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Research Article

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Characterizations of quasi-metric and *G*-metric completeness involving *w*-distances and fixed points

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Abstract: Involving w-distances we prove a fixed point theorem of Caristi-type in the realm of (non-necessarily T_1) quasi-metric spaces. With the help of this result, a characterization of quasi-metric completeness is obtained. Our approach allows us to retrieve several key examples occurring in various fields of mathematics and computer science and that are modeled as non-T_1 quasi-metric spaces. As an application, we deduce a characterization of complete G-metric spaces in terms of a weak version of Caristi's theorem that involves a G-metric version of w-distances.

Keywords: quasi-metric, complete, w-distance, fixed point, G-metric

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1 Introduction

It has long been widely recognized that Caristi's fixed point theorem [1, Theorem (2.1)'] constitutes one of the most prominent generalizations of the Banach contraction principle. Thus, Kirk showed in [2] that its validity characterizes the metric completeness. Furthermore, it has direct applications in functional analysis [3, Chapter 9], mathematical optimization [4], and, through a quasi-metric version, in the study of the complexity analysis of some algorithms via denotational semantics [5]. On the other hand, its equivalence with the celebrated Ekeland's variational principle [6,7] guarantees, at least indirectly, its applicability to a variety of issues about global analysis, optimal control, equilibrium problems, etc. Since there is a vast literature on these topics, we keep citing the recent contributions [8–10] with references therein. Generalizations and extensions of Caristi's theorem to b-metric spaces, quasi-metric spaces, partial metric spaces, and fuzzy metric spaces, among others, may be found in [11–17].

At this point, it is interesting to recall that the original proof of Caristi's theorem uses transfinite induction. Several mathematicians refined and improved such a proof, for instance, via Zermelo-Fraenkel Axioms or via Zorn's lemma-The Axiom of Choice (see [18, Section 1] and [19, Section 6] for detailed accounts on this subject). In this context, Khamsi [20, page 3] asked the question of finding a pure metric proof of Caristi's theorem (see also [21, page 13]). If we understand as "a pure metric proof" the one reasonably

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suggested by Kozlowski in [22, page 134], then, and as far as we know, a purely metric proof already implicitly appeared in a remarkable generalization of Caristi's theorem, in terms of w-distances, obtained by Kada et al. in [23, Theorem 2] as well as in [24, Theorem 2.3] within the framework of partial metric spaces. Later on, Kozlowski [22] and Du [25] also presented purely metric proofs of Caristi's theorem (see also [26, Theorem 2]). Let us note that these proofs have in common a like starting point and the approaches of some parts of such proofs follow similar patterns.

It is also interesting to mention the recent contribution from Darko et al. [27], where the authors use the concept of wt-distance (a b-metric counterpart of the notion of w-distance) to generalize a known fixed-point theorem of Ćirić [28] as well as recent results from [29] and [30]. They also consider Fisher's quasi-contraction in the framework of wt-distance.

In this article, we obtain a generalization of Kada-Suzuki-Takahashi's theorem cited earlier to the realm of (non-necessarily T_1) quasi-metric spaces, with a purely metric proof that is inspired by the proof of [26, Theorem 2]. From this result, we characterize those quasi-metric spaces that are complete in the sense of [31,32] (a very general type of quasi-metric completeness). We emphasize that our non-T_1 approach allows us to recover several fundamental examples in the basic theory of asymmetric functional analysis (see, e.g., [33, Section 2.1.6]), in some aspects of the calculus of variations (see, e.g., [35]) and in various branches of the theory of computation (see, e.g., [36–40]). The last part of the article is devoted to apply the obtained results in the quasi-metric setting to deduce a characterization of complete G-metric spaces in terms of a weak version of Caristi's theorem that involves a G-metric version of w-distances.

Two antecedents of our study are contained in articles by Park [41] and by Al-Homidan et al. [42], respectively, where the authors obtained characterizations of complete T_1 quasi-metric spaces from versions of Kada-Suzuki-Takahashi's theorem for T_1 quasi-metric spaces and whose proofs make use of a suitable quasi-order and the notion of maximal element.

2 Preliminaries

In this brief section, we recap several pertinent concepts and properties on quasi-metric spaces that will be useful throughout the article. Our main reference for these spaces is [33] and for general topology is [34].

By $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{N}_0$ we denote the set of real numbers, the set of non-negative real numbers, the set of positive integer numbers, and the set of non-negative integer numbers, respectively.

A quasi-metric on a set X is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following two conditions for any $x, y, z \in X$:

(qm1) $q(x, y) = q(y, x) = 0$ if and only if $x = y$;

(qm2) $q(x, z) \le q(x, y) + q(y, z)$.

If q satisfies (qm2) and the following condition, stronger than (qm1), we say that q is a T_1 quasi-metric on X:

(qm1') $q(x, y) = 0$ if and only if $x = y$.

A (T_1) quasi-metric space is a pair (X, q) such that X is a set and q is a (T_1) quasi-metric on X.

If q is a quasi-metric on a set X, the family $\{B_q(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base of open sets for a T_0 topology $\mathfrak{T}_q$ on X, where for each $x \in X$ and $\varepsilon > 0$, $B_q(x, \varepsilon) = \{y \in X : q(x, y) < \varepsilon\}$. Note that if q is a T_1 quasi-metric, then $\mathfrak{T}_q$ is a T_1 topology on X.

Given a (T_1) quasi-metric q on X, the function $q^* : X \times X \to \mathbb{R}^+$ defined by $q^*(x, y) = q(y, x)$ for all $x, y \in X$, is also a (T_1) quasi-metric on X, whereas the function $q^s : X \times X \to \mathbb{R}^+$ defined by $q^s(x, y) = \max\{q(x, y), q^*(x, y)\}$ for all $x, y \in X$, is a metric on X.

It is clear from the definition of $\mathfrak{T}_q$ that a sequence $(x_n)_{n\in\mathbb{N}}$ in X is $\mathfrak{T}_q$-convergent to some $x \in X$ if and only if $q(x, x_n) \to 0$ as $n \to \infty$. Similarly, a sequence $(x_n)_{n\in\mathbb{N}}$ in X is $\mathfrak{T}_{q^*}$-convergent to some $x \in X$ if and only if $q(x_n, x) \to 0$ as $n \to \infty$.

The following is a basic but paradigmatic instance of a non-T_1 quasi-metric space.

Example 1. Let $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by $q(x, y) = \max\{x - y, 0\}$. Then, q is a non-T_1 quasi-metric on X, and the topology $\mathfrak{T}_q$ is the so-called lower topology on $\mathbb{R}$. Note also that q^s is the usual metric on $\mathbb{R}$.

Due to the absence of symmetry, we can define various different types of Cauchy sequence and of completeness in the framework of quasi-metric spaces that, nevertheless, coincide with the usual notions of Cauchy sequence and completeness when dealing with a metric space (see, e.g., [33,43]).

Here, we will consider the following two notions of Cauchy sequence and of complete quasi-metric space:

A sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric space (X, q) is left Cauchy provided that for each $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that $q(x_n, x_m) < \varepsilon$ whenever $n_\varepsilon \leq n \leq m$, and it is Cauchy provided that is a Cauchy sequence in the metric space (X, q^s).

A quasi-metric space (X, q) is q^*-right complete provided that every left Cauchy sequence is $\mathfrak{T}_{q^*}$-convergent, and it is q^*-half complete provided that every Cauchy sequence is $\mathfrak{T}_{q^*}$-convergent.

In classical terminology (see, e.g., [33,43]), the notion of q^*-right completeness of (X, q) corresponds to the notion of right K-completeness of (X, q^*), while the notion of q^*-half completeness of (X, q) corresponds to the notion of sequential completeness of (X, q^*).

Obviously, every q^*-right complete quasi-metric space (X, q) is q^* -half complete. The converse does not hold, in general; in fact, the quasi-metric space (X, q) of Example 1 is q^*-half complete because (X, q^s) is a complete metric space, but it is not q^*-right complete because the sequence $(n)_{n\in\mathbb{N}}$ is left Cauchy but it is not $\mathfrak{T}_{q^*}$-convergent.

3 *Q*-functions and *w*-distances

In [41], Park extended the notion of w-distance to the setting of quasi-metric spaces. Later, Al-Homidan et al. [42] introduced and discussed, in the realm of T_1 quasi-metric spaces, the notion of Q-function as a generalization of Park's notion. In the sequel, we remind such notions.

Let (X, q) be a quasi-metric space and let $W : X \times X \to \mathbb{R}^+$. Consider the following conditions:

(**w1**) $W(x, y) \leq W(x, z) + W(z, y)$, for all $x, y, z \in X$.

(**w2**) For each $x \in X$, the function $W(x, \cdot) : X \to \mathbb{R}^+$ is $\mathfrak{T}_{q^*}$-lower semicontinuous ($\mathfrak{T}_{q^*}$-lsc, in short).

(**Q**) If $x, y \in X$, $(y_n)_{n\in\mathbb{N}}$ is a sequence in X that $\mathfrak{T}_{q^*}$-converges to y and there is a constant $M > 0$ such that $Q(x, y_n) \leq M$ for all $n \in \mathbb{N}$, then $Q(x, y) \leq M$.

(**w3**) For each $\varepsilon > 0$, there exists $\delta > 0$ such that $W(x, y) \leq \delta$ and $W(x, z) \leq \delta$ imply $q(y, z) \leq \varepsilon$.

The function W is said to be a w-distance on (X, q) if it satisfies conditions (**w1**), (**w2**), and (**w3**), and it is said to be a Q-function on (X, q) if it satisfies conditions (**w1**), (**Q**) and (**w3**).

Next, we show that actually the notions of w-distance and Q-function coincide.

Proposition 2. *Let (X, q) be a quasi-metric space. Then, a function $F : X \times X \to \mathbb{R}^+$ is a Q-function on (X, q) if and only if it is a w-distance on (X, q).*

Proof. It was noted in [42, page 128] that every w-distance on (X, q) is a Q-function on (X, q).

Now suppose that F is a Q-function on (X, q), which is not a w-distance on (X, q). Then, there is $x \in X$ for which the function $F(x, \cdot) : X \to \mathbb{R}^+$ is not $\mathfrak{T}_{q^*}$-lsc. Therefore, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in X that $\mathfrak{T}_{q^*}$-converges to some $y \in X$, and an $\varepsilon > 0$ and a subsequence $(y_{k_n})_{n\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ such that $F(x, y) \geq \varepsilon + F(x, y_{k_n})$, for all $n \in \mathbb{N}$. Put $M = F(x, y) - \varepsilon/2$. Then, $M > 0$ and $F(x, y_{k_n}) < M$ for all $n \in \mathbb{N}$. Since $(y_{k_n})_{n\in\mathbb{N}}$ is $\mathfrak{T}_{q^*}$-convergent to y and F is a Q-function, we deduce that $F(x, y) \leq M$, a contradiction. Hence, F is a w-distance on (X, q). □

Remark 3. Note that although the authors of [42] worked in the realm of T_1 quasi-metric spaces, Proposition 2 remains valid for every quasi-metric space.

It is well known that any metric d on a set X is a w-distance on the metric space (X, d) (see [23, Example 1]). However, there are quasi-metric spaces (X, q) for which the quasi-metric q is not a w-distance on (X, q) [31, Proposition 2.3]. Despite this, the use of w-distances instead of the original quasi-metric one yields better results in extending Caristi's theorem to the frame of non-T_1 quasi-metric spaces as we shall show in Theorem 12 in the next section.

We underline that there are many interesting examples of w-distances on quasi-metric spaces (see, e.g., [41,42,31]). Below are two of them, which are typical (cf. [42, Examples 2.1(a) and 2.1(b)]).

Example 4. Let q be the quasi-metric on $\mathbb{R}$ given by $q(x, x) = 0$ for all $x \in \mathbb{R}$, and $q(x, y) = |y|$ otherwise. Since $q(x, 0) = 0$ for all $x \in \mathbb{R}$, we deduce that $(\mathbb{R}, q)$ is a non-T_1 quasi-metric space. Now, let $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ defined by $W(x, y) = |y|$ for all $x, y \in \mathbb{R}$. Then, W is a Q-function on $(\mathbb{R}, q)$ [42, Example 1(a)], so it is a w-distance on $(\mathbb{R}, q)$ by Proposition 2. We shall show this fact directly for the sake of completeness. To this end, it suffices to verify condition (**w2**). Indeed, fix $x \in \mathbb{R}$ and let $(y_n)_{n\in\mathbb{N}}$ be a non-eventually constant sequence in $\mathbb{R}$ that $\mathfrak{T}_{q^*}$-converges to some $y \in \mathbb{R}$. Then, $q(y_n, y) \to 0$ as $n \to \infty$, and $q(y_n, y) = |y|$ eventually, so $y = 0$. Hence, $W(x, y) = 0$, and thus $W(x, \cdot)$ is $\mathfrak{T}_{q^*}$-lsc. We conclude that W is a w-distance on $(\mathbb{R}, q)$.

Example 5. Let q be the quasi-metric on $\mathbb{R}$ given by $q(x, y) = x - y$ if $y \leq x$, and $q(x, y) = 2(y - x)$ otherwise. Clearly, $(\mathbb{R}, q)$ is a T_1 quasi-metric space. Now, let $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ be defined by $W(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Then, W is a Q-function on $(\mathbb{R}, q)$ [42, Example 1(b)], so it is a w-distance on $(\mathbb{R}, q)$ by Proposition 2. We shall show directly this fact for the sake of completeness. To this end, it suffices to verify condition (**w2**). Indeed, fix $x \in \mathbb{R}$ and let $(y_n)_{n\in\mathbb{N}}$ be a non-eventually constant sequence in $\mathbb{R}$ that $\mathfrak{T}_{q^*}$-converges to some $y \in \mathbb{R}$. Then, $q(y_n, y) \to 0$ as $n \to \infty$, so, by the definition of q, $|y_n - y| \to 0$ as $n \to \infty$. Since $W(x, y) \leq W(x, y_n) + |y_n - y|$, we deduce that $W(x, \cdot)$ is $\mathfrak{T}_{q^*}$-lsc, so W is a w-distance on $(\mathbb{R}, q)$.

We conclude this section with a novel example, based on the notion of a partial function, which will be used in illustrating our w-distance version of Caristi's theorem.

It is interesting to point out that partial functions constitute an adequate instrument for modeling, through appropriate quasi-metrics, some typical procedures in symbolic computation as well as in complexity analysis of algorithms (see, e.g., [44,45]).

In our context, by a partial function, we mean a mapping f whose domain is an initial segment of $\mathbb{N}$ and takes values in $\mathbb{R}^+$. The set of all partial functions will be expressed as PF. Therefore, $f \in \mathrm{PF}$ if and only if there is $k \in \mathbb{N}$ such that $f : \{1, \ldots, k\} \to \mathbb{R}^+$. The number k is called the length of f and is denoted by $\ell(f)$.

Example 6. On the set PF of partial functions, we define a relation $\sqsubseteq_{\mathrm{PF}}$ as follows:

$$f \sqsubseteq_{\mathrm{PF}} g \iff \ell(f) = \ell(g) \quad \text{and} \quad f(n) \leq g(n) \quad \text{for all } n \in \{1, \ldots, \ell(f)\}.$$

It is clear that $\sqsubseteq_{\mathrm{PF}}$ is a partial order on PF (i.e., a reflexive, antisymmetric, and transitive relation).

Now, let q_{PF} be the non-T_1 quasi-metric on PF given by $q_{\mathrm{PF}}(f, g) = 0$ if $f \sqsubseteq_{\mathrm{PF}} g$, and $q_{\mathrm{PF}}(f, g) = 1$ otherwise. It is well known, and easily checked, that the topology induced by q_{PF} agrees with the famous Alexandroff topology on PF, that is, any topology where the intersection of an arbitrary family of open sets is open. Note that $(q_{\mathrm{PF}})^s$ is the discrete metric on PF, i.e., $(q_{\mathrm{PF}})^s(f, g) = 1$ whenever $f \neq g$, and hence the quasi-metric space $(\mathrm{PF}, q_{\mathrm{PF}})$ is $(q_{\mathrm{PF}})^*$-half complete because every Cauchy sequence in $(\mathrm{PF}, q_{\mathrm{PF}})$ is eventually constant.

Let f_0 be the element of PF such that $\ell(f_0) = 1$ and $f_0(1) = 1$.

Define a function $W_{\mathrm{PF}} : \mathrm{PF} \times \mathrm{PF} \to \mathbb{R}^+$ as follows:

$W_{\mathrm{PF}}(f_0, f_0) = 0$ and $W_{\mathrm{PF}}(f, g) = \ell(g)$ otherwise.

We are going to show that W_{PF} is a w-distance on $(\mathrm{PF}, q_{\mathrm{PF}})$, i.e., that it satisfies conditions (**w1**), (**w2**) and (**w3**). Indeed,

For (**w1**), let $f, g, h \in \mathrm{PF}$. Since $W_{\mathrm{PF}}(f, g) = W_{\mathrm{PF}}(h, g)$, we immediately obtain that $W_{\mathrm{PF}}(f, g) \leq W_{\mathrm{PF}}(f, h) + W_{\mathrm{PF}}(h, g)$.

For (**w2**), fix $f \in \mathrm{PF}$ and let $(g_j)_{j\in\mathbb{N}}$ be a sequence in PF that $\mathfrak{T}_{(q_{\mathrm{PF}})^*}$-converges to a $g \in \mathrm{PF}$. Then, $q_{\mathrm{PF}}(g_j, g) \to 0$ as $j \to \infty$, so there is $j_0 \in \mathbb{N}$ such that $q(g_j, g) = 0$ for all $j \geq j_0$. This implies that $g_j \sqsubseteq_{\mathrm{PF}} g$, and hence, $\ell(g_j) = \ell(g)$ for all $j \geq j_0$. If $f = g = f_0$, we have $W_{\mathrm{PF}}(f, g) = 0$. Otherwise, we obtain $W_{\mathrm{PF}}(f, g) = W_{\mathrm{PF}}(f, g_j)$ for all $j \geq j_0$. Consequently, $W_{\mathrm{PF}}(f, \cdot)$ is $\mathfrak{T}_{(q_{\mathrm{PF}})^*}$-lsc.

For (**w3**), fix $\varepsilon > 0$. Put $\delta = \min\{1/2, \varepsilon\}$. Let $f, g, h \in PF$ such that $W_{\mathrm{PF}}(f, g) \leq \delta$ and $W_{\mathrm{PF}}(f, h) \leq \delta$. Then, $f = g = h = f_0$, so $q_{\mathrm{PF}}(f, g) = 0 \leq \varepsilon$.

4 Main results

The following characterization of q^*-right complete quasi-metric spaces is an adaptation of [26, Theorem 2] to our context.

Theorem 7. *For a quasi-metric space (X, q), the following statements are equivalent.*
(1) *(X, q) is q^*-right complete.*
(2) *If T is a self-mapping T of X such that there is a $\mathfrak{T}_{q_*}$-nearly lsc function $\phi : X \to \mathbb{R}^+$ fulfilling, for every $x \in X$,*

$$q(x, Tx) \leq \phi(x) - \phi(Tx),$$

then, there exists $u \in X$ satisfying $\phi(u) = \phi(Tu)$.

Remark 8. We recall that, according to [26], given a quasi-metric space (X, q), a function $f : X \to \mathbb{R}$ is $\mathfrak{T}_q$-nearly lsc provided that whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence of distinct points in X that $\mathfrak{T}_q$-converges to some $x \in X$, we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$. Furthermore, the notions of $\mathfrak{T}_q$-nearly lsc and $\mathfrak{T}_q$-lsc coincide whenever (X, q) is a T_1 quasi-metric space.

Note that if in the preceding theorem (X, q) is a T_1 quasi-metric space, then u is a fixed point of T because from $\phi(u) = \phi(Tu)$, we deduce that $q(u, Tu) = 0$, so $u = Tu$ [46, Theorem 2.12]. However, the following modification of [26, Example 2] provides an instance of a self-mapping T of a non-T_1 quasi-metric space that has no fixed points but for which there is a function $\phi : X \to \mathbb{R}$ satisfying the conditions of (2) in the preceding theorem.

Example 9. Let X be the set of all ordinals less than the first uncountable ordinal number ω_1. Consider the non-T_1 quasi-metric q on X given by $q(x, y) = 0$ if $x \leq y$, and $q(x, y)$=1 otherwise. It is clear that (X, q) is $q*$-right complete because every left Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ is $\mathfrak{T}_{q^*}$-convergent to $x := \sup\{x_n : n \in \mathbb{N}\}$. Now consider the self-mapping of X given by $Tx = x + 1$ for all $x \in X$. Then, T has no fixed points but one has $q(x, Tx) = 0 = \phi(x) - \phi(Tx)$, for all $x \in X$, where $\phi(x) = 0$ for all $x \in X$.

Our next theorem shows that the use of w-distances instead of the quasi-metric q provides two important advantages with respect to the part (1) $\Rightarrow$ (2) in Theorem 7. By one hand, the result remains valid for the more general class of q^*-half complete quasi-metric spaces, and, on the other hand, the existence of fixed point is guaranteed.

Definition 10. Let (X, q) be a quasi-metric space. A self-mapping T of X is called a W-Caristi mapping (on (X, q)) if there exist a w-distance W on (X, q) and a $\mathfrak{T}_{q*}$-lsc function $\phi : X \to \mathbb{R}^+$ such that

$$W(x, Tx) \leq \phi(x) - \phi(Tx),$$

for all $x \in X$.

Before to establish our w-distance version of Caristi's theorem, we give an auxiliary lemma which will help us to simplify its proof. (As usual, given a [non-empty] set X, the family of all non-empty subsets of X will be denoted by 2^X.)

Lemma 11. *Let X be a (non-empty) set, $\mathcal{F} : X \mapsto 2^X$ a multivalued mapping, and ϕ a function from X to $\mathbb{R}^+$. Then, for each $x \in X$, there is a sequence $(x_n)_{n\in\mathbb{N}_0}$ in X such that $x_0 = x$, $x_{n+1} \in \mathcal{F}x_n$, and*

$$\phi(x_{n+1}) < 2^{-n} + \inf\phi(\mathcal{F}x_n)$$

for all $n \in \mathbb{N}_0$.

If, in addition, there is a function $W : X \times X \to \mathbb{R}^+$ satisfying the triangle inequality and verifying

$$W(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1})$$

for all $n \in \mathbb{N}_0$, then, for each $\delta > 0$, there is $n_\delta \in \mathbb{N}_0$ such that

$$W(x_n, x_m) < \delta,$$

whenever $m > n \geq n_\delta$.

Proof. Let $x \in X$. Put $x_0 = x$. Since $\mathcal{F}x_0 \neq \emptyset$, there exists $x_1 \in \mathcal{F}x_0$ such that $\phi(x_1) < 1 + \inf\phi(\mathcal{F}x_0)$.

Analogously, there exists $x_2 \in \mathcal{F}x_1$ such that $\phi(x_2) < 2^{-1}\inf\phi(\mathcal{F}x_1)$.

Thus, we inductively deduce the existence of a sequence $(x_n)_{n\in\mathbb{N}_0}$ in X such that $x_{n+1} \in \mathcal{F}x_n$ and $\phi(x_{n+1}) < 2^{-n} + \inf\phi(\mathcal{F}x_n)$, for all $n \in \mathbb{N}_0$.

Now, suppose that there is a function $W : X \times X \to \mathbb{R}^+$ satisfying the triangle inequality and verifying $W(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1})$, for all $n \in \mathbb{N}_0$.

Then, $(\phi(x_n))_{n\in\mathbb{N}_0}$ is a non-increasing sequence in $\mathbb{R}^+$, and hence, it is a Cauchy sequence in $\mathbb{R}^+$ when endowed with the usual metric. Consequently, given $\delta > 0$, there is $n_\delta \in \mathbb{N}_0$ such that $\phi(x_n) - \phi(x_m) < \delta$, for all $n, m \geq n_\delta$.

Since W satisfies the triangle inequality, we deduce that

$$W(x_n, x_m) \leq \sum_{k=n}^{m-1} W(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} (\phi(x_k) - \phi(x_{k+1})) = \phi(x_n) - \phi(x_m) < \delta,$$

whenever $m > n \geq n_\delta$. □

Theorem 12. *Every W-Caristi mapping on a q^*-half complete quasi-metric space (X, q) has a fixed point.*

Proof. Let T be a W-Caristi mapping on a q^*-half complete quasi-metric space (X, q). Then, there exist a w-distance W on (X, q) and a $\mathfrak{T}_{q*}$-lsc function $\phi : X \to \mathbb{R}^+$ such that

$$W(x, Tx) \leq \phi(x) - \phi(Tx)$$

for all $x \in X$.

Define a multivalued mapping $\mathcal{F} : X \mapsto 2^X$ by

$$\mathcal{F}x = \{y \in X : W(x, y) \leq \phi(x) - \phi(y)\}$$

for all $x \in X$.

Note that $\mathcal{F}$ is well-defined because $Tx \in \mathcal{F}x$, and thus $\mathcal{F}x \in 2^X$ for all $x \in X$.

Fix now an $x \in X$. By the first part of Lemma 11, there is a sequence $(x_n)_{n\in\mathbb{N}_0}$ in X such that $x_0 = x$, $x_{n+1} \in \mathcal{F}x_n$ and $\phi(x_{n+1}) < 2^{-n} + \inf\phi(\mathcal{F}x_n)$, for all $n \in \mathbb{N}_0$.

Since $x_{n+1} \in \mathcal{F}x_n$, it follows from the definition of the multivalued mapping $\mathcal{F}$ that $W(x_n, x_{n+1}) \leq \phi(x_n) - \phi(x_{n+1})$, for all $n \in \mathbb{N}_0$.

Therefore, by the second part of Lemma 11, we obtain that, for each $\delta > 0$, there is $n_\delta \in \mathbb{N}_0$ such that $W(x_n, x_m) < \delta$, whenever $m > n \geq n_\delta$.

Choose an arbitrary $\varepsilon > 0$. Let $\delta := \delta(\varepsilon)$ for which condition (**w3**) is fulfilled. Then, for every $j, k > n_\delta$, we have $W(x_{n_\delta}, x_j) < \delta$ and $W(x_{n_\delta}, x_k) < \delta$, so $q(x_j, x_k) \le \varepsilon$ and $q(x_k, x_j) \le \varepsilon$.

This implies that $(x_n)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in the metric space (X, q^s). Since (X, q) is q^*-sequentially complete, there exists $u \in X$ such that $q(x_n, u) \to 0$ as $n \to \infty$.

Next, we shall prove that $w(u, Tu) = 0$. To this end, we shall show four claims.

Claim 1. $W(x_n, u) \to 0$ as $n \to \infty$.

Indeed, given $\delta > 0$, there is $n_\delta \in \mathbb{N}_0$ such that $W(x_n, x_m) < \delta$, whenever $m > n \ge n_\delta$. Fix $n \ge n_\delta$. By condition (**w2**), there is $m > n$ such that $W(x_n, u) < \delta + W(x_n, x_m)$. Hence,

$$W(x_n, u) < 2\delta$$

for all $n \ge n_\delta$.

Claim 2. $u \in \bigcap_{n\in\mathbb{N}_0} \mathcal{F}x_n$.

Indeed, fix $n \in \mathbb{N}_0$. Choose an arbitrary $\delta > 0$. By Claim 1 and the fact that ϕ is $\mathfrak{T}_{q*}$-lsc, we deduce the existence of an $m > n$ such that $W(x_m, u) < \delta$ and $\phi(u) - \phi(x_m) < \delta$.

Taking into account that $x_m \in \mathcal{F}x_n$, we obtain

$$W(x_n, u) \le W(x_n, x_m) + W(x_m, u) \le \phi(x_n) - \phi(x_m) + \delta \le \phi(x_n) - \phi(u) + 2\delta.$$

Since δ is arbitrary, we conclude that $W(x_n, u) \le \phi(x_n) - \phi(u)$, so $u \in \mathcal{F}x_n$.

Claim 3. $\phi(u) = \inf_{n\in\mathbb{N}_0} \phi(x_n)$.

Indeed, by Claim 2, $u \in \mathcal{F}x_n$ for all $n \in \mathbb{N}_0$. So, by the definition of $\mathcal{F}$, $\phi(u) \le \phi(x_n)$ for all $n \in \mathbb{N}_0$. Thus, $\phi(u) \le \inf_{n\in\mathbb{N}_0} \phi(x_n)$.

On the other hand, we have

$$\phi(x_{n+1}) < 2^{-n} + \inf \phi(\mathcal{F}x_n) \le 2^{-n} + \phi(u)$$

for all $n \in \mathbb{N}_0$, and hence $\inf_{n\in\mathbb{N}_0} \phi(x_n) \le \phi(u)$.

Claim 4. $\phi(Tu) = \inf_{n\in\mathbb{N}_0} \phi(x_n)$.

Indeed, since T is W-Caristi mapping, we have $Tu \in \mathcal{F}u$, and, by Claim 2, we also have that $u \in \mathcal{F}x_n$ for all $n \in \mathbb{N}_0$. Therefore,

$$W(x_n, Tu) \le W(x_n, u) + W(u, Tu) \le \phi(x_n) - \phi(Tu)$$

for all $n \in \mathbb{N}_0$, so that $Tu \in \bigcap_{n\in\mathbb{N}_0} \mathcal{F}x_n$. As in the proof of Claim 2, we obtain that $\phi(Tu) \le \inf_{n\in\mathbb{N}_0} \phi(x_n)$, and also

$$\phi(x_{n+1}) < 2^{-n} + \inf \phi(\mathcal{F}x_n) \le 2^{-n} + \phi(Tu)$$

for all $n \in \mathbb{N}_0$, so $\inf_{n\in\mathbb{N}_0} \phi(x_n) \le \phi(Tu)$.

From Claims 3 and 4, we have that $\phi(u) = \phi(Tu)$, and, consequently, $w(u, Tu) = 0$.

Finally, we prove that $u = Tu$. Indeed, by Claim 1, the triangle inequality, and the fact that $w(u, Tu) = 0$, we deduce that $W(x_n, Tu) \to 0$ as $n \to \infty$.

Now, choose an arbitrary $\varepsilon > 0$. Let $\delta > 0$ for which condition (**w3**) is fulfilled. Take $n \in \mathbb{N}_0$ such that $W(x_n, u) \le \delta$ and $W(x_n, Tu) \le \delta$. From condition (**w3**), it follows that $q^s(u, Tu) \le \varepsilon$. Hence, $u = Tu$, and thus, u is a fixed point of T. Furthermore, $W(u, u) = 0$. □

We illustrate the preceding with the next (promised) example.

Example 13. Let $(\mathrm{PF}, q_{\mathrm{PF}})$ be the quasi-metric space of Example 6, and let T be the self-mapping of PF defined as follows:

For each $f \in \mathrm{PF}$ with $\ell(f) \ge 2$, Tf is the only element of PF such that $\ell(Tf) = \ell(f) - 1$ and $Tf(n) = f(n)$ for all $n \in \{1, \ldots, \ell(Tf)\}$, and for each $f \in \mathrm{PF}$ with $\ell(f) = 1$, put $Tf = f_0$.

We are going to prove that T is a W_{PF}-Caristi mapping on $(\mathrm{PF}, q_{\mathrm{PF}})$, where W_{PF} is the w-distance constructed in Example 6.

Indeed, define a function $\phi : PF \to \mathbb{R}^+$ by $\phi(f) = (\ell(f))^2$ for all $f \in \mathrm{PF}\backslash\{f_0\}$, and $\phi(f_0) = 0$.

Clearly, ϕ is $\mathfrak{T}_{(q_{\mathrm{PF}})*}$-lsc. Indeed, let $(g_j)_{j\in\mathbb{N}}$ be a sequence in PF that $\mathfrak{T}_{(q_{\mathrm{PF}})*}$-converges to a $g \in \mathrm{PF}\setminus\{f_0\}$. Then, $q_{\mathrm{PF}}(g_j, g) \to 0$ as $j \to \infty$, so there is $j_0 \in \mathbb{N}$ such that $q(g_j, g) = 0$ for all $j \geq j_0$. This implies that $g_j \sqsubseteq_{\mathrm{PF}} g$, and hence $\ell(g_j) = \ell(g)$ for all $j \geq j_0$, so $\phi(g) = \phi(g_j)$, for all $j \geq j_0$.

Since $W_{\mathrm{PF}}(f_0, Tf_0) = W_{\mathrm{PF}}(f_0, f_0) = 0$, we obtain $W_{\mathrm{PF}}(f_0, Tf_0) = \phi(f_0) - \phi(Tf_0)$.

Now let $f \in PF\setminus\{f_0\}$. If $\ell(f) \geq 2$, we obtain

$$W_{\mathrm{PF}}(f, Tf) = \ell(Tf) = \ell(f) - 1 < 2\ell(f) - 1 = (\ell(f))^2 - (\ell(f) - 1)^2 = \phi(f) - \phi(Tf),$$

and if $\ell(f) = 1$, we obtain

$$W_{\mathrm{PF}}(f, Tf) = W_{\mathrm{PF}}(f, f_0) = \ell(f_0) = 1 = (\ell(f))^2 = \phi(f) - \phi(Tf).$$

We have shown that T is a w_{PF}-Caristi mapping on the $(q_{\mathrm{PF}})*$-half complete quasi-metric space $(\mathrm{PF}, q_{\mathrm{PF}})$. Hence, we can apply Theorem 12 to conclude that T has a fixed point. In fact f_0 is the unique fixed point of T.

Finally, we shall show that $(\mathrm{PF}, q_{\mathrm{PF}})$ is not $(q_{\mathrm{PF}})*$-right complete, and thus we cannot apply Theorem 7.

Indeed, consider the sequence $(g_j)_{j\in\mathbb{N}}$ in PF such that $\ell(g_j) = 1$ and $g_j(1) = j$ for all $j \in \mathbb{N}$. Since $q_{\mathrm{PF}}(g_i, g_j) = 0$ whenever $i \leq j$, we deduce that $(g_j)_{j\in\mathbb{N}}$ is a $(q_{\mathrm{PF}})*$-right Cauchy sequence in $(\mathrm{PF}, q_{\mathrm{PF}})$. Suppose that there is $g \in \mathrm{PF}$ such that $q_{\mathrm{PF}}(g_j, g) \to 0$ as $j \to \infty$. Then, there is $j_0 \in \mathbb{N}$ such that $q_{\mathrm{PF}}(g_j, g) = 0$ for all $j \geq j_0$. Hence, $\ell(g_j) = \ell(g) = 1$ and $g_j(1) \leq g(1)$ for all $j \geq j_0$, so $j \leq g(1)$ for all $j \geq j_0$, a contradiction. Consequently, $(\mathrm{PF}, q_{\mathrm{PF}})$ is not $(q_{\mathrm{PF}})^*$-right complete.

We conclude this section with our main result.

Theorem 14. *A quasi-metric space (X, q) is q^*-half complete if and only if every W-Caristi mapping on it has a fixed point.*

Proof. The "only if" part follows from Theorem 12.

To prove the "if" part suppose that (X, q) is not q^*-half complete. Then, there exists a non-τ_{q^*}-convergent sequence $(x_n)_{n\in\mathbb{N}}$, which is a Cauchy sequence in the metric space (X, q^s).

Therefore, for each $n \in \mathbb{N}$, we can inductively find a $k_n \in \mathbb{N}$ such that $k_1 > 1$, $k_{n+1} > \max\{k_n, n + 1\}$, and $q^s(x_i, x_j) < 2^{-n}$ for all $i, j \geq k_n$. So, in particular, $q^s(x_{k_n}, x_{k_m}) < 2^{-n}$ whenever $m \geq n$.

Put $F := \{x_{k_n} : n \in \mathbb{N}\}$ and define a function $W : X \times X \to \mathbb{R}^+$ by $W(x, y) = q^s(x, y)$ if $x, y \in F$ and $W(x, y) = 1$ otherwise.

We check that W is a w-distance on (X, q).

We first note that $W(x, y) < 1/2$ for all $x, y \in F$.

Condition (**w1**) is clearly fulfilled.

For (**w2**), fix $x \in X$ and let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X and $y \in X$ such that $q(y_n, y) \to 0$ as $n \to \infty$. If there is a subsequence (z_n) of (y_n) such that $z_n \in F$ for all $n \in \mathbb{N}$, we deduce, by the triangle inequality, that $q(x_{nk}, y) \to 0$ as $n \to \infty$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, q^s), we deduce that $q(x_n, y) \to 0$ as $n \to \infty$, a contradiction.

Consequently, there is $n_0 \in \mathbb{N}$ such that $y_n \in X\setminus F$ for all $n \geq n_0$. Thus, $W(x, y_n) = 1$ for all $n \geq n_0$. Since $W(x, y) \leq 1$, we conclude that $q(x, \cdot)$ is $\mathfrak{T}_{q^*}$-lsc.

Finally, for (**w3**), choose an arbitrary $\varepsilon > 0$. Put $\delta = \min\{1/2, \varepsilon/2\}$. Let $x, y, z \in X$ such that $W(x, y) \leq \delta$ and $W(x, z) \leq \delta$. Then, $W(x, y) \leq 1/2$ and $W(x, z) \leq 1/2$, so $x, y, z \in F$. Therefore, $q^s(x, y) \leq \delta \leq \varepsilon/2$ and $q^s(x, z) \leq \delta \leq \varepsilon/2$. By the triangle inequality we conclude that $q^s(y, z) \leq \varepsilon$, and thus $q(y, z) \leq \varepsilon$.

Now define a function $\phi : X \to \mathbb{R}^+$ and a self-mapping T of X as follows:

$$\phi\left(x_{k_n}\right) = 2^{-(n-1)} \quad \text{for all } n \in \mathbb{N},$$

$$\phi(x) = 2 \quad \text{for all } x \in X\setminus F,$$

$$Tx_{kn} = x_{k_{n+1}} \quad \text{for all } n \in \mathbb{N},$$

and

$$Tx = x_{k_1} \quad \text{for all } x \in X\backslash F.$$

Obviously T has no fixed points. We shall show that, nevertheless, T is a W-Caristi mapping on (X, q) (with respect to the w-distance W and the function ϕ defined earlier).

We first check that ϕ is $\mathfrak{T}_{q^*}$-lsc. As previously mentioned, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X and $y \in X$ such that $q(y_n, y) \to 0$ as $n \to \infty$. If there is a subsequence (z_n) of (y_n) such that $z_n \in F$ for all $n \in \mathbb{N}$, we deduce that $q(x_n, y) \to 0$ as $n \to \infty$, a contradiction.

Consequently, there is $n_0 \in \mathbb{N}$ such that $y_n \in X\backslash F$ for all $n \geq n_0$. Thus, $\phi(y_n) = 2$ for all $n \geq n_0$. Since $\phi(y) \leq 2$, we conclude that ϕ is $\mathfrak{T}_{q^*}$ -lsc.

Now, let $x \in X$. If $x \in F$, we obtain $x := x_{k_n}$ for some $n \in \mathbb{N}$. Therefore,

$$W(x, Tx) = W\left(x_{k_n}, x_{k_{n+1}}\right) = q^s\left(x_{k_n}, x_{k_{n+1}}\right) < 2^{-n} = \phi\left(x_{k_n}\right) - \phi\left(x_{k_{n+1}}\right) = \phi(x) - \phi(Tx).$$

If $x \in X\backslash F$ we obtain

$$W(x, Tx) = W\left(x, x_{k_1}\right) = 1 = \phi(x) - \phi\left(x_{k_1}\right) = \phi(x) - \phi(Tx).$$

Thus, we have reached a contradiction that concludes the proof. □

5 An application to *G*-metric spaces

In this section, we apply Theorem 7 to obtain a characterization of complete G-metric spaces.

The concept of a G-metric space was introduced and analyzed by Mustafa and Sims in [47] motivated by the existence of several mistakes in the study of the topological structure of the so-called D-metric spaces. In fact, Mustafa-Sims' study constituted the starting point for the development of an intensive research in fixed point theory for this kind of spaces and other related ones (cf. [48–55] and references therein). In particular, our basic reference for G-metric spaces will be [49, Chapter 3].

Let us recall that a G-metric on a set X is a function $G : X \times X \times X \to \mathbb{R}^+$ that satisfies the following conditions for any $x, y, z, a \in X$:

(gm1) $G(x, y, z) = 0$ if $x = y = z$.

(gm2) $G(x, x, y) > 0$ if $x \neq y$.

(gm3) $G(x, x, y) \leq G(x, y, z)$ if $y \neq z$.

(gm4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all 3).

(gm5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality).

By a G-metric space, we mean a pair (X, G) such that X is a set and G is a G-metric on X.

In [49, p. 34–35], one can find numerous instances of G-metric spaces.

The following properties may be found in [49, Chapter 3].

Each G-metric G on a set X induces a topology $\mathfrak{T}_G$ on X, which has as a base the family of open balls $\{B_G(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_G(x, \varepsilon) = \{y \in X : G(x, y, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Furthermore, the topological space $(X, \mathfrak{T}_G)$ is metrizable.

A G-metric space (X, G) is complete provided that every G-Cauchy sequence is $\mathfrak{T}_G$-convergent, where a sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be G-Cauchy if for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_k) < \varepsilon$ for all $n, m, k \geq n_0$.

Given a G-metric space (X, G), the function $q_G : X \times X \to \mathbb{R}^+$ given by $q_G(x, y) = G(x, y, y)$ for all $x, y \in X$ is a T_1 quasi-metric on X. (This quasi-metric is denoted by q'_G in [49].)

Let (X, G) be a G-metric space. From [49, Lemma 3.3.1], we deduce the following important properties:

(**P1**) The topologies $\mathfrak{T}_G$, $\mathfrak{T}_{q_G}$, and $\mathfrak{T}_{(q_G)^*}$ coincide on X.

(**P2**) (X, G) is complete if and only if (X, q_G) is $(q_G)^*$-half complete.

Saadati et al. introduced in [56] a G-metric version of the notion of w-distance with the aim of obtaining fixed point results for complete ordered G-metric spaces. We modify the notion given in [56] as follows:

Definition 15. Let (X, G) be a G-metric space. We say that a function $WG : X \times X \to \mathbb{R}^+$ is a wG-distance on (X, G) if it verifies the following conditions:

($\boldsymbol{wG_1}$) $WG(x, y, z) \leq WG(x, a, a) + WG(a, y, z)$, for all $x, y, z, a \in X$.

($\boldsymbol{wG_2}$) For each $x \in X$, $WG(x, \cdot, \cdot) : X \times X \to \mathbb{R}^+$ is lsc in the sense that if $(y_n)_{n\in\mathbb{N}}$ is a sequence in X that $\mathfrak{T}_G$-converges to some $y \in X$, then for each $\varepsilon > 0$, $WG(x, y, y) < \varepsilon + WG(x, y_n, y_n)$, eventually.

($\boldsymbol{wG_3}$) For each $\varepsilon > 0$, there is $\delta > 0$ such that $WG(x, y, y) \leq \delta$ and $WG(x, z, z) \leq \delta$ imply $G(y, z, z) \leq \varepsilon$.

Remark 16. Note that by exchanging y with z in condition ($\boldsymbol{wG_3}$) , we also have $WG(z, y, y) \leq \varepsilon$.

Remark 17. It is not hard to check that every G-metric G on a set X is a wG-metric on (X, G). In fact, condition ($\boldsymbol{wG_1}$) follows directly from condition (gm5). Condition ($\boldsymbol{wG_2}$) follows from conditions (gm4) and (gm5) and the fact that $G(y, y, y_n) \to 0$ as $n \to \infty$, whenever the sequence $(y_n)_{n\in\mathbb{N}}\mathfrak{T}_G$-converges to y. Finally, to show condition ($\boldsymbol{wG_3}$) choose an $\varepsilon > 0$ and suppose that $G(x, y, y) \leq \varepsilon/3$ and $G(x, z, z) \leq \varepsilon/3$. Then, $G(y, z, z) \leq G(y, x, x) + G(x, z, z) \leq 2G(y, y, x) + G(x, z, z) \leq \varepsilon$.

As an immediate consequence of the preceding remark, we obtain that if (X, G) is a G-metric space, the quasi-metric q_G is a w-distance on (X, q_G).

Proposition 18. *Let WG be a wG-distance on a G-metric space (X, G). Then, the function $w : X \times X \to \mathbb{R}^+$ defined as $w(x, y) = WG(x, y, y)$ for all $x, y \in X$, is a w-distance on the quasi-metric space (X, q_G).*

Proof. Let $x, y, z \in X$. We proceed to check the conditions of the definition of a w-distance.

(**w1**): From condition ($\boldsymbol{wG_1}$), we obtain

$$w(x, y) = WG(x, y, y) \leq WG(x, z, z) + WG(z, y, y) = w(x, z) + w(z, y).$$

(**w2**): Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X that $\mathfrak{T}_{(q_G)^*}$-converges to some $y \in X$. Then, $(y_n)_{n\in\mathbb{N}}\mathfrak{T}_G$-converges to y by property (**P1**). Given $\varepsilon > 0$, it follows from condition ($\boldsymbol{wG_1}$) that

$$w(x, y) = WG(x, y, y) < \varepsilon + WG(x, y_n, y_n) = \varepsilon + w(x, y_n)$$

eventually.

(**w3**): Given $\varepsilon > 0$, suppose that $w(x, y) \leq \delta$ and $w(x, z) \leq \delta$, where this δ is the one associated with ε in condition ($\boldsymbol{wG_3}$) . Then, we have $WG(x, y, y) \leq \delta$ and $WG(x, z, z) \leq \delta$. Therefore, $WG(y, z, z) \leq \varepsilon$, i.e., $w(y, z) \leq \varepsilon$. □

Proposition 19. *Let (X, G) be a G-metric space and let q_G be the quasi-metric induced by G. If w is a w-distance on (X, q_G), then the function $WG : X \times X \to \mathbb{R}^+$ defined by $WG(x, y, z) = w(x, y)$ for all $x, y, z \in X$, is a wG-distance on (X, G).*

Proof. Let $x, y, z, a \in X$. We proceed to check the conditions of the definition of a wG-distance.

($\boldsymbol{wG_1}$) : Since, by definition, $WG(x, y, z) = w(x, y)$, $WG(x, a, a) = w(x, a)$, and $WG(a, y, z) = w(a, y)$, from condition (**w1**), we obtain

$$WG(x, y, z) \leq WG(x, a, a) + WG(a, y, z).$$

($\boldsymbol{wG_2}$) : Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X that $\mathfrak{T}_G$-converges to some $y \in X$. By property (**P1**) it $\mathfrak{T}_{(q_G)^*}$-converges to y. Thus, by condition (**w2**), given $\varepsilon > 0$ we have, in particular, $w(x, y) < \varepsilon + w(x, y_n)$ eventually. Since, by definition, $WG(x, y, y) = w(x, y)$ and $WG(x, y_n, y_n) = w(x, y_n)$, we deduce that

$$WG(x, y, y) < \varepsilon + WG(x, y_n, y_n)$$

eventually.

($\boldsymbol{wG_3}$): Given $\varepsilon > 0$, suppose that $WG(x, y, y) \leq \delta$ and $WG(x, z, z) \leq \delta$, where this δ is the one associated with ε in condition ($\boldsymbol{wG_3}$) . Since, by definition, $WG(x, y, y) = w(x, y)$ and $WG(x, z, z) = w(x, z)$, we deduce that $q_G(y, z) \leq \varepsilon$, i.e., $G(y, z, z) \leq \varepsilon$. (Note that we also obtain $q_G(z, y) \leq \varepsilon$, i.e., $G(z, y, y) \leq \varepsilon$.) □

Remark 20. It is easy to construct wG-distances. For instance, let (X, d) be a metric space. Then, the function $G : X \times X \times X \to \mathbb{R}^+$ given by

$$G(x, y, z) = d(x, y) + d(x, z) + d(y, z)$$

for all $x, y, z \in X$ is a G-metric on X (see [47]). Since q_G is a w-distance on (X, q_G) and $q_G(x, y) = G(x, y, y) = 2d(x, y)$, we deduce from Proposition 19 that the function WG defined by $WG(x, y, z) = 2d(x, y)$ for all $x, y, z \in X$ is a wG-distance on (X, G).

Definition 21. Let (X, G) be a G-metric space. A self-mapping T of X is said to be a WG-Caristi mapping (on (X, G)) if there exist a wG-distance WG on (X, G) and a $\mathfrak{T}_G$-lsc function $\phi : X \to \mathbb{R}^+$ such that

$$WG(x, Tx, Tx) \leq \phi(x) - \phi(Tx)$$

for all $x \in X$.

Theorem 22. *A G-metric space (X, G) is complete if and only if every WG-Caristi mapping on it has a fixed point.*

Proof. Let T be a WG-Caristi mapping on the complete G-metric space (X, G). By property (**P1**) and Proposition 18, we deduce that T is a W-Caristi mapping on the quasi-metric space (X, q_G), which is q^*-half complete by property (**P2**). Therefore, T has a fixed point by Theorem 12.

Conversely, let T be a W-Caristi mapping on the quasi-metric space (X, q_G). By property (**P1**) and Proposition 19, we deduce that T is a WG-Caristi mapping on (X, G), so, by hypothesis, it has a fixed point. Hence, (X, q_G) is q^*-half complete by Theorem 14. We conclude that (X, G) is complete by property (**P2**). □

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