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Coupled common fixed point results involving (ϕ, ψ)-contractions in ordered generalized metric spaces with application to integral equations

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Abstract

We establish some coupled coincidence and coupled common fixed point theorems for the mixed g-monotone mappings satisfying (ϕ, ψ)-contractive conditions in the setting of ordered generalized metric spaces. Presented theorems extend and generalize the very recent results of Choudhury and Maity (Math. Comput. Model. 54(1-2):73-79, 2011). To illustrate our results, an example and an application to integral equations have also been given.

MSC: 54H10; 54H25

Keywords: partially ordered set; mixed g-monotone property; coupled coincidence point; coupled common fixed point

1 Introduction and preliminaries

Mustafa and Sims [1] introduced the notion of G-metric spaces. The structure of G-metric spaces is a generalization of metric spaces. Mustafa *et al.* [2] initiated the theory of fixed points in G-metric spaces and established the Banach contraction principle in this generalized structure. Afterwards, different authors proved several fixed point results in this space. References [3–17] are some examples of these works.

Definition 1.1 [1] Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2 [1] Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and then we say that the sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in G-metric space (X,G) then, for any $\varepsilon > 0$, there exists a positive integer N such that $G(x,x_n,x_m) < \varepsilon$ for all $n,m \geq N$.

In [1], the authors have shown that the G-metric induces a Hausdorff topology, and the convergence described in the definition above is relative to this topology. This topology being Hausdorff, a sequence can converge at most to a point.

Definition 1.3 [1] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if for any $\varepsilon > 0$, there is a positive integer N such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq N$, that is, if $G(x_n,x_m,x_l) \to 0$, as $n,m,l \to \infty$.

Lemma 1.4 [1] *If (X,G) is a G-metric space, then the following are equivalent*:
(1) $\{x_n\}$ *is G-convergent to x*,
(2) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$,
(3) $G(x_n,x,x) \to 0$ *as* $n \to \infty$,
(4) $G(x_m,x_n,x) \to 0$ *as* $m,n \to \infty$.

Lemma 1.5 [1] *If (X,G) is a G-metric space, then the following are equivalent*:
(1) *the sequence $\{x_n\}$ is G-Cauchy*,
(2) *for every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n,x_m,x_m) < \varepsilon$ for all $n,m \geq N$*.

Lemma 1.6 [1] *If (X,G) is a G-metric space, then $G(x,y,y) \leq 2G(y,x,x)$ for all $x,y \in X$*.

Lemma 1.7 *If (X,G) is a G-metric space, then $G(x,x,y) \leq G(x,x,z) + G(z,z,y)$ for all $x,y,z \in X$*.

Definition 1.8 [1] Let (X,G), (X',G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G'-convergent to $f(x)$.

Lemma 1.9 [1] *Let (X,G) be a G-metric space, then the function $G(x,y,z)$ is jointly continuous in all three of its variables*.

Definition 1.10 [1] A G-metric space (X,G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X,G) is convergent in X.

Definition 1.11 [10] Let (X,G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y, respectively, $\{F(x_n,y_n)\}$ is G-convergent to $F(x,y)$.

Recently, fixed point theorems under different contractive conditions in metric spaces endowed with a partial order have been established by various authors. One can see the works noted in the references [7, 10–15, 18–38]. Bhaskar and Lakshmikantham [18] introduced the notion of coupled fixed points and proved some coupled fixed point theorems for a mapping satisfying mixed monotone property. The work [18] was illustrated by proving the existence and uniqueness of the solution for a periodic boundary value problem.

Lakshmikantham and Ćirić [19] extended the notion of mixed monotone property due to Bhaskar and Lakshmikantham [18] by introducing the notion of mixed g-monotone property in partially ordered metric spaces.

Definition 1.12 [18] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Definition 1.13 [19] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say that F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad gx_1 \leq gx_2 \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \leq gy_2 \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Definition 1.14 [18] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.15 [19] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.16 [19] An element $(x, y) \in X \times X$ is called a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.17 [19] The mappings $F : X \times X \to X$ and $g : X \to X$ are called commutative if

$$gF(x, y) = F(gx, gy)$$

for all $x, y \in X$.

Let $(X, \leq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Choudhury and Maity [10] established some coupled fixed point theorems for the mixed monotone mapping $F : X \times X \to X$ under a contractive condition of the form

$$G\big(F(x, y), F(u, v), F(w, z)\big) \leq \frac{k}{2}\big[G(x, u, w) + G(y, v, z)\big] \tag{1.1}$$

for $x, y, z, u, v, w \in X$ with $x \geq u \geq w$ and $y \leq v \leq z$, where $k \in [0, 1)$.

Different authors extended and generalized the results of Choudhury and Maity [10] under different contractive conditions in G-metric spaces. One can refer to the references [11–15, 17, 31].

Presented work extends and generalizes the work of Choudhury and Maity [10] for a pair commuting mappings. We first prove the existence of coupled coincidence points and then, prove the existence and uniqueness of coupled common fixed points for our main results.

2 Main results

Before we prove our main results, we need the following.

Denote by Φ the class of all functions $\phi : [0,\infty) \to [0,\infty)$ with the following properties:

(ϕ_{i}) ϕ is continuous and non-decreasing;
(ϕ_{ii}) $\phi(t) = 0$ if $t = 0$;
(ϕ_{iii}) $\phi(t+s) \leq \phi(t) + \phi(s)$ for all $t, s \in [0,\infty)$.

Denote by Ψ the class of all functions $\psi : [0,\infty) \to [0,\infty)$ with the following properties:

(ψ_{i}) $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$;
(ψ_{ii}) $\lim_{t\to 0+} \psi(t) = 0$.

Some examples of $\phi(t)$ are kt (where $k > 0$), $\frac{t}{t+1}$, $\frac{t}{t+2}$ and examples of $\psi(t)$ are kt (where $k > 0$), $\frac{\ln(2t+1)}{2}$.

Now, we give our results.

Theorem 2.1 *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$, $g : X \to X$ be two mappings. Assume that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} \phi\bigl(G\bigl(F(x,y),F(u,v),F(w,z)\bigr)\bigr) \leq{}& \frac{1}{2}\phi\bigl(G(gx,gu,gw)+G(gy,gv,gz)\bigr) \\ &- \psi\left(\frac{G(gx,gu,gw)+G(gy,gv,gz)}{2}\right) \end{aligned} \tag{2.1}$$

for all $x, y, u, v, w, z \in X$ with $gx \geq gu \geq gw$ and $gy \leq gv \leq gz$.

Assume that F and g satisfy the following conditions:

(1) *$F(X \times X) \subseteq g(X)$,*
(2) *F has the mixed g-monotone property,*
(3) *F is continuous,*
(4) *g is continuous and commutes with F.*

Suppose that there exist $x_0, y_0 \in X$ with $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then F and g have a coupled coincidence point in X, that is, there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Proof Suppose that $x_0, y_0 \in X$ are such that $gx_0 \leq F(x_0, y_0)$, $gy_0 \geq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$, $gy_1 = F(y_0, x_0)$. Again we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$, $gy_2 = F(y_1, x_1)$.

Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n), \qquad gy_{n+1} = F(y_n, x_n) \quad \text{for all } n \geq 0. \tag{2.2}$$

We shall prove for all $n \geq 0$, that

$$gx_n \leq gx_{n+1}, \tag{2.3}$$

$$gy_n \geq gy_{n+1}. \tag{2.4}$$

Since $gx_0 \leq F(x_0, y_0)$, $gy_0 \geq F(y_0, x_0)$ and $gx_1 = F(x_0, y_0)$, $gy_1 = F(y_0, x_0)$, we have $gx_0 \leq gx_1$, $gy_0 \geq gy_1$, that is, (2.3) and (2.4) hold for $n = 0$.

Suppose that (2.3) and (2.4) hold for some $n > 0$, that is, $gx_n \leq gx_{n+1}$, $gy_n \geq gy_{n+1}$. As F has the mixed g-monotone property, from (2.2), we have

$$gx_{n+1} = F(x_n, y_n) \leq F(x_{n+1}, y_n) \leq F(x_{n+1}, y_{n+1}) = gx_{n+2},$$

and

$$gy_{n+1} = F(y_n, x_n) \geq F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = gy_{n+2}.$$

Then by mathematical induction, it follows that (2.3) and (2.4) hold for all $n \geq 0$.

If for some n, we have $(gx_{n+1}, gy_{n+1}) = (gx_n, gy_n)$, then $gx_{n+1} = F(x_n, y_n) = gx_n$ and $gy_{n+1} = F(y_n, x_n) = gy_n$, that is, F and g have a coincidence point. So now onwards, we suppose that $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$ for all $n \geq 0$; that is, we suppose that either $gx_{n+1} = F(x_n, y_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n) \neq gy_n$.

Since $gx_n \geq gx_{n-1}$ and $gy_n \leq gy_{n-1}$, from (2.1) and (2.2), we have

$$\begin{aligned}
&\phi\bigl(G(gx_{n+1}, gx_{n+1}, gx_n)\bigr) \\
&\quad = \phi\bigl(G\bigl(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})\bigr)\bigr) \\
&\quad \leq \frac{1}{2}\phi\bigl(G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})\bigr) \\
&\qquad - \psi\left(\frac{G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{2}\right).
\end{aligned} \tag{2.5}$$

Similarly, we have

$$\begin{aligned}
&\phi\bigl(G(gy_{n+1}, gy_{n+1}, gy_n)\bigr) \\
&\quad \leq \frac{1}{2}\phi\bigl(G(gy_n, gy_n, gy_{n-1}) + G(gx_n, gx_n, gx_{n-1})\bigr) \\
&\qquad - \psi\left(\frac{G(gy_n, gy_n, gy_{n-1}) + G(gx_n, gx_n, gx_{n-1})}{2}\right).
\end{aligned} \tag{2.6}$$

Adding (2.5) and (2.6), we have

$$\begin{aligned}
&\phi\bigl(G(gx_{n+1}, gx_{n+1}, gx_n)\bigr) + \phi\bigl(G(gy_{n+1}, gy_{n+1}, gy_n)\bigr) \\
&\quad \leq \phi\bigl(G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})\bigr) \\
&\qquad - 2\psi\left(\frac{G(gy_n, gy_n, gy_{n-1}) + G(gx_n, gx_n, gx_{n-1})}{2}\right).
\end{aligned} \tag{2.7}$$

By (ϕ_{iii}), we have

$$\begin{aligned}&\phi\bigl(G(gx_{n+1},gx_{n+1},gx_n)+G(gy_{n+1},gy_{n+1},gy_n)\bigr)\\&\quad\le\phi\bigl(G(gx_{n+1},gx_{n+1},gx_n)\bigr)+\phi\bigl(G(gy_{n+1},gy_{n+1},gy_n)\bigr).\end{aligned}\tag{2.8}$$

From (2.7) and (2.8), we have

$$\begin{aligned}&\phi\bigl(G(gx_{n+1},gx_{n+1},gx_n)+G(gy_{n+1},gy_{n+1},gy_n)\bigr)\\&\quad\le\phi\bigl(G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})\bigr)\\&\qquad-2\psi\left(\frac{G(gy_n,gy_n,gy_{n-1})+G(gx_n,gx_n,gx_{n-1})}{2}\right)\qquad(2.9)\\&\quad\le\phi\bigl(G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1})\bigr).\qquad(2.10)\end{aligned}$$

Using (2.10) and the fact that ϕ is non-decreasing, we get

$$\begin{aligned}&G(gx_{n+1},gx_{n+1},gx_n)+G(gy_{n+1},gy_{n+1},gy_n)\\&\quad\le G(gx_n,gx_n,gx_{n-1})+G(gy_n,gy_n,gy_{n-1}).\end{aligned}$$

Let $R_n=G(gx_{n+1},gx_{n+1},gx_n)+G(gy_{n+1},gy_{n+1},gy_n)$, then the sequence $\{R_n\}$ is decreasing. Therefore, there exists some $R\ge 0$ such that

$$\lim_{n\to\infty}R_n=\lim_{n\to\infty}\bigl[G(gx_{n+1},gx_{n+1},gx_n)+G(gy_{n+1},gy_{n+1},gy_n)\bigr]=R.\tag{2.11}$$

We claim that $R=0$.

On the contrary, suppose that $R>0$.

Taking limit as $n\to\infty$ on both sides of (2.9) and using the properties of ϕ and ψ, we have

$$\begin{aligned}\phi(R)&=\lim_{n\to\infty}\phi(R_n)\\&\le\lim_{n\to\infty}\left[\phi(R_{n-1})-2\psi\left(\frac{R_{n-1}}{2}\right)\right]\\&=\phi(R)-2\lim_{Rn-1\to R}\psi\left(\frac{R_{n-1}}{2}\right)<\phi(R),\quad\text{a contradiction.}\end{aligned}$$

Thus, $R=0$, that is,

$$\lim_{n\to\infty}R_n=\lim_{n\to\infty}\bigl[G(gx_{n+1},gx_{n+1},gx_n)+G(gy_{n+1},gy_{n+1},gy_n)\bigr]=0.\tag{2.12}$$

Next, we shall show that$\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

If possible, suppose that at least one of $\{gx_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon>0$, for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k)>m(k)\ge k$ such that

$$r_k=G(gx_{n(k)},gx_{n(k)},gx_{m(k)})+G(gy_{n(k)},gy_{n(k)},gy_{m(k)})\ge\varepsilon.\tag{2.13}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (2.13). Then

$$G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) < \varepsilon. \tag{2.14}$$

By (2.13), (2.14) and (G5), we have

$$\begin{aligned}
\varepsilon \leq r_k &= G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&\leq G(gx_{n(k)}, gx_{n(k)} gx_{n(k)-1}) + G(gx_{n(k)-1}, gx_{n(k)-1}, gx_{m(k)}) \\
&\quad + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) + G(gy_{n(k)-1}, gy_{n(k)-1}, gy_{m(k)}) \\
&< G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)-1}) + G(gy_{n(k)}, gy_{n(k)}, gy_{n(k)-1}) + \varepsilon.
\end{aligned} \tag{2.15}$$

Letting $k \to \infty$ in (2.15) and using (2.12), we have

$$\lim_{n\to\infty} r_k = \lim_{n\to\infty} \bigl[G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})\bigr] = \varepsilon. \tag{2.16}$$

Again by (G5) and Lemma 1.6, we have

$$\begin{aligned}
&G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) \\
&\quad \leq G(gx_{n(k)}, gx_{n(k)}, gx_{n(k)+1}) + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)}) \\
&\quad \leq 2G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{n(k)}) \\
&\qquad + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) + G(gx_{m(k)+1}, gx_{m(k)+1}, gx_{m(k)}).
\end{aligned} \tag{2.17}$$

Similarly, we have

$$\begin{aligned}
&G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&\quad \leq 2G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{n(k)}) \\
&\qquad + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}) + G(gy_{m(k)+1}, gy_{m(k)+1}, gy_{m(k)}).
\end{aligned} \tag{2.18}$$

Summing (2.17) and (2.18), we have

$$\begin{aligned}
r_k &= G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) \\
&\leq 2R_{n(k)} + R_{m(k)} + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1}).
\end{aligned}$$

Since ϕ is non-decreasing and by (ϕ_{iii}), we have

$$\begin{aligned}
\phi(r_k) &\leq \phi\bigl(2R_{n(k)} + R_{m(k)} + G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1}) \\
&\qquad + G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})\bigr) \\
&\leq 2\phi(R_{n(k)}) + \phi(R_{m(k)}) \\
&\qquad + \phi\bigl(G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1})\bigr) + \phi\bigl(G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})\bigr).
\end{aligned} \tag{2.19}$$

Since $n(k) > m(k)$, $gx_{n(k)} \geq gx_{m(k)}$ and $gy_{n(k)} \leq gy_{m(k)}$, then from (2.1) and (2.2), we have

$$\begin{aligned}
&\phi\big(G(gx_{n(k)+1}, gx_{n(k)+1}, gx_{m(k)+1})\big) \\
&\quad = \phi\big(G\big(F(x_{n(k)}, y_{n(k)}), F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})\big)\big) \\
&\quad \leq \frac{1}{2}\phi\big(G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})\big) \\
&\qquad - \psi\left(\frac{G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)}) + G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)})}{2}\right) \\
&\quad = \frac{1}{2}\phi(r_k) - \psi\left(\frac{r_k}{2}\right).
\end{aligned} \tag{2.20}$$

Similarly, we have

$$\begin{aligned}
&\phi\big(G(gy_{n(k)+1}, gy_{n(k)+1}, gy_{m(k)+1})\big) \\
&\quad \leq \frac{1}{2}\phi\big(G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)})\big) \\
&\qquad - \psi\left(\frac{G(gy_{n(k)}, gy_{n(k)}, gy_{m(k)}) + G(gx_{n(k)}, gx_{n(k)}, gx_{m(k)})}{2}\right) \\
&\quad = \frac{1}{2}\phi(r_k) - \psi\left(\frac{r_k}{2}\right).
\end{aligned} \tag{2.21}$$

Using (2.19)-(2.21), we have

$$\phi(r_k) \leq 2\phi(R_{n(k)}) + \phi(R_{m(k)}) + \phi(r_k) - 2\psi\left(\frac{r_k}{2}\right).$$

Letting $k \to \infty$ in the last inequality, and using (2.12), (2.16) and the properties of ϕ and ψ, we have

$$\begin{aligned}
\phi(\varepsilon) &\leq 2\phi(0) + \phi(0) + \phi(\varepsilon) - 2\lim_{k\to\infty} \psi\left(\frac{r_k}{2}\right) \\
&= \phi(\varepsilon) - 2\lim_{rk\to\varepsilon} \psi\left(\frac{r_k}{2}\right) < \phi(\varepsilon), \quad \text{a contradiction.}
\end{aligned}$$

Therefore, both $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. Now, since the space (X, G) is a complete G-metric space, there exist x, y in X such that the sequences $\{gx_n\}$ and $\{gy_n\}$ are respectively G-convergent to x and y, then, using Lemma 1.4, we have

$$\lim_{n\to\infty} G(gx_n, gx_n, x) = \lim_{n\to\infty} G(gx_n, x, x) = 0, \tag{2.22}$$

$$\lim_{n\to\infty} G(gy_n, gy_n, y) = \lim_{n\to\infty} G(gy_n, y, y) = 0. \tag{2.23}$$

Using the G-continuity of g, Definition 1.8 and Lemma 1.4, we have

$$\lim_{n\to\infty} G(ggx_n, ggx_n, gx) = \lim_{n\to\infty} G(ggx_n, gx, gx) = 0, \tag{2.24}$$

$$\lim_{n\to\infty} G(ggy_n, ggy_n, gy) = \lim_{n\to\infty} G(ggy_n, gy, gy) = 0. \tag{2.25}$$

Since $gx_{n+1} = F(x_n, y_n)$ and $gy_{n+1} = F(y_n, x_n)$, hence the commutativity of F and g implies that

$$ggx_{n+1} = gF(x_n, y_n) = F(gx_n, gy_n), \tag{2.26}$$

$$ggy_{n+1} = gF(y_n, x_n) = F(gy_n, gx_n). \tag{2.27}$$

Since the mapping F is G-continuous, and the sequences $\{gx_n\}$ and $\{gy_n\}$ are respectively G-convergent to x and y, hence using Definition 1.11, the sequence $\{F(gx_n, gy_n)\}$ is G-convergent to $F(x, y)$. Then, by uniqueness of the limit, and using (2.24), (2.26), we finally get $F(x, y) = gx$. Similarly, we can show that $F(y, x) = gy$. Hence (x, y) is a coupled coincidence point of F and g. □

Taking g to be an identity mapping in Theorem 2.1, we have the following corollary.

Corollary 2.1 *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping. Assume that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\phi\big(G\big(F(x,y), F(u,v), F(w,z)\big)\big) \leq \frac{1}{2}\phi\big(G(x,u,w) + G(y,v,z)\big) - \psi\left(\frac{G(x,u,w) + G(y,v,z)}{2}\right) \tag{2.28}$$

for all $x, y, u, v, w, z \in X$ with $x \geq u \geq w$ and $y \leq v \leq z$.

Assume that F satisfies the following conditions:

(1) *F has the mixed monotone property,*

(2) *F is continuous.*

Suppose that there exist $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Taking ϕ and g to be identity mappings in Theorem 2.1, we have the following corollary.

Corollary 2.2 *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ be a mapping. Assume there exists $\psi \in \Psi$ such that*

$$G\big(F(x,y), F(u,v), F(w,z)\big) \leq \frac{1}{2}\big(G(x,u,w) + G(y,v,z)\big) - \psi\left(\frac{G(x,u,w) + G(y,v,z)}{2}\right) \tag{2.29}$$

for all $x, y, u, v, w, z \in X$ with $x \geq u \geq w$ and $y \leq v \leq z$.

Assume that F satisfies the following conditions:

(1) *F has the mixed monotone property,*

(2) *F is continuous.*

Suppose there exist $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Taking ϕ to be an identity mapping and $\psi(t) = (1-k)t$, $0 \leq k < 1$ in Theorem 2.1, we have the following result.

Corollary 2.3 *Let $(X, \leq)$ be a partially ordered set and suppose there exists a G-metric G on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$, $g : X \to X$ be two mappings. Assume there exists a real number $k \in [0,1)$ such that*

$$G\big(F(x,y), F(u,v), F(w,z)\big) \leq \frac{k}{2}\big[G(gx, gu, gw) + G(gy, gv, gz)\big] \tag{2.30}$$

for all x, y, u, v, w, z in X with $gx \geq gu \geq gw$, $gy \leq gv \leq gz$.

Assume that F and g satisfy the following conditions:

(1) *$F(X \times X) \subseteq g(X)$,*

(2) *F has the mixed g-monotone property,*

(3) *F is continuous,*

(4) *g is continuous and commutes with F.*

Suppose that there exist $x_0, y_0 \in X$ with $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then F and g have a coupled coincidence point in X, that is, there exist $x, y \in X$ such that $gx = F(x,y)$ and $gy = F(y,x)$.

Remark 2.1 Corollary 2.3 is an extension of Theorem 3.1 of Choudhury and Maity [10] for a pair of commuting mappings. Further, taking g to be the identity mapping in Corollary 2.3, we obtain Theorem 3.1 of Choudhury and Maity [10].

In the next theorem, we omit the continuity hypotheses of F. We need the following definition.

Definition 2.1 Let $(X, \leq)$ be a partially ordered set and suppose there exists a G-metric G on X. We say that $(X, G, \leq)$ is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \leq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y \leq y_n$ for all n.

Theorem 2.2 *Let $(X, \leq)$ be a partially ordered set, and suppose there exists a G-metric G on X. Let $F : X \times X \to X$, $g : X \to X$ be two mappings. Assume there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\begin{aligned} \phi\big(G\big(F(x,y), F(u,v), F(w,z)\big)\big) \leq{}& \frac{1}{2}\phi\big(G(gx, gu, gw) + G(gy, gv, gz)\big) \\ & - \psi\left(\frac{G(gx, gu, gw) + G(gy, gv, gz)}{2}\right) \end{aligned} \tag{2.31}$$

for all $x, y, u, v, w, z \in X$ with $gx \geq gu \geq gw$ and $gy \leq gv \leq gz$.

Assume that $(X, G, \leq)$ is regular. Suppose that $(g(X), G)$ is G-complete, F has the mixed g-monotone property and $F(X \times X) \subseteq g(X)$. Suppose that there exist $x_0, y_0 \in X$ with $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then F and g have a coupled coincidence point in X, that is, there exist $x, y \in X$ such that $gx = F(x,y)$ and $gy = F(y,x)$.

Proof Proceeding exactly as in Theorem 2.1, we have that $\{gx_n\}$ and $\{gy_n\}$ are G-Cauchy sequences in the G-complete G-metric space $(g(X), G)$. Then there exist $x, y \in X$ such that

$gx_n \to gx$ and $gy_n \to gy$, that is,

$$\begin{aligned} &\lim_{n\to\infty} G(gx_n, gx, gx) = \lim_{n\to\infty} G(gx_n, gx_n, gx) = 0, \\ &\lim_{n\to\infty} G(gy_n, gy, gy) = \lim_{n\to\infty} G(gy_n, gy_n, gy) = 0. \end{aligned} \tag{2.32}$$

Since $\{gx_n\}$ is non-decreasing and $\{gy_n\}$ is non-increasing, using the regularity of $(X, G, \leq)$, we have $gx_n \leq gx$ and $gy \leq gy_n$ for all $n \geq 0$. Using (2.31), we get

$$\begin{aligned} \phi\bigl(G\bigl(F(x,y), gx_{n+1}, gx_{n+1}\bigr)\bigr) &= \phi\bigl(G\bigl(F(x,y), F(x_n, y_n), F(x_n, y_n)\bigr)\bigr) \\ &\leq \frac{1}{2}\phi\bigl(G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)\bigr) \\ &\quad - \psi\left(\frac{G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)}{2}\right). \end{aligned} \tag{2.33}$$

Letting $n \to \infty$ in (2.33), then using (2.32) and the properties of ϕ and ψ, we obtain that

$$\begin{aligned} \phi\Bigl(\lim_{n\to\infty} G\bigl(F(x,y), gx_{n+1}, gx_{n+1}\bigr)\Bigr) &\leq \frac{1}{2}\phi\Bigl(\lim_{n\to\infty}\bigl(G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)\bigr)\Bigr) \\ &\quad - \lim_{n\to\infty} \psi\left(\frac{G(gx, gx_n, gx_n) + G(gy, gy_n, gy_n)}{2}\right) \\ &= 0, \end{aligned}$$

which gives us

$$\lim_{n\to\infty} G\bigl(F(x,y), gx_{n+1}, gx_{n+1}\bigr) = 0. \tag{2.34}$$

On the other hand, by condition (G5) we have

$$G\bigl(F(x,y), gx, gx\bigr) \leq G\bigl(F(x, y), gx_{n+1}, gx_{n+1}\bigr) + G(gx_n, gx, gx).$$

Letting $n \to \infty$, using (2.32) and (2.34), we have $G(F(x,y), gx, gx) = 0$. So $F(x,y) = gx$. Similarly, we can obtain that $gy = F(y,x)$. Thus, we proved that (x,y) is a coupled coincidence point of F and g. □

Taking ϕ to be the identity mapping and $\psi(t) = (1-k)t$, $0 \leq k < 1$ in Theorem 2.2, we have the following result.

Corollary 2.4 *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a G-metric G on X. Let $F : X \times X \to X$, $g : X \to X$ be two mappings. Assume that there exists a real number $k \in [0,1)$ such that*

$$G\bigl(F(x,y), F(u,v), F(w,z)\bigr) \leq \frac{k}{2}\bigl[G(gx, gu, gw) + G(gy, gv, gz)\bigr] \tag{2.35}$$

for all x, y, u, v, w, z in X with $gx \geq gu \geq gw$, $gy \leq gv \leq gz$.

Assume that $(X, G, \leq)$ is regular. Suppose that $(g(X), G)$ is G-complete, F has the mixed g-monotone property and $F(X \times X) \subseteq g(X)$. Also, assume that there exist $x_0, y_0 \in X$ with

$gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, then F and g have a coupled coincidence point in X, that is, there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$.

Remark 2.2 Corollary 2.4 is an extension of the Theorem 3.2 of Choudhury and Maity [10] for a pair of commuting mappings. Further, taking g to be the identity mapping in Corollary 2.4, we can obtain Theorem 3.2 of Choudhury and Maity [10].

Taking ϕ and g to be identity mappings in Theorem 2.2, we have the following result.

Corollary 2.5 *Let $(X, \leq)$ be a partially ordered set, and suppose that there is a G-metric G on X such that (X, G) is a complete metric space. Let $F : X \times X \to X$ be a mapping having mixed a monotone property. Assume that there exists $\psi \in \Psi$ such that*

$$\begin{aligned} G\bigl(F(x,y), F(u,v), F(w,z)\bigr) \leq{}& \frac{1}{2}\bigl(G(x,u,w) + G(y,v,z)\bigr) \\ &- \psi\left(\frac{G(x,u,w) + G(y,v,z)}{2}\right) \end{aligned} \tag{2.36}$$

for all $x, y, u, v, w, z \in X$ with $x \geq u \geq w$ and $y \leq v \leq z$.

Assume that $(X, G, \leq)$ is regular. Suppose that there exist $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Next, we give an example in support of Theorem 2.2.

Example 2.1 Let $X = [0, 1]$. Then $(X, \leq)$ is a partially ordered set with the natural ordering of real numbers. Let $G : X \times X \times X \to R^+$ be defined by

$$G(x, y, z) = |x - y| + |y - z| + |z - x| \quad \text{for } x, y, z \in X.$$

Then (X, G) is a regular G-metric space.

Let $g : X \to X$ be defined as

$$g(x) = \frac{x}{2} \quad \text{for all } x \in X.$$

Let $F : X \times X \to X$ be defined as

$$F(x, y) = \begin{cases} \frac{x-y}{24}, & \text{if } x, y \in [0, 1], x \geq y, \\ 0, & \text{if } x < y. \end{cases}$$

Clearly, F obeys the mixed g-monotone property. Also, $F(X \times X) \subseteq g(X)$ and $(g(X), G)$ is complete.

Let $\phi, \psi : [0, \infty) \to [0, \infty)$ be defined by $\phi(t) = \frac{t}{2}$, $\psi(t) = \frac{t}{4}$ for $t \in [0, \infty)$.

Also, $x_0 = 0$ and $y_0 = c$ (> 0) are two points in X such that $g(x_0) = g(0) = 0 = F(0, c) = F(x_0, y_0)$ and $g(y_0) = g(c) = \frac{c}{2} \geq \frac{c}{24} = F(c, 0) = F(y_0, x_0)$.

Next, we verify inequality (2.31) of Theorem 2.2.

We take $x, y, u, v, w, z \in X$ such that $gx \geq gu \geq gw$ and $gy \leq gv \leq gz$; that is, $x \geq u \geq w$ and $y \leq v \leq z$. We discuss the following cases.

Case 1: $x \geq y$, $u \geq v$, $w \geq z$.
Then

$$\begin{aligned}
&\phi\big(G\big(F(x,y),F(u,v),F(w,z)\big)\big)\\
&\quad= \phi\left(G\left(\frac{x-y}{24},\frac{u-v}{24},\frac{w-z}{24}\right)\right)\\
&\quad= \frac{1}{2}\left\{\frac{|(x-y)-(u-v)|}{24}+\frac{|(u-v)-(w-z)|}{24}+\frac{|(w-z)-(x-y)|}{24}\right\}\\
&\quad= \frac{1}{48}\big\{\big|(x-u)-(y-v)\big|+\big|(u-w)-(v-z)\big|+\big|(w-x)-(z-y)\big|\big\}\\
&\quad\leq \frac{1}{48}\big\{(x-u)+(v-y)+(u-w)+(z-v)+(x-w)+(z-y)\big\}\\
&\quad= \frac{1}{24}\left\{\left(\frac{(x-u)}{2}+\frac{(u-w)}{2}+\frac{(x-w)}{2}\right)+\left(\frac{(v-y)}{2}+\frac{(z-v)}{2}+\frac{(z-y)}{2}\right)\right\}\\
&\quad= \frac{1}{24}\big\{G(gx,gu,gw)+G(gy,gv,gz)\big\}\\
&\quad\leq \frac{1}{8}\big\{G(gx,gu,gw)+G(gy,gv,gz)\big\}\\
&\quad= \frac{1}{2}\phi\big(G(gx,gu,gw)+G(gy,gv,gz)\big)-\psi\left(\frac{G(gx,gu,gw)+G(gy,gv,gz)}{2}\right).
\end{aligned}$$

Case 2: $x \geq y$, $u \geq v$, $w < z$.
Then

$$\begin{aligned}
&\phi\big(G\big(F(x,y),F(u,v),F(w,z)\big)\big)\\
&\quad= \phi\left(G\left(\frac{x-y}{24},\frac{u-v}{24},0\right)\right)\\
&\quad= \frac{1}{2}\left\{\frac{|(x-y)-(u-v)|}{24}+\frac{|(u-v)|}{24}+\frac{|(x-y)|}{24}\right\}\\
&\quad= \frac{1}{48}\big\{\big|(x-u)-(y-v)\big|+\big|(u-v)\big|+\big|(x-y)\big|\big\}\\
&\quad\leq \frac{1}{48}\big\{(x-u)+(v-y)+(u-v)+(x-y)\big\}\\
&\quad= \frac{1}{48}\big\{(x-u)+(v-y)+(u-w+w-v)+(x-w+w-y)\big\}\\
&\quad= \frac{1}{48}\big\{(x-u)+(v-y)+(u-w)+(w-v)+(x-w)+(w-y)\big\}\\
&\quad\leq \frac{1}{48}\big\{(x-u)+(v-y)+(u-w)+(z-v)+(x-w)+(z-y)\big\}\\
&\quad= \frac{1}{24}\left\{\left(\frac{(x-u)}{2}+\frac{(u-w)}{2}+\frac{(x-w)}{2}\right)+\left(\frac{(v-y)}{2}+\frac{(z-v)}{2}+\frac{(z-y)}{2}\right)\right\}\\
&\quad= \frac{1}{24}\big\{G(gx,gu,gw)+G(gy,gv,gz)\big\}\\
&\quad\leq \frac{1}{8}\big\{G(gx,gu,gw)+G(gy,gv,gz)\big\}
\end{aligned}$$

$$= \frac{1}{2}\phi\big(G(gx,gu,gw) + G(gy,gv,gz)\big) - \psi\left(\frac{G(gx,gu,gw) + G(gy,gv,gz)}{2}\right).$$

Case 3: $x \geq y$, $u < v$, $w < z$.
Then

$$\begin{aligned}
&\phi\big(G\big(F(x,y),F(u,v),F(w,z)\big)\big) \\
&\quad = \phi\left(G\left(\frac{x-y}{24},0,0\right)\right) \\
&\quad = \frac{1}{2}\left\{\frac{|(x-y)|}{24} + \frac{|(x-y)|}{24}\right\} = \frac{1}{2}\left\{\frac{(x-y)}{24} + \frac{(x-y)}{24}\right\} \\
&\quad = \frac{1}{48}\big\{(x-u+u-y) + (x-w+w-y)\big\} \\
&\quad = \frac{1}{48}\big\{(x-u) + (u-y) + (x-w) + (w-y)\big\} \\
&\quad \leq \frac{1}{48}\big\{(x-u) + (v-y) + (x-w) + (w-u+u-y)\big\} \\
&\quad = \frac{1}{48}\big\{(x-u) + (v-y) + (x-w) + (w-u) + (u-y)\big\} \\
&\quad = \frac{1}{48}\big\{(x-u) + (v-y) + (x-w) + (w-u) + (u-z+z-y)\big\} \\
&\quad \leq \frac{1}{48}\big\{(x-u) + (v-y) + (x-w) + (u-w) + (v-z) + (z-y)\big\} \\
&\quad = \frac{1}{24}\left\{\left(\frac{(x-u)}{2} + \frac{(u-w)}{2} + \frac{(x-w)}{2}\right) + \left(\frac{(v-y)}{2} + \frac{(z-v)}{2} + \frac{(z-y)}{2}\right)\right\} \\
&\quad = \frac{1}{24}\big\{G(gx,gu,gw) + G(gy,gv,gz)\big\} \\
&\quad \leq \frac{1}{8}\big\{G(gx,gu,gw) + G(gy,gv,gz)\big\} \\
&\quad = \frac{1}{2}\phi\big(G(gx,gu,gw) + G(gy,gv,gz)\big) - \psi\left(\frac{G(gx,gu,gw) + G(gy,gv,gz)}{2}\right).
\end{aligned}$$

Case 4: $x < y$, $u < v$, $w < z$.
Then $\phi(G(F(x,y),F(u,v),F(w,z))) = \phi(0) = 0$, and hence inequality (2.31) of Theorem 2.2 is obvious.

Similarly, the cases like $x < y$, $u \geq v$, $w \geq z$; $x < y$, $u < v$, $w \geq z$ and others follow immediately.

Thus, it is verified that the functions F, g, ϕ, ψ satisfy all the conditions of Theorem 2.2. Indeed, $(0,0)$ is the coupled coincidence point of F and g in X.

Next, we prove the existence and uniqueness of the coupled common fixed point for our main result.

Theorem 2.3 *In addition to the hypotheses of Theorem* 2.1, *suppose that for every* $(x,y),(x^*,y^*) \in X \times X$ *there exists a* $(u,v) \in X \times X$ *such that* $(F(u,v),F(v,u))$ *is comparable* *to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(a,b) \in X \times X$ such that $a = g(a) = F(a,b)$ and $b = g(b) = F(b,a)$.*

Proof From Theorem 2.1, the set of coupled coincidences is non-empty. In order to prove the theorem, we shall first show that if (x,y) and (x^*,y^*) are coupled coincidence points, that is, if $g(x) = F(x,y)$, $gy = F(y,x)$ and $gx^* = F(x^*,y^*)$, $gy^* = F(y^*,x^*)$, then

$$g(x) = gx^* \quad \text{and} \quad gy = gy^*. \tag{2.37}$$

By assumption, there is $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable with $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0,v_0)$, $gv_1 = F(v_0,u_0)$.

Then, similarly as in the proof of Theorem 2.1, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ such that $gu_{n+1} = F(u_n,v_n)$ and $gv_{n+1} = F(v_n,u_n)$.

Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$, $y_0^* = y^*$ and, on the same way, define the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gx_n^*\}$, $\{gy_n^*\}$. Then it is easy to show that

$$\begin{aligned} &gx_{n+1} = F(x_n,y_n), \qquad gy_{n+1} = F(y_n,x_n) \quad \text{and} \\ &gx_{n+1}^* = F\bigl(x_n^*,y_n^*\bigr), \qquad gy_{n+1}^* = F\bigl(y_n^*,x_n^*\bigr) \quad \text{for all } n \geq 0. \end{aligned}$$

Since $(F(x,y),F(y,x)) = (gx_1,gy_1) = (gx,gy)$ and $(F(u,v),F(v,u)) = (gu_1,gv_1)$ are comparable, then $gu_1 \geq gx$ and $gv_1 \leq gy$. It is easy to show that (gx,gy) and (gu_n,gv_n) are comparable, that is, $gu_n \geq gx$ and $gv_n \leq gy$ for all $n \geq 1$. Thus, from (2.1)

$$\begin{aligned} \phi\bigl(G(gu_{n+1},gu_{n+1},gx)\bigr) &= \phi(G\bigl(F(u_n,v_n),F(u_n,v_n),F(x,y)\bigr) \\ &\leq \frac{1}{2}\phi\bigl(G(gu_n,gu_n,gx) + G(gv_n,gv_n,gy)\bigr) \\ &\quad - \psi\left(\frac{G(gu_n,gu_n,gx) + G(gv_n,gv_n,gy)}{2}\right), \end{aligned} \tag{2.38}$$

and

$$\begin{aligned} \phi\bigl(G(gv_{n+1},gv_{n+1},gy)\bigr) &\leq \frac{1}{2}\phi\bigl(G(gv_n,gv_n,gy) + G(gu_n,gu_n,gx)\bigr) \\ &\quad - \psi\left(\frac{G(gv_n,gv_n,gy) + G(gu_n,gu_n,gx)}{2}\right). \end{aligned} \tag{2.39}$$

Adding (2.38) and (2.39), we get

$$\begin{aligned} &\phi\bigl(G(gu_{n+1},gu_{n+1},gx)\bigr) + \phi\bigl(G(gv_{n+1},gv_{n+1},gy)\bigr) \\ &\quad \leq \phi\bigl(G(gu_n,gu_n,gx) + G(gv_n,gv_n,gy)\bigr) \\ &\qquad - 2\psi\left(\frac{G(gu_n,gu_n,gx) + G(gv_n,gv_n,gy)}{2}\right). \end{aligned} \tag{2.40}$$

Also, by (ϕ_{iii}) we have

$$\begin{aligned}
&\phi\bigl(G(gu_{n+1},gu_{n+1},gx)+G(gv_{n+1},gv_{n+1},gy)\bigr)\\
&\quad\le \phi\bigl(G(gu_{n+1},gu_{n+1},gx)\bigr)+\phi\bigl(G(gv_{n+1},gv_{n+1},gy)\bigr).
\end{aligned}\tag{2.41}$$

From (2.40) and (2.41),

$$\begin{aligned}
&\phi\bigl(G(gu_{n+1},gu_{n+1},gx)+G(gv_{n+1},gv_{n+1},gy)\bigr)\\
&\quad\le \phi\bigl(G(gu_n,gu_n,gx)+G(gv_n,gv_n,gy)\bigr)\\
&\qquad -2\psi\left(\frac{G(gu_n,gu_n,gx)+G(gv_n,gv_n,gy)}{2}\right) &(2.42)\\
&\quad\le \phi\bigl(G(gu_n,gu_n,gx)+G(gv_n,gv_n,gy)\bigr). &(2.43)
\end{aligned}$$

Since ϕ is non-decreasing, from (2.43), it follows that

$$G(gu_{n+1},gu_{n+1},gx)+G(gv_{n+1},gv_{n+1},gy)\le G(gu_n,gu_n,gx)+G(gv_n,gv_n,gy).$$

Let $\alpha_n = G(gu_n,gu_n,gx)+G(gv_n,gv_n,gy)$, then $\{\alpha_n\}$ is a monotonic decreasing sequence, so there exists some $\alpha\ge 0$ such that $\lim_{n\to\infty}\alpha_n=\alpha$.

We shall show that $\alpha = 0$. Suppose, on the contrary, that $\alpha > 0$. Then taking limit as $n\to\infty$, in (2.42) and using the continuity of ϕ and the property (ψ_{i}), we have

$$\phi(\alpha)\le\phi(\alpha)-2\lim_{\alpha n\to\alpha}\psi\left(\frac{\alpha_n}{2}\right)<\phi(\alpha).$$

A contradiction. Thus, $\alpha = 0$, that is,

$$\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\bigl(G(gu_n,gu_n,gx)+G(gv_n,gv_n,gy)\bigr)=0.$$

Hence, it follows that $gu_n\to gx$, $gv_n\to gy$.

Similarly, one can show that $gu_n\to gx^*$, $gv_n\to gy^*$.

By uniqueness of limit, it follows that $gx = gx^*$ and $gy = gy^*$. Thus, we have proved (2.37).

Since $gx = F(x,y)$, $gy = F(y,x)$ and the pair (F,g) is commuting, it follows that

$$ggx = gF(x,y) = F(gx,gy)\quad\text{and}\quad ggy = gF(y,x) = F(gy,gx).\tag{2.44}$$

Denote $gx = z$, $gy = w$. Then from (2.44), we have

$$gz = F(z,w)\quad\text{and}\quad gw = F(w,z).\tag{2.45}$$

Thus, (z,w) is a coupled coincidence point.

Then from (2.37) with $x^* = z$ and $y^* = w$, it follows that $gz = gx$ and $gw = gy$, that is,

$$gz = z,\qquad gw = w.\tag{2.46}$$

From (2.45) and (2.46), we have

$$z = gz = F(z, w) \quad \text{and} \quad w = gw = F(w, z).$$

Therefore, (z, w) is the coupled common fixed point of F and g.

To prove the uniqueness, assume that (p, q) is another coupled common fixed point. Then by (2.37), we have $p = gp = gz = z$ and $q = gq = gw = w$. □

Theorem 2.4 *Under the hypotheses of Theorem* 2.2, *suppose*, *in addition*, *that for every* (x, y) *and* (x^*, y^*) *in* X, *there exists* $(u, v) \in X \times X$ *such that* $(F(u, v), F(v, u))$ *is comparable to* $(F(x, y), F(y, x))$ *and* $(F(x^*, y^*), F(y^*, x^*))$. *If* F *and* g *are commuting*, *then* F *and* g *have a unique coupled common fixed point*, *that is*, *there exists a unique* $(a, b) \in X \times X$ *such that* $a = g(a) = F(a, b)$ *and* $b = g(b) = F(b, a)$.

Proof Following the steps of Theorem 2.3, proof follows immediately. □

3 Application to integral equations

Motivated by the work of Aydi *et al.* [15], in this section, we study the existence of solutions to nonlinear integral equations using some of our main results.

Consider the integral equations in the following system:

$$\begin{aligned} x(t) &= p(t) + \int_0^T S(t, s)\big[f\big(s, x(s)\big) + k\big(s, y(s)\big)\big]\, ds, \\ y(t) &= p(t) + \int_0^T S(t, s)\big[f\big(s, y(s)\big) + k\big(s, x(s)\big)\big]\, ds. \end{aligned} \tag{3.1}$$

As defined by Luong *et al.* [31], let Θ denote the class of those functions $\theta : [0, \infty) \to [0, \infty)$, which satisfy the following conditions:

(I) θ is increasing;

(II) There exists $\psi \in \Psi$ such that $\theta(r) = \frac{r}{2} - \psi(\frac{r}{2})$, for all $r \in [0, \infty)$.

For example, $\theta_1(x) = \alpha x$, where $0 \leq \alpha \leq \frac{1}{2}$, $\theta_2(x) = \frac{x^2}{2(x+1)}$ are some members of Θ.

We shall analyze the system (3.1) under the following assumptions:

(i) $f, k : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous,

(ii) $p : [0, T] \to \mathbb{R}$ is continuous,

(iii) $S : [0, T] \times \mathbb{R} \to [0, \infty)$ is continuous,

(iv) there exists $\lambda > 0$ and $\theta \in \Theta$ such that for all $x, y \in \mathbb{R}$, $y \geq x$,

$$\begin{aligned} &0 \leq f(s, y) - f(s, x) \leq \lambda\theta(y - x), \\ &0 \leq k(s, x) - k(s, y) \leq \lambda\theta(y - x). \end{aligned}$$

(v) We suppose that

$$3\lambda \sup_{t \in [0,T]} \int_0^T S(t, s)\, ds \leq \frac{1}{2}.$$

(vi) There exist continuous functions $\alpha, \beta : [0, T] \to \mathbb{R}$ such that

$$\alpha(t) \le p(t) + \int_a^b S(t,s)\big(f\big(s,\alpha(s)\big) + k\big(s,\beta(s)\big)\big)\,ds,$$
$$\beta(t) \ge p(t) + \int_a^b S(t,s)\big(f\big(s,\beta(s)\big) + k\big(s,\alpha(s)\big)\big)\,ds.$$

Consider the space $X = C([0,T],\mathbb{R})$ of continuous functions defined on $[0,T]$ endowed with the (G-complete) G-metric given by

$$G(u,v,w) = \sup_{t\in[0,T]}\big|u(t) - v(t)\big| + \sup_{t\in[0,T]}\big|v(t) - w(t)\big| + \sup_{t\in[0,T]}\big|w(t) - u(t)\big|$$

for all $u, v, w \in X$.

Endow X with the partial order $\le$ given by: $x, y \in X$, $x \le y \Leftrightarrow x(t) \le y(t)$ for all $t \in [0,T]$. Also, we may adjust as in [25] to prove that $(X, G, \le)$ is regular.

Theorem 3.1 *Under assumptions* (i)-(vi), *the system* (3.1) *has a solution in* $X^2 = (C([0,T],\mathbb{R}))^2$.

Proof Consider the operator $F : X \times X \to X$ defined by

$$F(x,y)(t) = p(t) + \int_0^T S(t,s)\big[f\big(s,x(s)\big) + k\big(s,y(s)\big)\big]\,ds, \quad t \in [0,T], \text{for all } x, y \in X.$$

First, we shall prove that F has the mixed monotone property.

In fact, for $x_1 \le x_2$ and $t \in [0,T]$, we have

$$F(x_2,y)(t) - F(x_1,y)(t) = \int_0^T S(t,s)\big[f\big(s,x_2(s)\big) - f\big(s,x_1(s)\big)\big]\,ds.$$

Taking into account that $x_1(t) \le x_2(t)$ for all $t \in [0,T]$, so by (iv), $f(s,x_2(s)) \ge f(s,x_1(s))$. Then $F(x_2,y)(t) \ge F(x_1,y)(t)$ for all $t \in [0,T]$, that is,

$$F(x_1,y) \le F(x_2,y).$$

Similarly, for $y_1 \le y_2$ and $t \in [0,T]$, we have

$$F(x,y_1)(t) - F(x,y_2)(t) = \int_0^T S(t,s)\big[k\big(s,y_1(s)\big) - k\big(s,y_2(s)\big)\big]\,ds.$$

Having $y_1(t) \le y_2(t)$, so by (iv), $k(s,y_1(s)) \ge k(s,y_2(s))$. Then $F(x,y_1)(t) \ge F(x,y_2)(t)$ for all $t \in [0,T]$, that is, $F(x,y_1) \ge F(x,y_2)$. Therefore, F has the mixed monotone property.

In what follows, we estimate the quantity $G(F(x,y),F(u,v),F(w,z))$ for all $x,y,u,v,w,z \in X$, with $x \ge u \ge w$, $y \le v \le z$. Since F has the mixed monotone property, we have

$$F(w,z) \le F(u,v) \le F(x,y).$$

We obtain

$$\begin{aligned}
&G\bigl(F(x,y),F(u,v),F(w,z)\bigr)\\
&\quad= \sup_{t\in[0,T]}\bigl|F(x,y)(t)-F(u,v)(t)\bigr| + \sup_{t\in[0,T]}\bigl|F(u,v)(t)-F(w,z)(t)\bigr|\\
&\qquad+ \sup_{t\in[0,T]}\bigl|F(w,z)(t)-F(x,y)(t)\bigr|\\
&\quad= \sup_{t\in[0,T]}\bigl(F(x,y)(t)-F(u,v)(t)\bigr) + \sup_{t\in[0,T]}\bigl(F(u,v)(t)-F(w,z)(t)\bigr)\\
&\qquad+ \sup_{t\in[0,T]}\bigl(F(x,y)(t)-F(w,z)(t)\bigr).
\end{aligned}$$

Also, for all $t\in[0,T]$, from (iv), we have

$$\begin{aligned}
F(x,y)-F(u,v) &= \int_0^T (t,s)\bigl[f\bigl(s,x(s)\bigr)-f\bigl(s,u(s)\bigr)\bigr]\,ds\\
&\quad+\int_0^T S(t,s)\bigl[k\bigl(s,y(s)\bigr)-k\bigl(s,v(s)\bigr)\bigr]\,ds\\
&\le \lambda\int_0^T S(t,s)\bigl[\theta\bigl(x(s)-u(s)\bigr)+\theta\bigl(v(s)-y(s)\bigr)\bigr]\,ds.
\end{aligned}\tag{3.2}$$

Since the function θ is increasing and $x\ge u\ge w$, $y\le v\le z$, we have

$$\begin{aligned}
&\theta\bigl(x(s)-u(s)\bigr)\le\theta\Bigl(\sup_{t\in I}\bigl|x(t)-u(t)\bigr|\Bigr),\\
&\theta\bigl(v(s)-y(s)\bigr)\le\theta\Bigl(\sup_{t\in I}\bigl|v(t)-y(t)\bigr|\Bigr),
\end{aligned}$$

hence by (3.2), we obtain

$$\begin{aligned}
&\bigl|F(x,y)-F(u,v)\bigr|\\
&\quad\le\lambda\int_0^T S(t,s)\Bigl[\theta\Bigl(\sup_{t\in I}\bigl|x(t)-u(t)\bigr|\Bigr)+\theta\Bigl(\sup_{t\in I}\bigl|v(t)-y(t)\bigr|\Bigr)\Bigr]\,ds,
\end{aligned}\tag{3.3}$$

as all the quantities on the right hand side of (3.2) are non-negative, so (3.3) is justified.

Similarly, we can obtain that

$$\bigl|F(x,y)-F(w,z)\bigr|\le\lambda\int_0^T S(t,s)\Bigl[\theta\Bigl(\sup_{t\in I}\bigl|x(t)-w(t)\bigr|\Bigr)+\theta\Bigl(\sup_{t\in I}\bigl|z(t)-y(t)\bigr|\Bigr)\Bigr]\,ds,\tag{3.4}$$

$$\bigl|F(w,z)-F(u,v)\bigr|\le\lambda\int_0^T S(t,s)\Bigl[\theta\Bigl(\sup_{t\in I}\bigl|u(t)-w(t)\bigr|\Bigr)+\theta\Bigl(\sup_{t\in I}\bigl|z(t)-v(t)\bigr|\Bigr)\Bigr]\,ds.\tag{3.5}$$

Summing (3.3), (3.4) and (3.5), and then taking the supremum with respect to t we get, by using (v), we obtain that

$$\begin{aligned}
&G\bigl(F(x,y),F(u,v),F(w,z)\bigr)\\
&\quad\le\lambda\sup_{t\in[0,T]}\int_0^T S(t,s)\,ds\cdot\Bigl[\theta\Bigl(\sup_{t\in I}\bigl|x(t)-u(t)\bigr|\Bigr)+\theta\Bigl(\sup_{t\in I}\bigl|x(t)-w(t)\bigr|\Bigr)
\end{aligned}$$

$$\begin{aligned}
&+ \theta\Big(\sup_{t\in I}\big|u(t)-w(t)\big|\Big)\Big] \\
&+ \lambda \sup_{t\in[0,T]}\int_0^T S(t,s)\,ds \cdot \Big[\theta\Big(\sup_{t\in I}\big|v(t)-y(t)\big|\Big) + \theta\Big(\sup_{t\in I}\big|z(t)-y(t)\big|\Big) \\
&+ \theta\Big(\sup_{t\in I}\big|z(t)-v(t)\big|\Big)\Big].
\end{aligned} \tag{3.6}$$

Further, since θ is increasing, so we have

$$\begin{aligned}
&\theta\Big(\sup_{t\in I}\big|x(t)-u(t)\big|\Big) \le \theta\big(G(x,u,w)\big), \qquad \theta\Big(\sup_{t\in I}\big|x(t)-w(t)\big|\Big) \le \theta\big(G(x,u,w)\big), \\
&\theta\Big(\sup_{t\in I}\big|u(t)-w(t)\big|\Big) \le \theta\big(G(x,u,w)\big).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\theta\Big(\sup_{t\in I}\big|v(t)-y(t)\big|\Big) \le \theta\big(G(y,v,z)\big), \qquad \theta\Big(\sup_{t\in I}\big|z(t)-y(t)\big|\Big) \le \theta\big(G(y,v,z)\big), \\
&\theta\Big(\sup_{t\in I}\big|z(t)-v(t)\big|\Big) \le \theta\big(G(y,v,z)\big).
\end{aligned}$$

Then by (3.6), we have

$$\begin{aligned}
&G\big(F(x,y),F(u,v),F(w,z)\big) \\
&\quad\le \lambda \sup_{t\in[0,T]}\int_0^T S(t,s)\,ds \cdot 3\theta\big(G(x,u,w)\big) + \lambda \sup_{t\in[0,T]}\int_0^T S(t,s)\,ds \cdot 3\theta\big(G(y,v,z)\big) \\
&\quad= 3\lambda \sup_{t\in[0,T]}\int_0^T S(t,s)\,ds \cdot \big(\theta\big(G(x,u,w)\big) + \theta\big(G(y,v,z)\big)\big) \\
&\quad\le \frac{(\theta(G(x,u,w)) + \theta(G(y,v,z)))}{2}.
\end{aligned} \tag{3.7}$$

Since θ is increasing, we have

$$\theta\big(G(x,u,w)\big) \le \theta\big(G(x,u,w)+G(y,v,z)\big), \qquad \theta\big(G(y,v,z)\big) \le \theta\big(G(x,u,w)+G(y,v,z)\big)$$

and so

$$\begin{aligned}
\frac{(\theta(G(x,u,w)) + \theta(G(y,v,z)))}{2} &\le \theta\big(G(x,u,w)+G(y,v,z)\big) \\
&= \frac{G(x,u,w)+G(y,v,z)}{2} \\
&\quad - \psi\left(\frac{G(x,u,w)+G(y,v,z)}{2}\right),
\end{aligned} \tag{3.8}$$

by definition of θ. Thus, by (3.7) and (3.8), we finally get

$$G\big(F(x,y),F(u,v),F(w,z)\big) \le \frac{G(x,u,w)+G(y,v,z)}{2} - \psi\left(\frac{G(x,u,w)+G(y,v,z)}{2}\right)$$

which is just the contractive condition (2.33) in Corollary 2.5.

Let α, β be the functions appearing in assumption (vi), then by (vi), we get

$$\alpha \leq F(\alpha, \beta), \qquad \beta \geq F(\beta, \alpha).$$

Applying Corollary 2.5, we deduce the existence of $x, y \in X$ such that

$$x = F(x, y), \qquad y = F(y, x),$$

that is, (x, y) is a solution of the system (3.1). □

4 Conclusion

In the frame-work of ordered generalized metric spaces, we established some coupled coincidence and common coupled fixed point theorems for the mixed g-monotone mappings satisfying (ϕ, ψ)-contractive conditions. We accompanied our theoretical results by an applied example and an application to integral equations. Our results are extensions and generalizations of the very recent results of Choudhury *et al.* cited in [10], as well as of several results as in relevant items from the reference section of this paper and in the literature in general.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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