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Research Article

Common Fixed Point Theorems on Tricomplex Valued Metric Space

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In this paper, we introduce the notion of tricomplex valued metric space and prove some common fixed point theorems. The presented results generalize and expand some of the literature well-known results. We also explore some applications of our key results.

1. Introduction

Fixed point theory plays an important role in applications of many branches of mathematics. There has been a number of generalizations of the usual notion of a metric space (see [1–8] and the references therein). Serge [9] made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, and so on as elements of an infinite set of algebra. Subsequently during the 1930s, other researchers also contributed in this area [10–12]. However, unfortunately the next fifty years failed to witness any advancement in this field. Afterward, Price [13] developed the bicomplex algebra and function theory. Recently renewed interest in this subject finds some significant applications in different fields of mathematical sciences as well as other branches of science and technology. Also, one can see the attempts in [14]. An impressive body of work has been developed by a number of researchers. Among them, an important work on elementary functions of bicomplex numbers has been done by Luna-Elizarrarás et al. [15]. Choi et al. [16] proved some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. Jebril et al. [17] proved some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. In 2021, Beg et al. [18] proved the following fixed point theorems on bicomplex valued metric spaces.

Theorem 1. *Let* $(\mathscr{W}, \varphi)$ *be a complete bicomplex valued metric space with degenerated* $1+\varphi(\varpi,\vartheta)$ *and* $\|1+\varphi(\varpi,\vartheta)\| \neq 0$ *for all* $\varpi, \vartheta \in \mathscr{W}$ *and* $\mathscr{S}, \mathscr{T}: \mathscr{W} \longrightarrow \mathscr{W}$ *such that*

$$\varphi(\mathcal{S}\varpi, \mathcal{T}\vartheta) \prec_{i_2} \lambda\varphi(\varpi, \vartheta) + \frac{\mu\varphi(\varpi, \mathcal{S}\varpi)\varphi(\vartheta, \mathcal{T}\vartheta)}{1 + \varphi(\varpi, \vartheta)}, \quad (1)$$

for all $\varpi, \vartheta \in \mathcal{W}$, *where* λ *and* μ *are nonnegative real numbers with* $\lambda + \sqrt{2}\mu < 1$. *Then,* $\mathcal{S}$ *and* $\mathcal{T}$ *have a unique common fixed point.*

In this paper, inspired by Theorem 1, we prove some common fixed point theorems on tricomplex metric space with applications.

2. Preliminaries

Throughout this paper, we denote the set of real, complex, bicomplex, and tricomplex numbers, respectively, as $\mathbb{C}_0$, $\mathbb{C}_1$, $\mathbb{C}_2$, and $\mathbb{C}_3$. Price [13] defined the bicomplex number as follows:

$$\sigma = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 i_1 i_2, \quad (2)$$

where $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4 \in \mathbb{C}_0$, and independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$; we denote the set of bicomplex numbers $\mathbb{C}_2$ as follows:

$$\mathbb{C}_2 = \{\sigma\colon \sigma = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 i_1 i_2, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4 \in \mathbb{C}_0\}, \quad (3)$$

i.e.,

$$\mathbb{C}_2 = \{\sigma\colon \sigma = \kappa_1 + i_2\kappa_2, \kappa_1, \kappa_2 \in \mathbb{C}_1\}, \quad (4)$$

where $\kappa_1 = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 \in \mathbb{C}_1$ and $\kappa_2 = \mathfrak{a}_3 + \mathfrak{a}_4 i_1 \in \mathbb{C}_1$. Price [13] defined the tricomplex number as follows:

$$\xi = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 j_1 + \mathfrak{a}_5 i_3 + \mathfrak{a}_6 j_2 + \mathfrak{a}_7 j_3 + \mathfrak{a}_8 i_4, \quad (5)$$

where $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8 \in \mathbb{C}_0$ and independent units $i_1, i_2, i_3, i_4, j_1, j_2$, and j_3 are such that $i_1^2 = i_4^2 = -1, i_4 = i_1 j_3 = i_1\ i_2 i_3, j_2 = i_1 i_3 = i_3 i_1, j_2^2 = 1, j_1 = i_1 i_2 = i_2 i_1$, and $j_1^2 = 1$; we denote the set of tricomplex numbers $\mathbb{C}_3$ as follows:

$$\mathbb{C}_3 = \left\{ \begin{array}{c} \xi\colon \xi = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 j_1 + \mathfrak{a}_5 i_3 + \mathfrak{a}_6 j_2 + \mathfrak{a}_7 j_3 + \mathfrak{a}_8 i_4, \\ \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \mathfrak{a}_4, \mathfrak{a}_5, \mathfrak{a}_6, \mathfrak{a}_7, \mathfrak{a}_8 \in \mathbb{C}_0 \end{array} \right\}, \quad (6)$$

i.e.,

$$\mathbb{C}_3 = \{\xi\colon \xi = \sigma_1 + i_3\sigma_2, \sigma_1, \sigma_2 \in \mathbb{C}_2\}, \quad (7)$$

where $\sigma_1 = \kappa_1 + \kappa_2 i_2 \in \mathbb{C}_2$ and $\sigma_2 = \kappa_3 + \kappa_4 i_2 \in \mathbb{C}_2$. If $\xi = \sigma_1 + i_3\sigma_2$ and $\eta = \mathfrak{w}_1 + i_3\mathfrak{w}_2$ be any two tricomplex numbers, then the sum is $\xi \pm \eta = (\sigma_1 + i_3\sigma_2) \pm (\mathfrak{w}_1 + i_3\mathfrak{w}_2) = \sigma_1 \pm \mathfrak{w}_1 + i_3(\sigma_2 \pm \mathfrak{w}_2)$ and the product is $\xi.\eta = (\sigma_1 + i_3\sigma_2)\ (\mathfrak{w}_1 + i_3\mathfrak{w}_2) = (\sigma_1\mathfrak{w}_1 - \sigma_2\mathfrak{w}_2) + i_3(\sigma_1\mathfrak{w}_2 + \sigma_2\mathfrak{w}_1)$.

There are four idempotent elements in $\mathbb{C}_3$; they are $0, 1, \mathfrak{e}_1 = 1 + j_3/2, \mathfrak{e}_2 = 1 - j_3/2$ out of which $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are nontrivial such that $\mathfrak{e}_1 + \mathfrak{e}_2 = 1$ and $\mathfrak{e}_1\mathfrak{e}_2 = 0$. Every tricomplex number $\sigma_1 + i_3\sigma_2$ can be uniquely be expressed as the combination of $\mathfrak{e}_1$ and $\mathfrak{e}_2$, namely,

$$\xi = \sigma_1 + i_3\sigma_2 = (\sigma_1 - i_2\sigma_2)\mathfrak{e}_1 + (\sigma_1 + i_2\sigma_2)\mathfrak{e}_2. \quad (8)$$

This representation of ξ is known as the idempotent representation of tricomplex number, and the complex coefficients $\xi_1 = (\sigma_1 - i_2\sigma_2)$ and $\xi_2 = (\sigma_1 + i_2\sigma_2)$ are known as idempotent components of the bicomplex number ξ.

An element $\xi = \sigma_1 + i_3\sigma_2 \in \mathbb{C}_3$ is said to be invertible if there exists another element η in $\mathbb{C}_3$ such that $\xi\eta = 1$ and η is said to be inverse (multiplicative) of ξ. Consequently, ξ is said to be the inverse (multiplicative) of η. An element which has an inverse in $\mathbb{C}_3$ is said to be the nonsingular element of $\mathbb{C}_3$ and an element which does not have an inverse in $\mathbb{C}_3$ is said to be the singular element of $\mathbb{C}_3$.

An element $\xi = \sigma_1 + i_3\sigma_2 \in \mathbb{C}_3$ is nonsingular if and only if $|\sigma_1^2 + \sigma_2^2| \neq 0$ and singular if and only if $|\sigma_1^2 + \sigma_2^2| = 0$.

The inverse of ξ is defined as

$$\xi^{-1} = \eta = \frac{\sigma_1 - i_3\sigma_2}{\sigma_1^2 + \sigma_2^2}. \quad (9)$$

The norm $\|.\|$ of $\mathbb{C}_3$ is a positive real valued function and $\|.\|\colon \mathbb{C}_3 \longrightarrow \mathbb{C}_0^+$ is defined by

$$\begin{aligned} \|\xi\| &= \|\sigma_1 + i_3\sigma_2\| = \left\{|\sigma_1|^2 + |\sigma_2|^2\right\}^{1/2} \\ &= \left[\frac{|(\sigma_1 - i_2\sigma_2)|^2 + |(\sigma_1 + i_2\sigma_2)|^2}{2}\right]^{1/2} \\ &= \left(\mathfrak{a}_1^2 + \mathfrak{a}_2^2 + \mathfrak{a}_1^2 + \mathfrak{a}_3^2 + \mathfrak{a}_4^2 + \mathfrak{a}_5^2 + \mathfrak{a}_6^2 + \mathfrak{a}_7^2 + \mathfrak{a}_8^2\right)^{1/2}, \end{aligned} \quad (10)$$

where $\xi = \mathfrak{a}_1 + \mathfrak{a}_2 i_1 + \mathfrak{a}_3 i_2 + \mathfrak{a}_4 j_1 + \mathfrak{a}_5 i_3 + \mathfrak{a}_6 j_2 + \mathfrak{a}_7 j_3 + \mathfrak{a}_8 i_4 = \sigma_1 + i_3\sigma_2 \in \mathbb{C}_3$.

The linear space $\mathbb{C}_3$ with respect to defined norm is a norm linear space; also $\mathbb{C}_3$ is complete; therefore, $\mathbb{C}_3$ is the Banach space. If $\xi, \eta \in \mathbb{C}_3$, then $\|\xi\eta\| \leq 2\|\xi\|\|\eta\|$ holds instead of $\|\xi\eta\|\| \leq \|\|\xi\|\|\eta\|$; therefore, $\mathbb{C}_3$ is not the Banach algebra. The partial order relation $\prec_{i_3}$ on $\mathbb{C}_3$ is defined as follows: let $\mathbb{C}_3$ be the set of tricomplex numbers and $\xi = \sigma_1 + i_3\sigma_2$ and $\eta = \mathfrak{w}_1 + i_3\mathfrak{w}_2 \in \mathbb{C}_3$, then $\xi \prec_{i_3} \eta$ if and only if $\sigma_1 \prec_{i_2} \mathfrak{w}_1$ and $\sigma_2 \prec_{i_2} \mathfrak{w}_2$, i.e., $\xi \prec_{i_3} \eta$ if one of the following conditions is fulfilled:

(a) $\sigma_1 = \mathfrak{w}_1$, $\sigma_2 = \mathfrak{w}_2$
(b) $\sigma_1 \prec_{i_2} \mathfrak{w}_1$, $\sigma_2 = \mathfrak{w}_2$
(c) $\sigma_1 = \mathfrak{w}_1$, $\sigma_2 \prec_{i_2} \mathfrak{w}_2$
(d) $\sigma_1 \prec_{i_2} \mathfrak{w}_1$, $\sigma_2 \prec_{i_2} \mathfrak{w}_2$

In particular, we can write $\xi \lneqq_{i_3} \eta$ if $\xi \prec_{i_3} \eta$ and $\xi \neq \eta$, i.e., one of (b), (c), and (d) is fulfilled and we will write $\xi \prec_{i_3} \eta$ if only (d) is fulfilled.

For any two tricomplex numbers $\xi, \eta \in \mathbb{C}_3$, we can verify the following:

(1) $\xi \prec_{i_3} \eta iff \|\xi\|\| \leq \|\|\eta\|$
(2) $\|\xi + \eta\|\| \leq \|\|\xi\| + \|\eta\|$
(3) $\|\mathfrak{a}\xi\| = |\mathfrak{a}|\|\xi\|$, where $\mathfrak{a}$ is a nonnegative real number
(4) $\|\xi\eta\|\| \leq 2\|\|\xi\|\|\eta\|$ and the equality holds only when at least one of ξ and η is nonsingular

(5) $\|\xi^{-1}\|\| = \|\|\xi\|^{-1}$ if ξ is a nonsingular

(6) $\|\xi\|/\|\eta\| = \|\xi\|/\|\eta\|$ if η is a nonsingular

Now, let us recall some basic concepts and notations, which will be used in the sequel.

Definition 1. Let $\mathscr{W}$ be a nonempty set and $\varphi: \mathscr{W} \times \mathscr{W} \longrightarrow \mathbb{C}_3$ such that

(A1) $0 \prec_{i_3} \varphi(\varpi, \vartheta)$, for all $\varpi, \vartheta \in \mathscr{W}$ and $\varphi(\varpi, \vartheta) = 0$ if and only if $\varpi = \vartheta$

(A2) $\varphi(\varpi, \vartheta) = \varphi(\vartheta, \varpi)$ for all $\varpi, \vartheta \in \mathscr{W}$

(A3) $\varphi(\varpi, \vartheta) \prec_{i_3} \varphi(\varpi, \sigma) + \varphi(\sigma, \vartheta)$ for all $\varpi, \vartheta, \sigma \in \mathscr{W}$

Then, φ is called the tricomplex valued metric on $\mathscr{W}$ and $(\mathscr{W}, \varphi)$ is called the tricomplex valued metric space.

Example 1. Let $\mathscr{W} = [0, 1]$ and $\varphi: \mathscr{W} \times \mathscr{W} \longrightarrow \mathbb{C}_3$ be defined by $\varphi(\varpi, \vartheta) = |\varpi - \vartheta| i_2 i_3$. Then, $(\mathscr{W}, \varphi)$ is a tricomplex valued metric space.

Definition 2. Let $(\mathscr{W}, \varphi)$ be a tricomplex valued metric space. A sequence $\{\varpi_{\mathfrak{k}}\}$ in $\mathscr{W}$ is said to be a convergent and converges to $\varpi \in \mathscr{W}$ if for every $0 \prec_{i_3} \epsilon \in \mathbb{C}_3$, there exists $\mathfrak{k}_0 \in \mathbb{N}$ such that $\varphi(\varpi_{\mathfrak{k}}, \varpi) \prec_{i_3} \epsilon$, for all $\mathfrak{k} \geq \mathfrak{k}_0$, and it is denoted by $\lim_{\mathfrak{k} \longrightarrow \infty} \varpi_{\mathfrak{k}} = \varpi$.

Lemma 1. *Let $(\mathscr{W}, \varphi)$ be a tricomplex valued metric space. A sequence $\{\varpi_{\mathfrak{k}}\} \in \mathscr{W}$ converges to $\varpi \in \mathscr{W}$ iff $\lim_{\mathfrak{k} \longrightarrow \infty} \|\varphi(\varpi_{\mathfrak{k}}, \varpi)\| = 0$.*

Proof. Let $\{\varpi_{\mathfrak{k}}\}$ be a convergent sequence and converges to a point ϖ, and let $\epsilon > 0$ be any real number. Suppose

$$\mathfrak{r} = \frac{\varepsilon}{\sqrt{8}} + i_1 \frac{\varepsilon}{\sqrt{8}} + i_2 \frac{\varepsilon}{\sqrt{8}} + j_1 \frac{\varepsilon}{\sqrt{8}} + i_3 \frac{\varepsilon}{\sqrt{8}} + j_2 \frac{\varepsilon}{\sqrt{8}} + j_3 \frac{\varepsilon}{\sqrt{8}} + i_4 \frac{\varepsilon}{\sqrt{8}}. \tag{11}$$

Then, $0 \prec_{i_3} \mathfrak{r} \in \mathbb{C}_3$, and for this $\mathfrak{r}$, there exists $\mathfrak{k}_0 \in \mathbb{N}$ such that $\varphi(\varpi_{\mathfrak{k}}, \varpi) \prec_{i_3} \mathfrak{r}$ for all $\mathfrak{k} \geq \mathfrak{k}_0$. Therefore,

$$\left\|\varphi(\varpi_{\mathfrak{k}}, \varpi) < \right\| \|\mathfrak{r}\| = \varepsilon, \forall \mathfrak{k} \geq \mathfrak{k}_0. \tag{12}$$

Hence, $\lim_{\mathfrak{k} \longrightarrow \infty} \|\varphi(\varpi_{\mathfrak{k}}, \varpi)\| = 0$.

Conversely, let $\lim_{\mathfrak{k} \longrightarrow \infty} \|\varphi(\varpi_{\mathfrak{k}}, \varpi)\| = 0$. Then, for each $0 \prec_{i_3} \mathfrak{r} \in \mathbb{C}_3$, there exists a real number $\epsilon > 0$ such that for all $\xi \in \mathbb{C}_3$,

$$\|\xi\| < \varepsilon \Rightarrow \xi \prec_{i_3} \mathfrak{r}. \tag{13}$$

Then, for this $\epsilon > 0$, there exists $\mathfrak{k}_0 \in \mathbb{N}$ such that

$$\left\|\varphi(\varpi_{\mathfrak{k}}, \varpi)\right\| < \epsilon, \forall \mathfrak{k} \geq \mathfrak{k}_0. \tag{14}$$

Therefore,

$$\varphi(\varpi_{\mathfrak{k}}, \varpi) \prec_{i_3} \mathfrak{r}, \forall \mathfrak{k} \geq \mathfrak{k}_0. \tag{15}$$

Hence, $\{\varpi_{\mathfrak{k}}\}$ converges to a point ϖ. □

Definition 3. Let $(\mathscr{W}, \varphi)$ be a tricomplex valued metric space. A sequence $\{\varpi_{\mathfrak{k}}\}$ in $\mathscr{W}$ is said to be a Cauchy sequence in $(\mathscr{W}, \varphi)$ if for any $0 \prec_{i_3} \epsilon \in \mathbb{C}_3$, there exists $\mathfrak{h} \in \mathbb{N}$ such that $\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+\mathfrak{m}}) \prec_{i_3} \varepsilon$ for all $\mathfrak{k}, \mathfrak{m} \in \mathbb{N}$ and $\mathfrak{k}, \mathfrak{m} \geq \mathfrak{h}$.

Definition 4. Let $(\mathscr{W}, \varphi)$ be a tricomplex valued metric space. Let $\{\varpi_{\mathfrak{k}}\}$ be any sequence in $\mathscr{W}$. Then, if every Cauchy sequence in $\mathscr{W}$ is convergent in $\mathscr{W}$, then $(\mathscr{W}, \varphi)$ is said to be a complete tricomplex valued metric space.

Lemma 2. *Let $(\mathscr{W}, \varphi)$ be a tricomplex valued metric space and $\{\varpi_{\mathfrak{k}}\}$ be a sequence in $\mathscr{W}$. Then, $\{\varpi_{\mathfrak{k}}\}$ is a Cauchy sequence in $\mathscr{W}$ iff $\lim_{\mathfrak{k} \longrightarrow \infty} \|\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+\mathfrak{m}})\| = 0$.*

Proof. Let $\{\varpi_{\mathfrak{k}}\}$ is a Cauchy sequence in $\mathscr{W}$. Let $\epsilon > 0$ be any real number. Suppose

$$\mathfrak{r} = \frac{\varepsilon}{\sqrt{8}} + i_1 \frac{\varepsilon}{\sqrt{8}} + i_2 \frac{\varepsilon}{\sqrt{8}} + j_1 \frac{\varepsilon}{\sqrt{8}} + i_3 \frac{\varepsilon}{\sqrt{8}} + j_2 \frac{\varepsilon}{\sqrt{8}} + j_3 \frac{\varepsilon}{\sqrt{8}} + i_4 \frac{\varepsilon}{\sqrt{8}}. \tag{16}$$

Then, $0 \prec_{i_3} \mathfrak{r} \in \mathbb{C}_2^+$, and for this $\mathfrak{r}$, there exists $\mathfrak{k}_0 \in \mathbb{N}$ such that $\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+\mathfrak{m}}) \prec_{i_3} \mathfrak{r}$, for all $\mathfrak{k} > \mathfrak{k}_0$. Therefore,

$$\left\|\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+\mathfrak{m}})\right\| < \|\mathfrak{r}\| = \varepsilon, \quad \text{for all } \mathfrak{k} > \mathfrak{k}_0. \tag{17}$$

And, this implies that

$$\lim_{\mathfrak{k} \longrightarrow \infty} \left\|\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+\mathfrak{m}})\right\| = 0. \tag{18}$$

Conversely, let $\lim_{\mathfrak{k} \longrightarrow \infty} \|\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+\mathfrak{m}})\| = 0$. Then, for each $0 \prec_{i_3} \mathfrak{r} \in \mathbb{C}_3^+$, there exists a real number $\epsilon > 0$ such that for all $\xi \in \mathbb{C}_2^+$,

$$\|\xi\| < \varepsilon \Rightarrow \xi \prec_{i_3} \mathfrak{r}. \tag{19}$$

Then, for this $\epsilon > 0$, there exists a natural number $\mathfrak{k}_0 \in \mathbb{N}$ such that

$$\left\|\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+\mathfrak{m}})\right\| < \varepsilon, \forall \mathfrak{k} > \mathfrak{k}_0. \tag{20}$$

Therefore,

$$\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+\mathfrak{m}}) \prec_{i_3} \mathfrak{r}, \forall \mathfrak{k} > \mathfrak{k}_0. \tag{21}$$

Hence, $\{\varpi_{\mathfrak{k}}\}$ is a Cauchy sequence. □

Definition 5. Let $\mathscr{S}$ and $\mathscr{T}$ be self mappings of nonvoid set $\mathscr{W}$. A point $\varpi \in \mathscr{W}$ is called a common fixed point of $\mathscr{S}$ and $\mathscr{T}$ if $\varpi = \mathscr{S}\varpi = \mathscr{T}\varpi$.

3. Main Result

In this section, we prove common fixed point theorem in a tricomplex valued metric space using rational type contraction condition.

Theorem 2. *If $\mathscr{S}$ and $\mathscr{T}$ are self mapping defined on a complete tricomplex valued metric space $(\mathscr{W}, \varphi)$ such that*

$$\varphi(\mathcal{S}\varpi, \mathcal{T}\vartheta) \prec_{i_3} \lambda\varphi(\varpi, \vartheta) + \frac{\mu\varphi(\varpi, \mathcal{S}\varpi)\varphi(\vartheta, \mathcal{T}\vartheta) + \gamma\varphi(\vartheta, \mathcal{S}\varpi)\varphi(\varpi, \mathcal{T}\vartheta)}{1 + \varphi(\varpi, \vartheta)}, \tag{22}$$

for all $\varpi, \vartheta \in \mathcal{W}$ where λ, μ, γ are nonnegative reals with $\lambda + 2\mu + 2\gamma < 1$, then $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. Let ϖ_0 be an arbitrary point in $\mathcal{W}$ and define $\varpi_{2\ell+1} = \mathcal{S}\varpi_{2\ell}$, $\varpi_{2\ell+2} = \mathcal{T}\varpi_{2\ell+1}$, $\mathfrak{k} = 0, 1, 2, \cdots$. Then,

$$\begin{aligned}\varphi(\varpi_{2\ell+1}, \varpi_{2\ell+2}) &= \varphi(\mathcal{S}\varpi_{2\ell}, \mathcal{T}\varpi_{2\ell+1}) \prec_{i_3} \lambda\varphi(\varpi_{2\ell}, \varpi_{2\ell+1}) \\ &\quad + \frac{\mu\varphi(\varpi_{2\ell}, \mathcal{S}\varpi_{2\ell})\varphi(\varpi_{2\ell+1}, \mathcal{T}\varpi_{2\ell+1}) + \gamma\varphi(\varpi_{2\ell}, \mathcal{T}\varpi_{2\ell+1})\varphi(\varpi_{2\ell+1}, \mathcal{S}\varpi_{2\ell})}{1 + \varphi(\varpi_{2\ell}, \varpi_{2\ell+1})}.\end{aligned} \tag{23}$$

Since $\varpi_{2\ell+1} = \mathcal{S}\varpi_{2\ell}$ implies $\varphi(\varpi_{2\ell+1}, \mathcal{S}\varpi_{2\ell}) = 0$, therefore,

$$\begin{aligned}\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2}) &\prec_{i_3} \lambda\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1}) \\ &\quad + \frac{\mu\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})}{1 + \varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})},\end{aligned} \tag{24}$$

which implies that

$$\begin{aligned}\left\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\right\| &\le \lambda\left\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\right\| \\ &\quad + \frac{2\mu\left\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\right\|\left\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\right\|}{\left\|1 + \varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\right\|}.\end{aligned} \tag{25}$$

Since $\|1 + \varphi(\varpi_{2\ell}, \varpi_{2\ell+1})\| \ge \|\varphi(\varpi_{2\ell}, \varpi_{2\ell+1})\|$, therefore,

$$\left\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\right\| \le \lambda\left\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\right\| + 2\mu\left\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\right\|, \tag{26}$$

so that

$$\left\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\right\| \le \frac{\lambda}{1 - 2\mu}\left\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\right\|. \tag{27}$$

Also,

$$\begin{aligned}\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+3}) &= \varphi\left(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \int \varpi_{2\mathfrak{k}+2}\right) \\ &= \varphi(\mathcal{S}\varpi_{2\mathfrak{k}+2}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) \prec_{i_3} \lambda\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1}) + \frac{\mu\varphi(\varpi_{2\mathfrak{k}+2}, \mathcal{S}\varpi_{2\mathfrak{k}+2})\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) + \gamma\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{S}\varpi_{2\mathfrak{k}+2})\varphi(\varpi_{2\mathfrak{k}+2}, \mathcal{T}\varpi_{2\mathfrak{k}+1})}{1 + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})}.\end{aligned} \tag{28}$$

Since $\varpi_{2\ell+2} = \mathcal{T}\varpi_{2\ell+1}$ implies $\varphi(\varpi_{2\mathfrak{k}+2}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) = 0$, therefore,

$$\begin{aligned}\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+3}) &\prec_{i_3} \lambda\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1}) \\ &\quad + \frac{\mu\varphi(\varpi_{2\mathfrak{k}+2}, \mathcal{S}\varpi_{2\mathfrak{k}+2})\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{T}\varpi_{2\mathfrak{k}+1})}{1 + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})},\end{aligned} \tag{29}$$

which implies that

$$\begin{aligned}\left\|\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+3})\right\| &\le \lambda\left\|\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})\right\| \\ &\quad + \frac{2\mu\left\|\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+3})\right\|\left\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\right\|}{\left\|1 + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})\right\|}.\end{aligned} \tag{30}$$

As $\|1 + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})\| \ge \|\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})\|$, therefore,

$$\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+3}) \le \frac{\lambda}{1 - 2\mu}\left\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\right\|. \tag{31}$$

Putting $\mathfrak{h} = \lambda/1 - 2\mu$, we have (for all $\mathfrak{k}$)

$$\begin{aligned}\left\|\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+1})\right\| &\le \mathfrak{h}\left\|\varphi(\varpi_{\mathfrak{k}-1}, \varpi_{\mathfrak{k}})\right\| \le \mathfrak{h}^2\left\|\varphi(\varpi_{\mathfrak{k}-2}, \varpi_{\mathfrak{k}-1})\right\| \\ &\le \ldots \le \mathfrak{h}^{\mathfrak{k}}\left\|\varphi(\varpi_0, \varpi_1)\right\|.\end{aligned} \tag{32}$$

Therefore, for any $\mathfrak{m} > \mathfrak{k}$, we have

$$\begin{aligned}\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{m}}) &\le \left\|\varphi(\varpi_{\mathfrak{k}}, \varpi_{\mathfrak{k}+1})\right\| + \left\|\varphi(\varpi_{\mathfrak{k}+1}, \varpi_{\mathfrak{k}+2})\right\| \\ &\quad + \left\|\varphi(\varpi_{\mathfrak{k}+2}, \varpi_{\mathfrak{k}+3})\right\| \cdots + \left\|\varphi(\varpi_{\mathfrak{m}-1}, \varpi_{\mathfrak{m}})\right\| \\ &\le \left[\mathfrak{h}^{\mathfrak{k}} + \mathfrak{h}^{\mathfrak{k}+1} + \mathfrak{h}^{\mathfrak{k}+2} + \cdots + \mathfrak{h}^{\mathfrak{m}-1}\right]\left\|\varphi(\varpi_0, \varpi_1)\right\| \\ &\le \left[\frac{\mathfrak{h}^{\mathfrak{k}}}{1 - \mathfrak{h}}\right]\left\|\varphi(\varpi_0, \varpi_1)\right\|,\end{aligned} \tag{33}$$

which implies that

$$\left\|\varphi\left(\varpi_{\mathfrak{k}},\varpi_{\mathfrak{m}}\right)\right\|\leq\left[\frac{\mathfrak{h}^{\mathfrak{k}}}{1-\mathfrak{h}}\right]\left\|\varphi\left(\varpi_0,\varpi_1\right)\right\|\longrightarrow 0 \text{ as } \mathfrak{k}\longrightarrow\infty. \tag{34}$$

In view of Lemma 2, the sequence $\{\varpi_{\mathfrak{k}}\}$ is Cauchy. Since $\mathcal{W}$ is complete, there exists some $\mathfrak{l}\in\mathcal{W}$ such that $\varpi_{\mathfrak{k}}\longrightarrow\mathfrak{l}$ as $\mathfrak{k}\longrightarrow\infty$. On the contrary, let $\mathfrak{l}\neq\mathcal{S}\mathfrak{k}$ so that $0\prec_{i_3}\sigma=\varphi(\mathfrak{l},\mathcal{S}\mathfrak{k})$, then

$$\begin{aligned}\sigma=\varphi(\mathfrak{l},\mathcal{S}\mathfrak{k})&\prec_{i_3}\varphi\left(\mathfrak{l},\mathcal{T}\varpi_{2\mathfrak{k}+1}\right)+\varphi\left(\mathcal{T}\varpi_{2\mathfrak{k}+1},\mathcal{S}\mathfrak{k}\right)\prec_{i_3}\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+2}\right)\\&+\lambda\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+1}\right)+\frac{\mu\varphi(\mathfrak{l},\mathcal{S}\mathfrak{k})\varphi\left(\varpi_{2\mathfrak{k}+1},\mathcal{T}\varpi_{2\mathfrak{k}+1}\right)+\gamma\varphi\left(\varpi_{2\mathfrak{k}+1},\mathcal{S}\mathfrak{k}\right)\varphi\left(\mathfrak{l},\mathcal{T}\varpi_{2\mathfrak{k}+1}\right)}{1+\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+1}\right)}\prec_{i_3}\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+2}\right)\\&+\lambda\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+1}\right)+\frac{\mu\sigma\varphi\left(\varpi_{2\mathfrak{k}+1},\mathcal{T}\varpi_{2\mathfrak{k}+1}\right)+\gamma\varphi\left(\varpi_{2\mathfrak{k}+1},\mathcal{S}\mathfrak{k}\right)\varphi\left(\mathfrak{l},\mathcal{T}\varpi_{2\mathfrak{k}+1}\right)}{1+\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+1}\right)}.\end{aligned}\tag{35}$$

Also, for all $\mathfrak{k}$, we have

$$\begin{aligned}\varphi(\mathfrak{l},\mathcal{S}\mathfrak{k})\leq&\left\|\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+2}\right)\right\|+\lambda\left\|\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+1}\right)\right\|\\&+\frac{\mu\sigma\left\|\varphi\left(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2}\right)\right\|+2\gamma\left\|\varphi\left(\varpi_{2\mathfrak{k}+1},\mathcal{S}\mathfrak{k}\right)\right\|\left\|\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+2}\right)\right\|}{\left\|1+\varphi\left(\mathfrak{l},\varpi_{2\mathfrak{k}+1}\right)\right\|}.\end{aligned}\tag{36}$$

As $\mathfrak{k}\longrightarrow\infty$, we get

$$\|\varphi(\mathfrak{l},\mathcal{S}\mathfrak{k})\|=0. \tag{37}$$

Therefore, $\mathfrak{l}=\mathcal{S}\mathfrak{k}$. Similarly, we can derive that $\mathfrak{l}=\mathcal{T}\mathfrak{l}$. Let $\mathfrak{l}^*$ (in $\mathcal{W}$) be another common fixed point of $\mathcal{S}$ and $\mathcal{T}$, i.e., $\mathfrak{l}^*=\mathcal{S}\mathfrak{l}^*=\mathcal{T}\mathfrak{l}^*$. Then,

$$\begin{aligned}\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)=\varphi\left(\mathcal{S}\mathfrak{k},\mathcal{T}\mathfrak{l}^*\right)&\prec_{i_3}\lambda\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)+\frac{\mu\varphi(\mathfrak{l},\mathcal{S}\mathfrak{k})\varphi\left(\mathfrak{l}^*,\mathcal{T}\mathfrak{l}^*\right)+\gamma\varphi\left(\mathfrak{l}^*,\mathcal{S}\mathfrak{k}\right)\varphi\left(\mathfrak{l},\mathcal{T}\mathfrak{l}^*\right)}{1+\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)}\\&=\lambda\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)+\frac{\gamma\varphi\left(\mathfrak{l}^*,\mathfrak{l}\right)\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)}{1+\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)},\end{aligned}\tag{38}$$

which implies that

$$\left\|\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)\right\|\leq\lambda\left\|\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)\right\|+\frac{2\gamma\left\|\varphi\left(\mathfrak{l}^*,\mathfrak{l}\right)\right\|\left\|\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)\right\|}{\left\|1+\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)\right\|}. \tag{39}$$

Since $\|1+\varphi(\mathfrak{l},\mathfrak{l}^*)\|>\|\varphi(\mathfrak{l},\mathfrak{l}^*)\|$, therefore,

$$\left\|\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)\right\|\leq(\lambda+2\gamma)\left\|\varphi\left(\mathfrak{l},\mathfrak{l}^*\right)\right\|. \tag{40}$$

Therefore, $\mathfrak{l}=\mathfrak{l}^*$ (as $\lambda+2\gamma<1$). □

Remark 1 (see [13]). We have

$$\mathbb{C}_0\subseteq\mathbb{C}_1\subseteq\mathbb{C}_2\subseteq\mathbb{C}_3. \tag{41}$$

Example 2. Considering $\mathcal{W}=\mathbb{R}$, define a mapping $\varphi\colon\mathcal{W}\times\mathcal{W}\longrightarrow\mathbb{C}_3$ by $\varphi(\varpi,\vartheta)=i_2i_3|\varpi-\vartheta|$, for all $\varpi,\vartheta\in\mathcal{W}$, where $|.|$ is the usual real modulus. Then, $(\mathcal{W},\varphi)$ is a complete tricomplex valued metric space. Every real number is a tricomplex number but every tricomplex number is not necessarily a real number. Therefore, $(\mathcal{W},\varphi)$ is not a metric space. Now, we consider a self mapping $\mathcal{S},\mathcal{T}\colon\mathcal{W}\longrightarrow\mathcal{W}$ defined by

$$\begin{aligned}\mathcal{S}(\varpi)&=\frac{\varpi}{4},\\\mathcal{T}(\vartheta)&=\frac{\vartheta}{2},\end{aligned}\tag{42}$$

for all $\varpi,\vartheta\in\mathcal{W}$. Then,

$$\begin{aligned}\varphi(\mathcal{S}\varpi,\mathcal{T}\vartheta)&=\varphi\left(\frac{\varpi}{4},\frac{\vartheta}{2}\right)\\&=i_2i_3\left|\frac{\varpi}{4}-\frac{\vartheta}{2}\right|\\&=\frac{i_2i_3}{2}\left|\frac{\varpi}{2}-\vartheta\right|\prec_{i_3}\frac{i_2i_3}{2}|\varpi-\vartheta|\\&=\frac{1}{2}\varphi(\varpi,\vartheta).\end{aligned}\tag{43}$$

Thus, all the hypothesis of Theorem 2 are fulfilled with $\lambda=(1/2)<1$ and $\mu=\gamma=0$. Hence, $\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Example 3. Considering $\mathcal{W}=\mathbb{C}_3$, define a mapping $\varphi\colon\mathcal{W}\times\mathcal{W}\longrightarrow\mathbb{C}_3$ by $\varphi(\varpi,\vartheta)=(1+i_1i_3)|\varpi-\vartheta|$, for all $\varpi,\vartheta\in\mathcal{W}$. Then, $(\mathcal{W},\varphi)$ is a complete tricomplex valued metric space. Every real number is a tricomplex number but every tricomplex number is not necessarily a real number. Therefore, $(\mathscr{W}, \varphi)$ is not a metric space. Now, we consider a self mappings $\mathscr{S}, \mathscr{T}: \mathscr{W} \longrightarrow \mathscr{W}$ defined by

$$\begin{aligned} \mathscr{S}(\varpi) &= \frac{i_1 i_3 \varpi}{4}, \\ \mathscr{T}(\vartheta) &= \frac{i_1 i_3 \vartheta}{2}, \end{aligned} \tag{44}$$

for all $\varpi, \vartheta \in \mathscr{W}$. Then,

$$\begin{aligned} \varphi(\mathscr{S}\varpi, \mathscr{T}\vartheta) &= \varphi\left(\frac{i_1 i_3 \varpi}{4}, \frac{i_1 i_3 \vartheta}{2}\right) \\ &= (1 + i_1 i_3)\left|\frac{i_1 i_3 \varpi}{4} - \frac{i_1 i_3 \vartheta}{2}\right| \\ &= \frac{(1 + i_1 i_3)}{2}\left|\frac{\varpi}{2} - \vartheta\right| \prec_{i_3} \frac{(1 + i_1 i_3)}{2}|\varpi - \vartheta| \\ &= \frac{1}{2}\varphi(\varpi, \vartheta). \end{aligned} \tag{45}$$

Every real number is a tricomplex number but every tricomplex number is not necessarily a real number. Therefore, we cannot find common fixed point for such mappings on metric space. Thus, all the hypothesis of Theorem 2 are fulfilled with $\lambda = (1/2) < 1$ and $\mu = \gamma = 0$. Hence, $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point.

By setting $\mathscr{S} = \mathscr{T}$ in Theorem 2, one deduces the following.

Corollary 1. *If $\mathscr{T}: \mathscr{W} \longrightarrow \mathscr{W}$ is a self mapping defined on a complete tricomplex valued metric space $(\mathscr{W}, \varphi)$ such that*

$$\varphi(\mathscr{T}\varpi, \mathscr{T}\vartheta) \prec_{i_3} \lambda\varphi(\varpi, \vartheta) + \frac{\mu\varphi(\varpi, \mathscr{T}\varpi)\varphi(\vartheta, \mathscr{T}\vartheta) + \gamma\varphi(\vartheta, \mathscr{T}\varpi)\varphi(\varpi, \mathscr{T}\vartheta)}{1 + \varphi(\varpi, \vartheta)}, \tag{46}$$

for all $\varpi, \vartheta \in \mathscr{W}$, where λ, μ, γ are nonnegative reals with $\lambda + 2\mu + 2\gamma < 1$, then $\mathscr{T}$ has a unique fixed point.

Theorem 3. *Let $(\mathscr{W}, \varphi)$ be a complete tricomplex valued metric space and the mappings $\mathscr{S}, \mathscr{T}: \mathscr{W} \longrightarrow \mathscr{W}$ such that*

$$\varphi(\mathscr{S}\varpi, \mathscr{T}\vartheta) \prec_{i_3} \begin{cases} \lambda\varphi(\varpi, \vartheta) + \mu\dfrac{\varphi(\varpi, \mathscr{S}\varpi)\varphi(\vartheta, \mathscr{T}\vartheta) + \varphi(\vartheta, \mathscr{S}\varpi)\varphi(\varpi, \mathscr{T}\vartheta)}{\varphi(\mathscr{S}\varpi, \varpi) + \varphi(\mathscr{T}\vartheta, \vartheta)}, \\ +\gamma\dfrac{\varphi(\varpi, \mathscr{S}\varpi)\varphi(\varpi, \mathscr{T}\vartheta) + \varphi(\vartheta, \mathscr{S}\varpi)\varphi(\vartheta, \mathscr{T}\vartheta)}{\varphi(\mathscr{S}\varpi, \vartheta) + \varphi(\mathscr{T}\vartheta, \varpi)}, \\ \text{if} \quad \mathscr{D} \neq 0,\ \mathscr{D}_1 \neq 0, \\ 0, \\ \text{if} \quad \mathscr{D} = 0 \text{ or } \mathscr{D}_1 = 0, \end{cases} \tag{47}$$

for all $\varpi, \vartheta \in \mathscr{W}$, where $\mathscr{D} = \varphi(\mathscr{S}\varpi, \varpi) + \varphi(\mathscr{T}\vartheta, \vartheta)$ and $\mathscr{D}_1 = \varphi(\mathscr{S}\varpi, \vartheta) + \varphi(\mathscr{T}\vartheta, \varpi)$ and λ, μ, γ are nonnegative reals with $\lambda + 2\mu + \gamma < 1$. Then, $\mathscr{S}$, $\mathscr{T}$ have a unique common fixed point.

Proof. Let ϖ_0 be an arbitrary point in $\mathscr{W}$. Define $\varpi_{2\ell+1} = \mathscr{S}\varpi_{2\ell}$ and $\varpi_{2\ell+2} = \mathscr{T}\varpi_{2\ell+1}$, $\mathfrak{k} = 0, 1, 2, \ldots$ Now, we distinguish two cases. First, if (for $\mathfrak{k} = 0, 1, 2, \ldots$) $\varphi(\mathscr{S}\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}}) +$

$\varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+1}) \neq 0$ and $\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}}) \neq 0$, then

$$\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2}) = \varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) \prec_{i_3} \lambda\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1}) + \mu\frac{\varphi(\varpi_{2\mathfrak{k}}, \mathcal{S}\varpi_{2\mathfrak{k}})\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) + \varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{S}\varpi_{2\mathfrak{k}})\varphi(\varpi_{2\mathfrak{k}}, \mathcal{T}\varpi_{2\mathfrak{k}+1})}{\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+1})} + \gamma\frac{\varphi(\varpi_{2\mathfrak{k}}, \mathcal{S}\varpi_{2\mathfrak{k}})\varphi(\varpi_{2\mathfrak{k}}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) + \varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{S}\varpi_{2\mathfrak{k}})\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{T}\varpi_{2\mathfrak{k}+1})}{\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}})}. \tag{48}$$

Since $\varpi_{2\ell+1} = \mathcal{S}\varpi_{2\ell}$ and $\varpi_{2\ell+2} = \mathcal{T}\varpi_{2\ell+1}$, therefore,

$$\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2}) \prec_{i_3} \lambda\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1}) + \gamma\frac{\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+2}) + \varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+1})\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})}{\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+1}) + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}})}, \tag{49}$$

$$\text{or } \varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2}) \prec_{i_3} \lambda\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1}) + \mu\frac{\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})}{\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}}) + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})} + \gamma\frac{\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+2})}{\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}})}, \tag{50}$$

which implies that

$$\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\| \leq \lambda\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\| + 2\mu\frac{\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\|\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\|}{\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}}) + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})\|} + \gamma\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\|. \tag{51}$$

Since $\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}}) + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})\| > \|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}})\|$, therefore,

$$\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\| \leq \lambda\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\| + 2\mu\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\| + \gamma\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\|, \tag{52}$$

so that

$$\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\| \leq \frac{\lambda + \gamma}{1 - 2\mu}\|\varphi(\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1})\|. \tag{53}$$

Also,

$$\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+3}) = \varphi(\mathcal{S}\varpi_{2\mathfrak{k}+2}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) \prec_{i_3} \lambda\varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1}) + \mu\frac{\varphi(\varpi_{2\mathfrak{k}+2}, \mathcal{S}\varpi_{2\mathfrak{k}+2})\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) + \varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{S}\varpi_{2\mathfrak{k}+2})\varphi(\varpi_{2\mathfrak{k}+2}, \mathcal{T}\varpi_{2\mathfrak{k}+1})}{\varphi(\mathcal{S}\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+2}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+1})} + \gamma\frac{\varphi(\varpi_{2\mathfrak{k}+2}, \mathcal{S}\varpi_{2\mathfrak{k}+2})\varphi(\varpi_{2\mathfrak{k}+2}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) + \varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{S}\varpi_{2\mathfrak{k}+2})\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{T}\varpi_{2\mathfrak{k}+1})}{\varphi(\mathcal{S}\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})}. \tag{54}$$

Since $\varpi_{2\mathfrak{k}+3} = \mathcal{S}\varpi_{2\mathfrak{k}+2}$ and $\varpi_{2\mathfrak{k}+2} = \mathcal{T}\varpi_{2\mathfrak{k}+1}$, we get

$$\begin{aligned}\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3}) \prec_{i_3} &\lambda\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1}) \\ &+\mu\frac{\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3})\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})+\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+3})\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+2})}{\varphi(\varpi_{2\mathfrak{k}+3},\varpi_{2\mathfrak{k}+2})+\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1})} \\ &+\gamma\frac{\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3})\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+2})+\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+3})\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})}{\varphi(\varpi_{2\mathfrak{k}+3},\varpi_{2\mathfrak{k}+1})+\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+2})},\end{aligned} \tag{55}$$

$$\begin{aligned}\text{or } \varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3}) \prec_{i_3} &\lambda\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1}) \\ &+\mu\frac{\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3})\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})}{\varphi(\varpi_{2\mathfrak{k}+3},\varpi_{2\mathfrak{k}+2})+\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1})} \\ &+\gamma\frac{\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+3})\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})}{\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+3})},\end{aligned} \tag{56}$$

which implies that

$$\begin{aligned}\|\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3})\| \le &\lambda\|\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1})\| \\ &+2\mu\frac{\|\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3})\|\|\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})\|}{\|\varphi(\varpi_{2\mathfrak{k}+3},\varpi_{2\mathfrak{k}+2})+\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1})\|} \\ &+\gamma\|\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})\|.\end{aligned} \tag{57}$$

Since $\|\varphi(\varpi_{2\mathfrak{k}+3},\varpi_{2\mathfrak{k}+2})+\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1})\| > \|\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})\|$, therefore,

$$\begin{aligned}\|\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3})\| \le &\lambda\|\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1})\| \\ &+2\mu\frac{\|\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3})\|\|\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})\|}{\|\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+1})\|} \\ &+\gamma\|\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})\|,\end{aligned} \tag{58}$$

$$\text{or } \|\varphi(\varpi_{2\mathfrak{k}+2},\varpi_{2\mathfrak{k}+3})\| \le \frac{\lambda+\gamma}{1-2\mu}\|\varphi(\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+2})\|. \tag{59}$$

Now, with $\mathfrak{h} = \lambda + \gamma/1 - 2\mu$, we have (for all $\mathfrak{k}$)

$$\|\varphi(\varpi_{\mathfrak{k}},\varpi_{\mathfrak{k}+1})\| \le \mathfrak{h}\|\varphi(\varpi_{\mathfrak{k}-1},\varpi_{\mathfrak{k}})\| \le \cdots \le \mathfrak{h}^{\mathfrak{k}}\|\varphi(\varpi_0,\varpi_1)\|. \tag{60}$$

So, for any $\mathfrak{m} > \mathfrak{k}$, we have

$$\begin{aligned}\|\varphi(\varpi_{\mathfrak{k}},\varpi_{\mathfrak{m}})\| \le &\|\varphi(\varpi_{\mathfrak{k}},\varpi_{\mathfrak{k}+1})\| \\ &+\|\varphi(\varpi_{\mathfrak{k}+1},\varpi_{\mathfrak{k}+2})\|+\cdots+\|\varphi(\varpi_{\mathfrak{m}-1},\varpi_{\mathfrak{m}})\| \le \left[\mathfrak{h}^{\mathfrak{k}}+\mathfrak{h}^{\mathfrak{k}+1}+\cdots+\mathfrak{h}^{\mathfrak{m}-1}\right]\|\varphi(\varpi_0,\varpi_1)\| \le \left[\frac{\mathfrak{h}^{\mathfrak{k}}}{1-\mathfrak{h}}\right]\|\varphi(\varpi_0,\varpi_1)\|,\end{aligned} \tag{61}$$

and henceforth

$$\|(\varphi\varpi_{\mathfrak{k}},\varpi_{\mathfrak{m}})\| \le \left[\frac{\mathfrak{h}^{\mathfrak{k}}}{1-\mathfrak{h}}\right]\|\varphi(\varpi_0,\varpi_1)\| \longrightarrow 0 \, as \, \mathfrak{k} \longrightarrow \infty. \tag{62}$$

On using Lemma 2, we conclude that $\{\varpi_{\mathfrak{k}}\}$ is a Cauchy sequence. Since $\mathscr{W}$ is a complete, then there exists $\mathfrak{l} \in \mathscr{W}$ such that $\varpi_{\mathfrak{k}} \longrightarrow \mathfrak{l}$ as $\mathfrak{k} \longrightarrow \infty$. Now, we assert that $\mathfrak{l} = \mathcal{S}\mathfrak{k}$; otherwise, $0 \prec_{i_3} \sigma = \varphi(\mathfrak{l}, \mathcal{S}\mathfrak{k})$ and we have

$$\begin{aligned}\sigma = \varphi(\mathfrak{l},\mathcal{S}\mathfrak{k}) \prec_{i_3} \varphi(\mathfrak{l},\mathcal{T}\varpi_{2\mathfrak{k}+1}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1},\mathcal{S}\mathfrak{k}) \prec_{i_3} &\varphi(\mathfrak{l},\varpi_{2\mathfrak{k}+2}) + \lambda\varphi(\mathfrak{l},\varpi_{2\mathfrak{k}+1}) \\ &+\mu\frac{\varphi(\mathfrak{l},\mathcal{S}\mathfrak{k})\varphi(\varpi_{2\mathfrak{k}+1},\mathcal{T}\varpi_{2\mathfrak{k}+1})+\varphi(\varpi_{2\mathfrak{k}+1},\mathcal{S}\mathfrak{k})\varphi(\mathfrak{l},\mathcal{T}\varpi_{2\mathfrak{k}+1})}{\varphi(\mathcal{S}\mathfrak{k},\mathfrak{l})+\varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1},\varpi_{2\mathfrak{k}+1})} \\ &+\gamma\frac{\varphi(\mathfrak{l},\mathcal{S}\mathfrak{k})\varphi(\mathfrak{l},\mathcal{T}\varpi_{2\mathfrak{k}+1})+\varphi(\varpi_{2\mathfrak{k}+1},\mathcal{S}\mathfrak{k})\varphi(\varpi_{2\mathfrak{k}+1},\mathcal{T}\varpi_{2\mathfrak{k}+1})}{\varphi(\mathcal{S}\mathfrak{k},\varpi_{2\mathfrak{k}+1})+\varphi(\varpi_{2\mathfrak{k}+2},\mathfrak{l})},\end{aligned} \tag{63}$$

which implies that

$$\|\sigma\| = \|\varphi(\mathfrak{l}, \mathcal{S}\mathfrak{k})\| \leq \|\varphi(\mathfrak{l}, \varpi_{2\mathfrak{k}+2})\| + \lambda\|\varphi(\mathfrak{l}, \varpi_{2\mathfrak{k}+1})\| + \mu \frac{\sigma\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\| + 2\|\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{S}\mathfrak{k})\|\|\varphi(\mathfrak{l}, \varpi_{2\mathfrak{k}+2})\|}{\|\varphi(\mathcal{S}\mathfrak{k}, \mathfrak{l}) + \varphi(\varpi_{2\mathfrak{k}+2}, \varpi_{2\mathfrak{k}+1})\|} + \gamma \frac{\sigma\|\varphi(\mathfrak{l}, \varpi_{2\mathfrak{k}+2})\| + 2\|\varphi(\varpi_{2\mathfrak{k}+1}, \mathcal{S}\mathfrak{k})\|\|\varphi(\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+2})\|}{\|\varphi(\mathcal{S}\mathfrak{k}, \varpi_{2\mathfrak{k}+1}) + \varphi(\varpi_{2\mathfrak{k}+2}, \mathfrak{l})\|}. \tag{64}$$

Therefore, $\|\sigma\| = \|\varphi(\mathfrak{l}, \mathcal{S}\mathfrak{k})\| = 0$, i.e., $\mathfrak{l} = \mathcal{S}\mathfrak{k}$. Similarly, we can derive that $\mathfrak{l} = \mathcal{T}\mathfrak{l}$. Assume that $\mathfrak{l}^*$ in $\mathcal{W}$ is an another common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Then,

$$\mathcal{S}\mathfrak{l}^* = \mathcal{T}\mathfrak{l}^* = \mathfrak{l}^*. \tag{65}$$

Since $\mathcal{D} = \varphi(\mathcal{S}\mathfrak{k}, \mathfrak{l}) + \varphi(\mathcal{T}\mathfrak{l}^*, \mathfrak{l}^*) = 0$, then

$$\varphi(\mathfrak{l}, \mathfrak{l}^*) = \varphi(\mathcal{S}\mathfrak{k}, \mathcal{T}\mathfrak{l}^*) = 0. \tag{66}$$

Hence, $\mathfrak{l} = \mathfrak{l}^*$.

Second, we consider that the case $(\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+1})) \times (\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}})) = 0$ (for any $\mathfrak{k}$) implies $\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) = 0$. Now, if $\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}+1}) = 0$, then $\varpi_{2\mathfrak{k}} = \mathcal{S}\varpi_{2\mathfrak{k}} = \varpi_{2\mathfrak{k}+1} = \mathcal{T}\varpi_{2\mathfrak{k}} = \varpi_{2\mathfrak{k}+2}$. Thus, we have $\varpi_{2\mathfrak{k}+1} = \mathcal{S}\varpi_{2\mathfrak{k}} = \varpi_{2\mathfrak{k}}$, so there exist $\mathfrak{k}_1$ and $\mathfrak{m}_1$ such that $\mathfrak{k}_1 = \mathcal{S}\mathfrak{m}_1 = \mathfrak{m}_1$. Similarly, we can derive that there exist $\mathfrak{k}_2$ and $\mathfrak{m}_2$ such that $\mathfrak{k}_2 = \mathcal{T}\mathfrak{m}_2 = \mathfrak{m}_2$. As $\varphi(\mathcal{S}\mathfrak{m}_1, \mathfrak{m}_1) + \varphi(\mathcal{T}\mathfrak{m}_2, \mathfrak{m}_2) = 0$ (due to definition), it implies $\varphi(\mathcal{S}\mathfrak{m}_1, \mathcal{T}\mathfrak{m}_2) = 0$, so that $\mathfrak{k}_1 = \mathcal{S}\mathfrak{m}_1 = \mathcal{T}\mathfrak{m}_2 = \mathfrak{k}_2$ which in turn yields that $\mathfrak{k}_1 = \mathcal{S}\mathfrak{m}_1 = \mathcal{S}\mathfrak{k}_1$. Similarly, one can also have $\mathfrak{k}_2 = \mathcal{T}\mathfrak{k}_2$. As $\mathfrak{k}_1 = \mathfrak{k}_2$, it implies $\mathcal{S}\mathfrak{k}_1 = \mathcal{T}\mathfrak{k}_1 = \mathfrak{k}_1$; therefore, $\mathfrak{k}_1 = \mathfrak{k}_2$, is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Assume that $\mathfrak{k}_1^*$ in $\mathcal{W}$ is an another common fixed point of $\mathcal{S}$ and $\mathcal{T}$. Then,

$$\mathcal{S}\mathfrak{k}_1^* = \mathcal{T}\mathfrak{k}_1^* = \mathfrak{k}_1^*. \tag{67}$$

Since $\mathcal{D} = \varphi(\mathcal{S}\mathfrak{k}_1, \mathfrak{k}_1) + \varphi(\mathcal{T}\mathfrak{k}_1^*, \mathfrak{k}_1^*) = 0$, then

$$\varphi(\mathfrak{k}_1, \mathfrak{k}_1^*) = \varphi(\mathcal{S}\mathfrak{k}_1, \mathcal{T}\mathfrak{k}_1^*) = 0. \tag{68}$$

This implies that $\mathfrak{k}_1^* = \mathfrak{k}_1$.

If $\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \varpi_{2\mathfrak{k}+1}) + \varphi(\mathcal{T}\varpi_{2\mathfrak{k}+1}, \varpi_{2\mathfrak{k}}) = 0$ implies that $\varphi(\mathcal{S}\varpi_{2\mathfrak{k}}, \mathcal{T}\varpi_{2\mathfrak{k}+1}) = 0$, then also proof can be completed on the preceding lines. □

Example 4. Considering $\mathcal{W} = [0, \infty)$, define a mapping φ: $\mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{C}_3$ by $\varphi(\varpi, \vartheta) = (1 + i_3)|\varpi - \vartheta|$, for all $\varpi, \vartheta \in \mathcal{W}$, where $|.|$ is the usual real modulus. Then, $(\mathcal{W}, \varphi)$ is a complete tricomplex valued metric space. Now, we consider a self mapping $\mathcal{T}$: $\mathcal{W} \longrightarrow \mathcal{W}$ defined by

$$\mathcal{T}(\varpi) = \frac{\varpi}{6}, \quad \forall \varpi \in \mathcal{W}. \tag{69}$$

Let $\lambda = 1/6$, $\mu = 0$ and $\lambda = 0$, then $\lambda + 2\mu + 2\gamma = 1/6 < 1$. Let $\varpi, \vartheta \in \mathcal{W}$, then

$$\varphi(\mathcal{T}\varpi, \mathcal{T}\vartheta) = \frac{(1 + i_3)}{6}|\varpi - \vartheta| \prec_{i_3} \frac{(1 + i_3)}{2}|\varpi - \vartheta|. \tag{70}$$

Therefore,

$$\varphi(\mathcal{T}\varpi, \mathcal{T}\vartheta) \prec_{i_3} \lambda\varphi(\varpi, \vartheta) + \frac{\mu\varphi(\varpi, \mathcal{T}\varpi)\varphi(\vartheta, \mathcal{T}\vartheta) + \gamma\varphi(\vartheta, \mathcal{T}\varpi)\varphi(\varpi, \mathcal{T}\vartheta)}{1 + \varphi(\varpi, \vartheta)}. \tag{71}$$

Hence, the conditions of Corollary 1 are fulfilled. Therefore, 0 is the unique fixed point of $\mathcal{T}$.

Example 5. Let $\mathcal{W} = \mathcal{B}(0, \mathfrak{q})$, $\mathfrak{q} > 1$, for all $\varpi, \vartheta \in \mathcal{W}$. Define φ: $\mathcal{W} \times \mathcal{W} \longrightarrow \mathbb{C}_3$ by

$$\varphi(\varpi(\mathfrak{t}), \vartheta(\mathfrak{t})) = \frac{i_3}{2\pi}\left|\int_E \frac{\varpi(\mathfrak{t})}{\mathfrak{t}} - \int_E \frac{\vartheta(\mathfrak{t})}{\mathfrak{t}}\right|, \tag{72}$$

where $\mathcal{E}$ is a closed path in $\mathcal{W}$ containing a zero. Then, then we prove that $(\mathcal{W}, \varphi)$ is a complete tricomplex valued metric space. We have

$$\begin{aligned} \varphi(\varpi(\mathfrak{t}), \vartheta(\mathfrak{t})) &= \frac{i_3}{2\pi}\left|\int_E \frac{\varpi(\mathfrak{t})}{\mathfrak{t}} - \int_E \frac{\vartheta(\mathfrak{t})}{\mathfrak{t}}\right|, \\ &= \frac{i_3}{2\pi}\left|\int_E \frac{\varpi(\mathfrak{t})}{\mathfrak{t}} - \int_E \frac{\sigma(\mathfrak{t})}{\mathfrak{t}} + \int_E \frac{\sigma(\mathfrak{t})}{\mathfrak{t}} - \int_E \frac{\vartheta(\mathfrak{t})}{\mathfrak{t}}\right| \prec_{i_3} \frac{i_3}{2\pi}\left|\int_E \frac{\varpi(\mathfrak{t})}{\mathfrak{t}} - \int_E \frac{\sigma(\mathfrak{t})}{\mathfrak{t}}\right| + \frac{i_3}{2\pi}\left|\int_E \frac{\sigma(\mathfrak{t})}{\mathfrak{t}} - \int_E \frac{\vartheta(\mathfrak{t})}{\mathfrak{t}}\right| \\ &= \varphi(\varpi(\mathfrak{t}), \sigma(\mathfrak{t})) + \varphi(\sigma(\mathfrak{t}), \vartheta(\mathfrak{t})). \end{aligned} \tag{73}$$

Now, we define the mapping $\mathscr{S}, \mathscr{T}\colon \mathscr{W} \longrightarrow \mathscr{W}$ by

$$\mathscr{S}\varpi(\mathfrak{t}) = \mathfrak{t}, \mathscr{T}\vartheta(\mathfrak{t}) = e^{\mathfrak{t}} - 1. \tag{74}$$

Using the Cauchy integral formula when the mappings $\mathscr{S}$ and $\mathscr{T}$ are analytic, we get

$$\begin{aligned}
\varphi(\mathscr{S}\varpi(\mathfrak{t}), \mathscr{T}\vartheta(\mathfrak{t})) &= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\mathfrak{t}}{\mathfrak{t}} - \int_{\mathscr{C}}\frac{e^{\mathfrak{t}}-1}{\mathfrak{t}}\right| \\
&= 0, \\
\varphi(\varpi(\mathfrak{t}), \vartheta(\mathfrak{t})) &= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\varpi(\mathfrak{t})}{\mathfrak{t}} - \int_{\mathscr{C}}\frac{\vartheta(\mathfrak{t})}{\mathfrak{t}}\right|, \\
\varphi(\varpi(\mathfrak{t}), \mathscr{S}\varpi(\mathfrak{t})) &= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\varpi(\mathfrak{t})}{\mathfrak{t}} - \int_{\mathscr{C}}\frac{\mathfrak{t}}{\mathfrak{t}}\right| \\
&= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\varpi(\mathfrak{t})}{\mathfrak{t}}\right|, \\
\varphi(\vartheta(\mathfrak{t}), \mathscr{T}\vartheta(\mathfrak{t})) &= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\vartheta(\mathfrak{t})}{\mathfrak{t}} - \int_{\mathscr{C}}\frac{e^{\mathfrak{t}}-1}{\mathfrak{t}}\nu\right| \\
&= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\vartheta(\mathfrak{t})}{\mathfrak{t}}\right|, \\
\varphi(\varpi(\mathfrak{t}), \mathscr{S}\vartheta(\mathfrak{t})) &= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\varpi(\mathfrak{t})}{\mathfrak{t}} - \int_{\mathscr{C}}\frac{\mathfrak{t}}{\mathfrak{t}}\right| \\
&= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\varpi(\mathfrak{t})}{\mathfrak{t}}\right|, \\
\varphi(\varpi(\mathfrak{t}), \mathscr{T}\vartheta(\mathfrak{t})) &= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\varpi(\mathfrak{t})}{\mathfrak{t}} - \int_{\mathscr{C}}\frac{e^{\mathfrak{t}}-1}{\mathfrak{t}}\right| \\
&= \frac{i_3}{2\pi}\left|\int_{\mathscr{C}}\frac{\varpi(\mathfrak{t})}{\mathfrak{t}}\right|.
\end{aligned} \tag{75}$$

Clearly,

$$\varphi(\mathscr{S}\varpi, \mathscr{T}\vartheta) \prec_{i_3} \lambda\varphi(\varpi, \vartheta) + \frac{\mu\varphi(\varpi, \mathscr{S}\varpi)\varphi(\vartheta, \mathscr{T}\vartheta) + \gamma\varphi(\vartheta, \mathscr{S}\varpi)\varphi(\varpi, \mathscr{T}\vartheta)}{1 + \varphi(\varpi, \vartheta)}, \tag{76}$$

for all $\varpi, \vartheta \in \mathscr{W}$, where λ, μ, γ are nonnegative reals with $\lambda + 2\mu + 2\gamma < 1$. Therefore, all the conditions of Theorem 2 are fulfilled, then the mappings $\mathscr{S}$ and $\mathscr{T}$ have a unique common fixed point in $\mathscr{W}$.

Definition 6 (see [19]). Two self mappings $\mathscr{T}$ and $\mathscr{S}$ of a metric space $(\mathscr{W}, \varphi)$ are said to be commuting iff $\mathscr{T}\mathscr{S}\varpi = \mathscr{S}\mathscr{T}\varpi$ for all ϖ in $\mathscr{W}$.

Example 6. Considering $\mathscr{W} = \mathbb{R}$, define a mapping $\varphi\colon \mathscr{W} \times \mathscr{W} \longrightarrow \mathbb{C}_3$ by $\varphi(\varpi, \vartheta) = (1 + i_3)|\varpi - \vartheta|$, for all $\varpi, \vartheta \in \mathscr{W}$, where $|.|$ is the usual real modulus. Then, $(\mathscr{W}, \varphi)$ is a complete tricomplex valued metric space. Now, we consider a self mappings $\mathscr{T}, \mathscr{S}\colon \mathscr{W} \longrightarrow \mathscr{W}$ defined by

$$\begin{aligned}
\mathscr{T}(\varpi) &= 7\varpi, \\
\mathscr{S}(\vartheta) &= 11\vartheta.
\end{aligned} \tag{77}$$

Then,

$$\mathscr{T}(\mathscr{S}(\varpi)) = \mathscr{T}(11\varpi) = 11(7\varpi) = \mathscr{S}(7\varpi) = \mathscr{S}(\mathscr{T}(\varpi)). \tag{78}$$

So, $\mathscr{T}$ and $\mathscr{S}$ are commute. Now,

$$\varphi(\mathscr{T}\varpi, \mathscr{T}\vartheta) = \varphi(7\varpi, 7\vartheta) = 7(1 + i_3)|\varpi - \vartheta|, \tag{79}$$

$$\varphi(\mathscr{S}\varpi, \mathscr{S}\vartheta) = \varphi(11\varpi, 11\vartheta) = 11(1 + i)_3|\varpi - \vartheta|. \tag{80}$$

Therefore,

$$\varphi(\mathscr{T}\varpi, \mathscr{T}\vartheta) \prec_{i_3} \frac{2}{3}\varphi(\mathscr{S}\varpi, \mathscr{S}\vartheta) = \lambda\left(= \frac{2}{3}\right)\varphi(\mathscr{S}\varpi, \mathscr{S}\vartheta). \tag{81}$$

However,

$$\varphi(\mathscr{T}\varpi, \mathscr{T}\vartheta) = 7(1 + i_3)|\varpi - \vartheta| \not\precsim_{i_3} \lambda|\varpi - \vartheta|. \tag{82}$$

Thus, φ is not a contraction. Hence, $\mathscr{T}$ and $\mathscr{S}$ have a unique common fixed point.

We can prove that common fixed point theorems on tricomplex valued metric space under commuting mappings.

Definition 7 (see [19]). Two self mappings $\mathscr{T}$ and $\mathscr{S}$ of a metric space $(\mathscr{W}, \varphi)$ are said to be weakly compatible if the mappings commute at their coincidence points, i.e., $\mathscr{T}\varpi = \mathscr{S}\varpi$ for some $\varpi \in \mathscr{W}$ implies $\mathscr{T}\mathscr{S}\varpi = \mathscr{S}\mathscr{T}\varpi$.

We can prove common fixed point theorems on tricomplex valued metric space under weakly compatible mappings.

4. Applications

In this section, we give applications using Theorem 2 and Corollary 1.

Let $\mathscr{W} = C[\curlywedge_1, \curlywedge_2]$ be a set of all real continuous functions on $[\curlywedge_1, \curlywedge_2]$ equipped with metric $\varphi(\varpi, \vartheta) = (1 + i_3)(|\varpi(\sqcup) - \vartheta(\sqcup)|)$ for all $\varpi, \vartheta \in C[\curlywedge_1, \curlywedge_2]$ and $\sqcup \in [\curlywedge_1, \curlywedge_2]$, where $|.|$ is the usual real modulus. Then, $(\mathscr{W}, \varphi)$ is a complete tricomplex valued metric space. Now, we consider the system of nonlinear Fredholm integral equation as follows:

$$\varpi(\sqcup) = \mathfrak{b}(\sqcup) + \frac{1}{\curlywedge_2 - \curlywedge_1}\int_{\curlywedge_1}^{\curlywedge_2}\mathfrak{K}_1(\sqcup, \mathfrak{s}, \varpi(\mathfrak{s}))d\mathfrak{s}, \tag{83}$$

$$\varpi(\sqcup) = \mathfrak{b}(\sqcup) + \frac{1}{\curlywedge_2 - \curlywedge_1}\int_{\curlywedge_1}^{\curlywedge_2}\mathfrak{K}_2(\sqcup, \mathfrak{s}, \varpi(\mathfrak{s}))d\mathfrak{s}, \tag{84}$$

where $\sqcup, \mathfrak{s} \in [\curlywedge_1, \curlywedge_2]$. Assume that $\mathfrak{K}_1, \mathfrak{K}_2\colon [\curlywedge_1, \curlywedge_2] \times [\curlywedge_1, \curlywedge_2] \times \mathscr{W} \longrightarrow \mathbb{R}$ and $\mathfrak{b}\colon [\curlywedge_1, \curlywedge_2] \longrightarrow \mathbb{R}$ are continuous, where $\mathfrak{b}(\sqcup)$ is a given function in $\mathscr{W}$. We define a partial order $\sqcup_{i_3}$ in $\mathbb{C}_3$ as $\varpi \sqcup_{i_3} \vartheta$ iff $\varpi \leq \vartheta$.

Theorem 4. *Suppose that $(\mathscr{W},\varphi)$ is a complete tricomplex valued metric space equipped with metric $\varphi(\varpi,\vartheta) = (1+i_3)(|\varpi(\sqcup)-\vartheta(\sqcup)|)$ for all $\varpi,\vartheta\in\mathscr{W}$, $\sqcup\in[\curlywedge_1,\curlywedge_2]$ and $\mathscr{S},\mathscr{T}\colon\mathscr{W}\longrightarrow\mathscr{W}$ be continuous operator on $\mathscr{W}$ defined by*

$$\mathscr{S}\varpi(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\curlywedge_2-\curlywedge_1}\int_{\curlywedge_1}^{\curlywedge_2}\mathfrak{K}_1(\sqcup,\mathfrak{s},\varpi(\mathfrak{s}))d\mathfrak{s}, \tag{85}$$

$$\mathscr{T}\varpi(\sqcup) = \mathfrak{v}(\sqcup) + \frac{1}{\curlywedge_2-\curlywedge_1}\int_{\curlywedge_1}^{\curlywedge_2}\mathfrak{K}_2(\sqcup,\mathfrak{s},\varpi(\mathfrak{s}))d\mathfrak{s}. \tag{86}$$

If there exists $\lambda<1$ such that for all $\varpi,\vartheta\in\mathscr{W}$ with $\varpi\neq\vartheta$ and $\mathfrak{s},\sqcup\in[\curlywedge_1,\curlywedge_2]$ satisfying the following inequality:

$$\left|\mathfrak{K}_1(\sqcup,\mathfrak{s},\varpi(s)) - \mathfrak{K}_2(\sqcup,\mathfrak{s},\vartheta(\mathfrak{s}))\right| \le \lambda|\varpi(\sqcup)-\vartheta(\sqcup)|, \tag{87}$$

then the integral operators defined by (85) and (86) have a unique common solution.

Proof. Consider

$$\begin{aligned}(1+i_3)(|\mathscr{S}\varpi(\sqcup)-\mathscr{T}\vartheta(\sqcup)|) &= \frac{(1+i_3)}{|\curlywedge_2-\curlywedge_1|}\left(\left|\int_{\curlywedge_1}^{\curlywedge_2}\mathfrak{K}_1(\sqcup,\mathfrak{s},\varpi(\mathfrak{s}))d\mathfrak{s}\int_{\curlywedge_1}^{\curlywedge_2}\mathfrak{K}_2(\sqcup,\mathfrak{s},\vartheta(\mathfrak{s}))d\mathfrak{s}\right|\right)\\ &\le \frac{\lambda}{|\curlywedge_2-\curlywedge_1|}\int_{\curlywedge_1}^{\curlywedge_2}(1+i_3)|\varpi(\sqcup)-\vartheta(\sqcup)|d\mathfrak{s} \le \frac{\lambda(1+i_3)||\varpi(\sqcup)-\vartheta(\sqcup)||}{|\curlywedge_2-\curlywedge_1|}\int_{\curlywedge_1}^{\curlywedge_2}d\mathfrak{s}.\end{aligned} \tag{88}$$

Therefore,

$$\varphi(\mathscr{S}\varpi,\mathscr{T}\vartheta)\le\lambda\varphi(\varpi,\vartheta). \tag{89}$$

Hence, all the hypothesis of Theorem 2 are fulfilled with $\lambda<1$ and $\mu=\gamma=0$, and so the integral operators $\mathscr{S}$ and $\mathscr{T}$ defined by (85) and (86) have a unique common solution. □

Theorem 5 *Let $\mathscr{W}=\mathbb{C}^{\mathfrak{k}}$ be a complete tricomplex valued metric space with the metric*

$$\varphi(\varpi,\vartheta) = \sum_{i=1}^{\mathfrak{k}}\left(|\varpi_i-\vartheta_i| + i_3|\varpi_i-\vartheta_i|\right), \tag{90}$$

where $\varpi,\vartheta\in\mathscr{W}$. If

$$\sum_{j=1}^{\mathfrak{k}}\left|\lambda_{ij}\right| \prec_{i_3}\lambda<1,\quad \text{for all } i=1,2,\ldots\mathfrak{k}, \tag{91}$$

then the linear system

$$\begin{cases}\mathfrak{b}_1 = \mathfrak{a}_{11}\varpi_1+\mathfrak{a}_{12}\varpi_2+\ldots+\mathfrak{a}_{1\mathfrak{k}}\varpi_{\mathfrak{k}},\\ \mathfrak{b}_2 = \mathfrak{a}_{21}\varpi_1+\mathfrak{a}_{22}\varpi_2+\ldots+\mathfrak{a}_{2\mathfrak{k}}\varpi_{\mathfrak{k}},\\ \vdots\\ \mathfrak{b}_{\mathfrak{k}} = \mathfrak{a}_{\mathfrak{k}1}\varpi_1+\mathfrak{a}_{\mathfrak{k}2}\varpi_2+\ldots+\mathfrak{a}_{\mathfrak{k}\mathfrak{k}}\varpi_{\mathfrak{k}},\end{cases} \tag{92}$$

of $\mathfrak{k}$ linear equations with $\mathfrak{k}$ unknowns has a unique solution.

Proof. Define $\mathscr{T}\colon\mathscr{W}\longrightarrow\mathscr{W}$ by

$$\mathscr{T}(\varpi) = \mathscr{A}\varpi+\mathfrak{b}, \tag{93}$$

where $\varpi=(\varpi_1,\varpi_2,\varpi_3,\ldots,\varpi_{\mathfrak{k}})\in\mathbb{C}^{\mathfrak{k}}$, $\mathfrak{b}=(\mathfrak{b}_1,\mathfrak{b}_2,\ldots,\mathfrak{b}_{\mathfrak{k}})\in\mathbb{C}^{\mathfrak{k}}$, and

$$\mathscr{A} = \begin{pmatrix}\mathfrak{a}_{11} & \mathfrak{a}_{11} & \cdots & \mathfrak{a}_{1\mathfrak{k}}\\ \mathfrak{a}_{21} & \mathfrak{a}_{22} & \cdots & \mathfrak{a}_{2\mathfrak{k}}\\ \vdots & \vdots & \vdots & \vdots\\ \mathfrak{a}_{\mathfrak{k}1} & \mathfrak{a}_{\mathfrak{k}2} & \cdots & \mathfrak{a}_{\mathfrak{k}\mathfrak{k}}\end{pmatrix}. \tag{94}$$

Now,

$$\begin{aligned}\varphi(\mathscr{T}(\varpi),\mathscr{T}(\vartheta)) &= \sum_{j=1}^{\mathfrak{k}}\left|\lambda_{ij}\left(\varpi_j-\vartheta_j\right)\right| + i_3\left|\lambda_{ij}\left(\varpi_j-\vartheta_j\right)\right| \prec_{i_3}\sum_{j=1}^{\mathfrak{k}}\left|\lambda_{ij}\right|\left(\sum_{j=1}^{\mathfrak{k}}\left|\left(\varpi_j-\vartheta_j\right)\right| + i_3\left|\left(\varpi_j-\vartheta_j\right)\right|\right)\\ &= \lambda\varphi(\varpi,\vartheta)\\ &= \lambda\varphi(\varpi,\vartheta) + \frac{\mu\varphi(\varpi,\mathscr{T}\varpi)\varphi(\vartheta,\mathscr{T}\vartheta)+\gamma\varphi(\vartheta,\mathscr{T}\varpi)\varphi(\varpi,\mathscr{T}\vartheta)}{1+\varphi(\varpi,\vartheta)}.\end{aligned} \tag{95}$$

Hence, all the conditions of Corollary 1 are satisfied with $\lambda = 1/6$, $\mu = 0$, $\gamma = 0$, and $\lambda + 2\mu + 2\gamma < 1$ and so the linear system of equation has a unique solution. □

5. Conclusion and Future Work

In this paper, we introduce the notion of tricomplex valued metric space and proved some common fixed point theorems on tricomplex valued metric space. An illustrative example and applications on tricomplex valued metric space is given. In 2013, Patel et al. [20] proved common fixed points for a pair of maps on metric space. It is an interesting open problem to study the tricomplex valued metric space instead of metric space and obtain common fixed points for a pair of maps on tricomplex valued metric space. In 2013, Rouzkard et al. [21] proved existence and uniqueness theorems on ordered metric spaces via generalized distances. It is an interesting open problem to study the ordered tricomplex valued metric space instead of ordered metric space and obtain fixed point theorems on ordered tricomplex valued metric space. Recently, Altun et al. [22, 23] proved the best proximity point theorems on complete metric space. It is an interesting open problem to study the best proximity theorems on tricomplex valued metric space instead of best proximity point theorems on complete metric space.

Data Availability

No data were used to support this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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