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Coincidence and Fixed Points of Non-expansive Type Mappings on 2-Metric Spaces

R. K. Namdeo and Sweetee Dubey

Department of Mathematics
Dr. Hari Singh Gour Vishwavidyalaya
Sagar (M.P), 470003, India
rkn_math@yahoo.com, dubey_sweetee@rediffmail.com

Kenan Tas

Çankaya University
Department of Mathematics and Computer Science
Ankara, Turkey
kenan@cankaya.edu.tr

Abstract

In this paper a unique coincidence value is obtained for a class of self mappings satisfying non-expansive type condition on 2-metric spaces under various conditions and a fixed point theorem is also obtained.

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INTRODUCTION

The concept of 2-metric space has been introduced and studied by Gahler $([2]-[4])$. A number of authors have studied the contractive and contraction type mapping in 2-metric spaces. Recently, S.L.Singh et al. [5] proved a fixed point theorem in 2-metric space for non-expansive type mappings.

Gahler defined 2-metric space as follows:
A 2-metric on a set X with at least three points is a non-negative real-valued mapping d: X x X x $X \to \Re$ satisfying the following properties:
(G_1) To each pair of points a, b with $a \neq b$ in X there is a point $c \in X$ such

that $d(a,b,c) \neq 0$
(G_2) $d(a,b,c) = 0$, if at least two of the points are equal
(G_3) $d(a,b,c) = d(b,c,a) = d(a,c,b)$
(G_4) $d(a,b,c) \leq d(a,b,u) + d(a,u,c) + d(u,b,c)$ for all $a,b,c,u \in X$
The pair (X,d) is called a 2-metric space.

The sequence $\{x_n\}$ is convergent to $x \in X$ and x is the limit of this sequence if $\lim_{n\to\infty} d(x_n, x, u) = 0$ for each $u \in X$.

A sequence $\{x_n\}$ is called Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m, u) = 0$ for all $u \in X$. A 2-metric space in which every Cauchy sequence convergent is called complete.

Let f and g be two self maps of a 2-metric space (X,d).Then, f and g are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n, u) = 0$ for each $u \in X$, whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.

A map $T : X \to X$ is said to be non-expansive if $d(Tx, Ty, u) \leq d(x, y, u)$ for all $x, y, u \in X$. The non-expansive type definition used herein is an extension of that of Ciric [1](see also [6]).
In [5] , the following result is obtained:
Theorem 1. Let (X,d) be a 2-metric space and $T : X \to X$ satisfying the following non-expansive type condition:

$$\begin{aligned} d(Tx,Ty,u) &\leq a\max\{d(x,y,u), d(x,Tx,u), d(y,Ty,u), \\ \tfrac{1}{2} & [d(x,Ty,u) + d(y,Tx,u)]\} + b\max\{d(x,Tx,u), d(y,Ty,u)\} \\ &+ c[d(x,Ty,u) + d(y,Tx,u)] \qquad (1) \end{aligned}$$

for all $x, y, u \in X$ where a, b, c are real numbers such that $a + b + 2c = 1$ and $a \geq 0, b > 0, c > 0$.Then T has a unique fixed point and T is continuous at the fixed point.
We now prove the following result and that our condition (2) includes the above condition (1) of S.L.Singh et al. [5].
Theorem 2. Let (X,d) be a 2-metric space.Let T, f be self maps of X satisfying

$$\begin{aligned} d(Tx,Ty,u) &\leq a(x,y)d(fx,fy,u) \\ &+ b(x,y)\max\{d(fx,Tx,u), d(fy,Ty,u)\} \\ &+ c(x,y)[d(fx,Ty,u) + d(fy,Tx,u)] \qquad (2) \end{aligned}$$

where $a(x,y) \geq 0$, $\beta = \inf b(x,y) > 0$, $\gamma = \inf c(x,y) > 0$ and $\sup[a(x,y) + b(x,y) + 2c(x,y)] = 1, x, y \in X$.
With $T(X) \subseteq f(X)$ and either
(a) X is complete and f is surjective,or,
(b) X is complete,f is continuous and T and f are compatible ,or,
(c) $f(X)$ is complete ,or,
(d) $T(X)$ is complete.
Then f and T have a coincidence point in X.Further,the coincidence value is unique,that is $f_p = f_q$ whenever $f_p = T_p$ and $f_q = T_q$; $p, q \in X$.
Proof. Let $x = x_0$ be an arbitrary point in X.Since $T(X) \subseteq f(X)$ choose x_1 so that $y_1 = fx_1 = Tx_0$. In general , choose x_{n+1} such that $y_{n+1} = fx_{n+1} = Tx_n$ for $n = 0, 1, 2, ...$On applying inequality (2) and taking $a(x_n, x_{n+1}) = a, b(x_n, x_{n+1}) = b, c(x_n, x_{n+1}) = c$ we get

$$\begin{aligned} d(fx_{n+2}, fx_{n+1}, fx_n) &= d(Tx_{n+1}, Tx_n, fx_n) \leq a\ d(fx_n, fx_{n+1}, fx_n) \\ &+ b\ \max\{d(fx_n, Tx_n, fx_n), d(fx_{n+1}, Tx_{n+1}, fx_n)\} \\ &+ c\ [d(fx_n, Tx_{n+1}, fx_n) + d(fx_{n+1}, Tx_n, fx_n)] \\ &= b\ d(fx_{n+1}, Tx_{n+1}, fx_n) \\ &= b\ d(fx_{n+2}, fx_{n+1}, fx_n) \end{aligned}$$

Therefore we obtained $d(fx_{n+2}, fx_{n+1}, fx_n) \leq b\ d(fx_{n+2}, fx_{n+1}, fx_n)$.This implies that $(1-b)\ d(fx_{n+2}, fx_{n+1}, fx_n) \leq 0$. Since $1 - b > 0$ we get,

$$d(fx_{n+2}, fx_{n+1}, fx_n) = 0. \tag{3}$$

On applying inequality (2) again and using (G_4) and (3) , we get,

$$\begin{aligned} d(Tx_n, Tx_{n+1}, u) &\leq a\ d(fx_n, fx_{n+1}, u) \\ &+ b\ \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &+ c\ [d(fx_n, Tx_{n+1}, u) + d(fx_{n+1}, Tx_n, u)] \\ &= a\ d(fx_n, Tx_n, u) \\ &+ b\ \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &+ c\ d(fx_n, Tx_{n+1}, u) \\ &= a\ d(fx_n, Tx_n, u) \\ &+ b\ \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \end{aligned}$$

$$\begin{aligned} &+ \quad c\,[d(fx_n, Tx_{n+1}, Tx_n) + d(fx_n, Tx_n, u) + d(Tx_{n+1}, Tx_n, u)] \\ &= \quad a\,d(fx_n, Tx_n, u) \\ &+ \quad b\ \max\{d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)\} \\ &+ \quad c\,[d(fx_n, Tx_n, u) + d(fx_{n+1}, Tx_{n+1}, u)] \qquad (4) \end{aligned}$$

Suppose that for some n, $d(fx_{n+1}, Tx_{n+1}, u) > d(fx_n, Tx_n, u)$ then we have from inequality (4),

$$\begin{aligned} d(fx_{n+1}, Tx_{n+1}, u) &= d(Tx_n, Tx_{n+1}, u) < a\ d(fx_{n+1}, Tx_{n+1}, u) \\ &+ b\ d(fx_{n+1}, Tx_{n+1}, u) \\ &+ c\ [d(fx_{n+1}, Tx_{n+1}, u) + d(fx_{n+1}, Tx_{n+1}, u)] \\ &= (a + b + 2c)\ d(fx_{n+1}, Tx_{n+1}, u) \\ &\leq d(fx_{n+1}, Tx_{n+1}, u) \end{aligned}$$

a contradiction,hence we must have,
$d(fx_{n+1}, Tx_{n+1}, u) \leq d(fx_n, Tx_n, u)$, equivalently,

$$d(Tx_n, Tx_{n+1}, u) \leq d(Tx_{n-1}, Tx_n, u) \qquad (5)$$

On applying inequality (2) and evaluating a, b, c at (x_{n-1}, x_n),

$$\begin{aligned} d(y_n, y_{n+1}, u) &= d(Tx_{n-1}, Tx_n, u) \leq a\ d(fx_{n-1}, fx_n, u) \\ &+ b\ \max\{d(fx_{n-1}, Tx_{n-1}, u), d(fx_n, Tx_n, u)\} \\ &+ c[d(fx_{n-1}, Tx_n, u) + d(fx_n, Tx_{n-1}, u)] \\ &= a\ d(Tx_{n-2}, Tx_{n-1}, u) + b\ d(Tx_{n-2}, Tx_{n-1}, u) \\ &+ c\ d(Tx_{n-2}, Tx_n, u) \qquad (6) \end{aligned}$$

On applying inequality (2) again, and using (3) , (5) and (G_4), we get

$$\begin{aligned} d(Tx_{n-2}, Tx_n, u) &\leq \bar{a} d(fx_{n-2}, fx_n, u) \\ &+ \bar{b}\ \max\{d(fx_{n-2}, Tx_{n-2}, u), d(fx_n, Tx_n, u)\} \\ &+ \bar{c}\ [d(fx_{n-2}, Tx_n, u) + d(fx_n, Tx_{n-2}, u)] \\ &= \bar{a}\ d(Tx_{n-3}, Tx_{n-1}, u) \\ &+ \bar{b}\ \max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u)\} \\ &+ \bar{c}\ [d(Tx_{n-3}, Tx_n, u) + d(Tx_{n-1}, Tx_{n-2}, u)] \\ &\leq \bar{a}\ [d(Tx_{n-3}, Tx_{n-2}, Tx_{n-1}) + d(Tx_{n-3}, Tx_{n-2}, u) \\ &+ d(Tx_{n-2}, Tx_{n-1}, u)] \end{aligned}$$

$$
\begin{aligned}
&+ \quad \bar{b}\ \max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u)\} \\
&+ \quad \bar{c}\ [d(Tx_{n-3}, Tx_{n-2}, Tx_n) + d(Tx_{n-3}, Tx_{n-2}, u) \\
&+ \quad d(Tx_{n-2}, Tx_n, u) + d(Tx_{n-1}, Tx_{n-2}, u)] \\
&\leq \quad \bar{a}\ [d(Tx_{n-3}, Tx_{n-2}, Tx_{n-1}) + d(Tx_{n-3}, Tx_{n-2}, u) \\
&+ \quad d(Tx_{n-2}, Tx_{n-1}, u)] \\
&+ \quad \bar{b}\ \max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u)\} \\
&+ \quad \bar{c}\ [d(Tx_{n-3}, Tx_{n-2}, Tx_{n-1}) + d(Tx_{n-3}, Tx_{n-1}, Tx_n) \\
&+ \quad d(Tx_{n-2}, Tx_{n-1}, Tx_n) + d(Tx_{n-3}, Tx_{n-2}, u) \\
&+ \quad d(Tx_{n-2}, Tx_{n-1}, Tx_n) \\
&+ \quad d(Tx_n, Tx_{n-1}, u) + d(Tx_{n-2}, Tx_{n-1}, u) \\
&+ \quad d(Tx_{n-1}, Tx_{n-2}, u)] \\
&= \quad \bar{a}\ [d(Tx_{n-3}, Tx_{n-2}, u) + d(Tx_{n-2}, Tx_{n-1}, u)] \\
&+ \quad \bar{b}\ \max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-1}, Tx_n, u)\} \\
&+ \quad \bar{c}\ [d(Tx_{n-3}, Tx_{n-1}, Tx_n) + d(Tx_{n-3}, Tx_{n-2}, u) \\
&+ \quad d(Tx_n, Tx_{n-1}, u) + d(Tx_{n-2}, Tx_{n-1}, u) \\
&+ \quad d(Tx_{n-1}, Tx_{n-2}, u)] \\
&\leq \quad \bar{a}\ [d(Tx_{n-3}, Tx_{n-2}, u) + d(Tx_{n-3}, Tx_{n-2}, u)] \\
&+ \quad \bar{b}\ \max\{d(Tx_{n-3}, Tx_{n-2}, u), d(Tx_{n-3}, Tx_{n-2}, u)\} \\
&+ \quad \bar{c}\ [d(Tx_{n-3}, Tx_{n-1}, Tx_n) + d(Tx_{n-3}, Tx_{n-2}, u) \\
&+ \quad d(Tx_{n-3}, Tx_{n-2}, u) + d(Tx_{n-3}, Tx_{n-2}, u) \\
&+ \quad d(Tx_{n-3}, Tx_{n-2}, u)] \\
&\leq \quad 2\bar{a}\ d(Tx_{n-3}, Tx_{n-2}, u) + \bar{b}\ d(Tx_{n-3}, Tx_{n-2}, u) \\
&+ \quad 4\bar{c}\ d(Tx_{n-3}, Tx_{n-2}, u) \\
&= \quad [2(\bar{a} + \bar{b} + 2\bar{c}) - \bar{b}]\ d(Tx_{n-3}, Tx_{n-2}, u) \\
&\leq \quad (2 - \bar{b})\ d(Tx_{n-3}, Tx_{n-2}, u) \qquad (7)
\end{aligned}
$$

where $\bar{a}, \bar{b}, \bar{c}$ are evaluated at (x_{n-2}, x_n).
At the bottom line of the above inequality, $d(Tx_{n-3}, Tx_{n-1}, Tx_n) = 0$. Because, let $d(Tx_{n-3}, Tx_{n-1}, Tx_n) \neq 0$, then applying (4), we get

$$
\begin{aligned}
d(Tx_{n-3}, Tx_{n-1}, Tx_n) &= d(Tx_{n-1}, Tx_n, Tx_{n-3}) \\
&\leq a\ d(fx_{n-1}, Tx_{n-1}, Tx_{n-3}) \\
&+ b\ \max\{d(fx_{n-1}, Tx_{n-1}, Tx_{n-3}), d(fx_n, Tx_n, Tx_{n-3})\} \\
&+ c\ [d(fx_{n-1}, Tx_{n-1}, Tx_{n-3}) + d(fx_n, Tx_n, Tx_{n-3})] \\
&\leq a\ d(Tx_{n-2}, Tx_{n-1}, Tx_{n-3}) \\
&+ b\ \max\{d(Tx_{n-2}, Tx_{n-1}, Tx_{n-3}), d(Tx_{n-1}, Tx_n, Tx_{n-3})\} \\
&+ c\ \max\{d(Tx_{n-2}, Tx_{n-1}, Tx_{n-3}) + d(Tx_{n-1}, Tx_n, Tx_{n-3})\}
\end{aligned}
$$

$$\begin{aligned} &= (b+c)\ d(Tx_{n-1}, Tx_n, Tx_{n-3}) \\ &< \ d(Tx_{n-1}, Tx_n, Tx_{n-3}). \end{aligned}$$

a contradiction, showing $d(Tx_{n-1}, Tx_n, Tx_{n-3}) = 0$.

On using inequalities (5),(6) and (7), we get

$$\begin{aligned} ad(Tx_{n-1}, Tx_n, u) &= d(y_n, y_{n+1}, u) \\ &\leq a\ d(Tx_{n-2}, Tx_{n-1}, u) + b\ d(Tx_{n-2}, Tx_{n-1}, u) \\ &+ c[(2-\bar{b})\ d(Tx_{n-3}, Tx_{n-2}, u)] \\ &\leq a\ d(Tx_{n-3}, Tx_{n-2}, u) + b\ d(Tx_{n-3}, Tx_{n-2}, u) \\ &+ c(2-\bar{b})\ d(Tx_{n-3}, Tx_{n-2}, u) \\ &= (a+b+2c)\ d(Tx_{n-3}, Tx_{n-2}, u) \\ &- \bar{b}c\ d(Tx_{n-3}, Tx_{n-2}, u) \\ &\leq (1-\bar{b}c)\ d(Tx_{n-3}, Tx_{n-2}, u) \\ &\leq (1-\beta\gamma)\ d(Tx_{n-3}, Tx_{n-2}, u) \\ &\leq (1-\beta\gamma)^{n/2}\ d(y_0, y_1, u) \end{aligned} \tag{8}$$

Hence $\{y_n\}$ is a Cauchy sequence.
For cases (a) and (b) suppose that X is complete. Then Cauchy sequence $\{y_n\}$ will converge to a point p in X.
Case (a): Since f is surjective, then there will exist a point z in X such that $p = fz$.
Then by applying inequality (2) , we obtain

$$\begin{aligned} d(fx, Tz, u) &\leq d(fz, y_{n+1}, u) + d(fx, Tz, y_{n+1}) + d(Tz, y_{n+1}, u) \\ &\leq d(fz, y_{n+1}, u) + d(fx, Tz, y_{n+1}) + d(Tx_n, Tz, u) \\ &\leq d(fz, y_{n+1}, u) + d(fz, Tz, y_{n+1}) \\ &+ a(x,y)d(fx_n, fz, u) + b(x,y)\{d(fx_n, Tx_n, u), d(fz, Tz, u)\} \\ &+ c(x,y)[d(fx_n, Tz, u) + d(fz, Tx_n, u)] \\ &\leq \sup_{x,y\in X}[a(x,y)+c(x,y)]\max\{d(fx_n, fz, u), d(fz, fx_{n+1}, u)\} \\ &+ \sup_{x,y\in X}[b(x,y)+c(x,y)]\max[\max\{d(fx_n, Tx_n, u), d(fz, Tz, u)\}, d(fx_n, Tz, u) \\ &+ d(fz, Tx_n, u)] + d(fz, y_{n+1}, u) + d(fz, Tz, y_{n+1}) \end{aligned}$$

Taking the limit as $n \to \infty$ we have

$$d(fz, Tz, u) \leq \sup_{x,y\in X}(b+c)d(fz, Tz, u) < d(fz, Tz, u)$$

implies that $fz = Tz$.
Case (b): Since f is continuous and f and T are compatible , then $\lim_{n\to\infty} fy_n =$

fp and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_{n+1} = p$ and hence

$$\lim_{n\to\infty} d(fTx_n, Tfx_n, u) = 0 \tag{9}$$

Using (8), we get

$$\begin{aligned} d(fp, Tp, u) &\le d(fp, fy_{n+1}, Tp) + d(fp, fy_{n+1}, u) + d(fy_{n+1}, u, Tp) \\ &\le d(fp, fy_{n+1}, Tp) + d(fp, fy_{n+1}, u) + d(Tp, Tfx_n, u) \\ &\le ad(ffx_n, fp, u) + b\max\{d(ffx_n, Tfx_n, u), d(fp, Tp, u)\} \\ &+ c[d(ffx_n, Tp, u) + d(fp, Tfx_n, u)] \\ &\le \sup_{x,y\in X} a(x,y) d(ffx_n, fp, u) \\ &+ \sup_{x,y\in X} [b(x,y) + c(x,y)] \max[\max\{d(ffx_n, Tfx_n, u) \\ &, d(fp, Tp, u)\}, d(ffx_n, Tp, u) + d(fp, Tfx_n, u)] \end{aligned} \tag{10}$$

Now, we have

$$\begin{aligned} d(ffx_n, Tfx_n, u) &\le d(ffx_n, fTx_n, u) + d(fTx_n, Tfx_n, u) \\ &+ d(ffx_n, fTx_n, Tfx_n) \end{aligned}$$

Using the continuity of f and the compatibility of f and T , it follows that

$$\lim_{n\to\infty} d(ffx_n, Tfx_n, u) = 0, \lim_{n\to\infty} d(ffx_n, fTx_n, u) = 0 \tag{11}$$

$$\lim_{n\to\infty} ffx_n = fp \Rightarrow \lim_{n\to\infty} Tfx_n = fp.$$

Taking the limit as $n \to \infty$ and using the inequality (9) and (10) we get

$$d(fp, Tp, u) \le \sup_{x,y\in X} [b(x,y) + c(x,y)] d(fp, Tp, u)$$

implies $fp = Tp$.
Case (c): In this case $p \in f(X)$. Let $z \in f^{-1}p$ then $p = fz$,and the proof is completed by case (a).
Case (d): In this case $p \in T(X) \subseteq f(X)$ and proof is completed by case (c).
To establish uniqueness, suppose that q is another coincidence point of f and

T.Then from (2) with a, b and c evaluated at (p, q), we have

$$\begin{aligned} d(Tp, Tq, u) &\leq ad(fp, fq, u) + b\max\{d(fp, Tp, u), d(fq, Tq, u)\} \\ &+ c[d(fp, Tq, u) + d(fq, Tp, u)] \end{aligned}$$

and so

$$d(Tp, Tq, u) \leq (a + 2c)d(Tp, Tq, u).$$

Hence $Tp = Tq$.
Now we show that our condition (2) includes the condition (1) of [4].
Define ,

$$M(x, y, u) = \max\{d(x, y, u), d(u, Tx, u), d(y, Ty, u), \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)]\}$$

and $f = I$, the identity mapping on X. For each x, y of X such that

$$M(x, y, u) = d(x, y, u)$$

define

$$a(x, y) = a, b(x, y) = b, c(x, y) = c.$$

For each x, y such that

$$M(x, y, u) = \max\{d(x, Tx, u), d(y, Ty, u)\},$$

define

$$a(x, y) = 0, b(x, y) = a + b, c(x, y) = c.$$

For each x, y such that

$$M(x, y, u) = \frac{1}{2}[d(x, Ty, u) + d(y, Tx, u)]$$

define

$$a(x, y) = 0, b(x, y) = b, c(x, y) = \frac{a}{2} + c.$$

Corollary. Let (X, d) be a complete 2-metric space and T a self map of X satisfying (2) with $f = I$ the identity map on X. Then T has a unique fixed point and at this fixed point T is continuous.

Proof. The existence and uniqueness of the fixed point come from Theorem (2) by setting $f = I$. To prove continuity at the unique fixed point p, we apply inequality (2),where a, b, c are evaluating at (y_n, p).

$$\begin{aligned} d(Ty_n, p, u) = d(Ty_n, Tp, u) &\leq a\ d(y_n, p, u) + b\ \max\{d(y_n, Ty_n, u), d(p, Tp, u)\} \\ &+ c\ [d(y_n, Tp, u) + d(p, Ty_n, u)] \end{aligned}$$

Taking the limit as $n \to \infty$ yields

$$\lim_{n\to\infty} d(Ty_n, p, u) \leq (b + c) \lim_{n\to\infty} d(p, Ty_n, u) \quad < \quad \lim_{n\to\infty} d(p, Ty_n, u)$$

a contradiction, which implies,
$\lim_{n\to\infty} Ty_n = p = Tp$.

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