{\zeta(\wp)} - \frac{r(k-\lambda_1+1)}{\zeta(k-\lambda_1)}\right| \zeta(\wp)\psi(\wp)\wp^{\alpha-1}$$
$$- \sum_{\wp=k}^{n-1} \left|\frac{r(\wp+1)}{\zeta(\wp)} - \frac{r(k+\lambda_2+1)}{\zeta(k+\lambda_2)}\right| h(\wp)\psi(\wp)\wp^{\alpha-1}.$$

3. Conclusions

In this work, we explore new generalizations of the integral Steffensen inequality given in [38,39,43] by the utilization of the α-conformable derivatives and integrals, A few of these results are generalised to time scales. We also obtained the discrete and continuous case of our main results, in order to gain some fresh inequalities as specific cases.

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Conflict of interest

The authors declare no conflict of interest.

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