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Research Article

Einstein Aggregation Operators for Pythagorean Fuzzy Soft Sets with Their Application in Multiattribute Group Decision-Making

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The Pythagorean fuzzy soft set (PFSS) is the most proficient and manipulative leeway of the Pythagorean fuzzy set (PFS), which contracts with parameterized values of the alternatives. It is a generalized form of the intuitionistic fuzzy soft set (IFSS), which provides healthier and more accurate evaluations through decision-making (DM). The main determination of this research is to prolong the idea of Einstein's aggregation operators for PFSS. We introduce the Einstein operational laws for Pythagorean fuzzy soft numbers (PFSNs). Based on Einstein operational laws, we construct two novel aggregation operators (AOs) such as Pythagorean fuzzy soft Einstein-weighted averaging (PFSEWA) and Pythagorean fuzzy soft Einstein-weighted geometric (PFSEWG) operators. In addition, important possessions of proposed operators, such as idempotency, boundedness, and homogeneity, are discussed. Furthermore, to validate the practicability of the anticipated operators, a multiple attribute group decision-making (MAGDM) method is developed. We intend innovative AOs considering the Einstein norms for PFSS to elect the most subtle business. Pythagorean fuzzy soft numbers (PFSNs) support us to signify unclear data in real-world perception. Furthermore, a numerical description is planned to certify the efficacy and usability of the projected method in the DM practice. The recent approach's pragmatism, usefulness, and tractability are validated through comparative exploration with the support of some prevalent studies.

1. Introduction

MAGDM is the most applicable method for finding the adequate alternative from all conceivable alternatives. Conventionally, it is anticipated that all data retrieving alternatives according to attributes and their conforming weights are stated in crisp numbers. On the other hand, maximum judgments are taken in situations where the objectives are usually indefinite or ambiguous in real-life circumstances. To over-come such ambiguities and anxieties, Zadeh offered the concept of the fuzzy set (FS) [1], a prevailing tool to handle the obscurities and uncertainties in DM considering the membership values of the alternatives. Experts mostly consider a membership and a nonmembership value in the DM process that FS cannot handle. Atanassov [2] introduced the generalization of the FS, the idea of the intuitionistic fuzzy set (IFS) to overcome the inadequacy mentioned above. In 2011, Wang and Liu [3] presented numerous operations on IFS, such as

Einstein product and Einstein sum, and constructed two AOs. They also discussed some essential properties of these operators and utilized their proposed operator to resolve multiattribute decision-making (MADM) for the IFS information. Atanassov [4] presented a generalized form of IFS in the light of ordinary interval values, called interval-valued IFS. Garg and Kaur [5] prolonged the impression of IFS and offered a novel idea of the cubic intuitionistic fuzzy set.

The models mentioned above have been well recognized by the specialists. Still, the existing IFS cannot handle the inappropriate and vague data. For example, if decision-makers choose membership (MD) and nonmembership (NMD) 0.9 and 0.6, respectively, then $0.9 + 0.6 \geq 1$. The IFS theory mentioned above cannot be applied to this data. To resolve the limitation described above, Yager [6, 7] presented the notion of the PFS by improving the basic circumstance $a + b \leq 1$ to $a^2 + b^2 \leq 1$ and developed some results associated with the score function and accuracy function. Rahman et al. [8] presented the Einstein geometric aggregation operator and introduced a MAGDM methodology utilizing the proposed operator. Zhang and Xu [9] developed some basic operational laws and prolonged the technique for preference by similarity to the ideal solution (TOPSIS) method to resolve multicriteria decision-making (MCDM) complications under a PFS setting. Wei and Lu [10] proposed the power AOs for PFS and discussed their fundamental properties. They also offered a DM technique to resolve MADM complications using their presented operators. Wang and Li [11] protracted Bonferroni's mean AOs for PFS considering the interaction. Ilbahar et al. [12] introduced the Pythagorean fuzzy proportional risk assessment technique to assess professional health risk. Zhang [13] proposed a novel DM approach based on similarity measures to resolve PFS information's multicriteria group decision-making (MCGDM) problems. Peng and Yang [14] introduced the division and subtraction operations for Pythagorean fuzzy numbers (PFNs), proved their basic properties, and presented a superiority and inferiority ranking approach under the PFS to overcome the MAGDM complications. Garg [15, 16] introduced operational laws based on Einstein norms for PFNs, proposed weighted AOs, and ordered weighted AOs for PFS. Garg [17] introduced logarithmic operational laws for the PFS and constructed various weighted AOs based on the presented logarithm operational laws.

Gao et al. [18] settled interaction AOs under a PFS environment and gave the MADM approach to solving real-life problems. Wang et al. [19] protracted the interactive Hamacher AOs for the PFS and settled a DM method. Wang and Li [20] utilized the interval-valued PFS, presented some novel PFS operators, and offered a DM approach to resolve the MCGDM complications. Moreover, to deal with the MCDM complexities, Gao et al. [18] constructed hybrid AOs for PFS and presented a DM methodology utilizing these operators. Peng and Yuan [21] extended the AOs for PFS and introduced the generalized AOs for PFS with their desirable properties. They also constructed a MADM approach established on their advanced operators. Zulqarnain et al. [22] developed novel algorithms for multipolar neutrosophic soft sets. They utilized their established algorithms in medical diagnoses. Zulqarnain et al. [23] protracted the generalized TOPSIS method under a

neutrosophic setting to solve MCDM problems. Arora and Garg [24] presented basic operational laws for linguistic IFS and suggested some AOs under the considered scenario. To examine the ranking of normal IFS and interval valued IFS, Garg [25] gave novel algorithms for solving the MADM problems. Ma and Xu [26] modified the existing score function and accuracy function for PFNs and defined novel AOs for PFNs. All of the previously mentioned methods have excessive applications in several fields, but due to their ineffectiveness, these methods have many limitations with parameterization. Molodtsov [27] presented the basic notion of soft sets (SS) and deliberated some basic operations with their belongings. Maji et al. [28] presented the idea of SS and demarcated several basic operations. In [29], the authors developed a DM approach for SS. Maji et al. [30] demonstrated the theory of IFSS and offered some basic operations with their essential properties. Zulqarnain et al. [31] presented the TOPSIS method using a correlation coefficient for interval-valued IFSS. Zulqarnain et al. [32] utilized the TOPSIS method for the prediction of diabetes patients.

Nowadays, the conception and application consequences of soft sets and the earlier-mentioned several research developments are evolving speedily. Peng et al. [33] established the concept of PFSS by merging two current models, PFS and SS. They also debated some fundamental operations with their essential possessions. Athira et al. [34] established entropy measures for the PFSS. They also offered Euclidean distance and hamming distance for the PFSS and utilized their methods for DM [35]. Naeem et al. [36] developed the TOPSIS and VIKOR methods for PFSNs and presented an approach for the stock exchange investment problem. Zulqarnain et al. [37, 38] introduced the AOs and interaction AOs under the PFSS environs. They also constructed the DM methods based on their operators and utilized them in green supplier chain management. Siddique et al. [39] settled a DM technique based on a score matrix for PFSS. Zulqarnain et al. [40] presented the correlation coefficient (CC) for PFSS and proposed the TOPSIS approach based on developed CC to resolve MADM problems. Zulqarnain et al. [41, 42] introduced the Einstein-ordered weighted AOs for PFSS and settled the DM approaches using their established operators.

The PFSS can potentially disclose unconvinced and obscure information in practical applications. This article establishes a new strategy for coping with DM issues under the PFSS environs. PFSS is an innovative hybrid configuration of PFS. Enriched organization approaches captivate investigators to interpret confusing and deficient data. PFSS performs a vital part in DM by congregation various sources into a single value in terms of findings. According to the best-known familiarity, the advent of hybridization of PFS and SS is not separate from PFS's perspective. Thus, to motivate modern exploration on PFSS, we will state the AOs based on rough data, with the subsequent elementary objectives of the study:

(1) PFSS is proficient in conducting complex problems competently, considering the properties in the DM progression. With this advantage in mind, we put up the Einstein AOs for PFSS

(2) In some cases, it has been noted that the prevailing AOs do not seem keen to flag precise DM techniques. To handle these specific troubles, these AOs must be amended. We presented an advanced algorithm for the Pythagorean fuzzy soft numbers (PFSNs) founded on the Einstein norm

(3) The PFSEWA and PFSEWG operators are built using Einstein operational laws with some basic properties

(4) An innovative MAGDM method is established based on the anticipated PFSEWA and PFSEWG operators to tackle the DM problem

(5) A comparative analysis of the settled MAGDM technique and current approaches has been offered to deliberate realism and dominance

The configuration of the subsequent study is prearranged as follows: in Section 2, we recalled some elementary notions such as FS, IFS, PFS, SS, FSS, IFSS, PFSS, and Einstein norms. Section 3 defined some basic operational laws for PFSNs based on Einstein norms, developed PFSEWA and PFSEWG operators, and discussed their essential properties. Section 4 settled the MAGDM methodology built on planned operators and gave a numerical illustration for finding the most delicate business to invest money in. In Section 5, a comparison with some prevailing methods has been provided.

2. Preliminaries

This section comprises some elementary definitions, such as SS, IFS, PFS, FSS, IFSS, and PFSS, which will deliver the basis for the structure of the following manuscript.

Definition 1. (see [27]). Let X and $\mathbb{N}$ be the universe of discourse and set of attributes, respectively. Let $\mathscr{P}(X)$ be the power set of X and $\mathscr{A} \subseteq \mathbb{N}$. A pair $(\Omega, \mathscr{A})$ is called a SS over X, and its mapping is expressed as follows:

$$\Omega : \mathscr{A} \longrightarrow \mathscr{P}(X). \tag{1}$$

Also, it can be defined as follows:

$$(\Omega, \mathscr{A}) = \{\Omega(t) \in \mathscr{P}(X): t \in \mathbb{N}, \Omega(t) = \varnothing \text{ if } t \notin \mathscr{A}\}. \tag{2}$$

Definition 2. (see [6]). Let X be a collection of objects and then a PFS, A over X is defined as

$$A = \{(x, a_A(x), b_A(x)) | x \in X\}, \tag{3}$$

where $a_A(x), b_A(x): X \longrightarrow [0, 1]$ represents the MD and NMD such as $0 \le a_A(x)^2 + b_A(x)^2 \le 1$ and $I = 1 - a(x)^2 - b_A(x)^2$ expressed the indeterminacy.

Definition 3. (see [30]). Let X and $\mathbb{N}$ be the universe of discourse and set of attributes, respectively; then, a pair $(\Omega, \mathbb{N})$ is called an IFSS over X.

Let $\Omega : \mathbb{N} \longrightarrow IK^X$ be a mapping and IK^X be a collection of intuitionistic fuzzy subsets. Also, it is defined as follows:

$$(\Omega, A) = \{t, (a_A(t), b_A(t)) | t \in A\}, \tag{4}$$

where $a_A(t), b_A(t): A \longrightarrow [0, 1]$ are MD and NMD and $0 \le a_A(t) + b_A(t) \le 1$.

The above IFSS cannot contract with the state when the combination of MD and NMD is more than one, so to contract with such circumstances, Yager [6, 7] reformed the state of IFSS to MD + NMD ≤ 1 presenting a general concept with its features. $(MD)^2 + (NMD)^2 \le 1$.

Definition 4. (see [33]). Let X and $\mathbb{N}$ be the universe of discourse and set of attributes, respectively; then, a pair $(\Omega, \mathbb{N})$ is called a PFSS over X.

Let Ω be a mapping such that $\Omega : \mathbb{N} \longrightarrow \wp K^X$ and $\wp K^X$ be a collection of Pythagorean fuzzy subsets. Also, it is defined as follows:

$$(\Omega, A) = \{t, (a_A(t), b_A(t)) | t \in A\}, \tag{5}$$

where $a_A(t), b_A(t): A \longrightarrow [0, 1]$ are MD and NMD, respectively, and $0 \le a_A(t)^2 + b_A(t)^2 \le 1$. $\mathfrak{I} = \sqrt{1 - a_A(t)^2 - b_A(t)^2}$ expressed the indeterminacy.

If $\mathscr{H}_{ij} = \langle a_{ij}, b_{ij} \rangle$ is a PFSN, then to compute the alternatives, Zulqarnain et al. [37] offered the score and accuracy functions for $\mathscr{H}_{ij}$ as

$$S(\mathscr{H}_{ij}) = a_{ij}^2 - b_{ij}^2, \tag{6}$$

where $S(\mathscr{H}_{ij}) \in [-1, 1]$. It is reported that the score function cannot discriminate the PFSNs in some cases. For example, if $\mathscr{H}_{11} = \langle 0.3162, 0.44720.4472 \rangle$ and $\mathscr{H}_{12} = \langle 0.5477, 0.6324 \rangle$, then $S(\mathscr{H}_{11}) = -0.1$ and $S(\mathscr{H}_{12}) - 0.1$. In that case, the use of the score function for bargaining is incredible. An accuracy function has been developed that combines MD and NMD to handle this error.

$$A(\mathscr{H}_{ij}) = a_{ij}^2 + b_{ij}^2, \tag{7}$$

where $A(\mathscr{H}_{ij}) \in [-1, 1]$.

Thus, to compare two PFSNs $\mathscr{H}_{ij}$ and $\mathscr{R}_{ij}$, the following comparison laws are defined:

(1) If $S(\mathscr{H}_{ij}) > S(\mathscr{R}_{ij})$, then $\mathscr{H}_{ij} > \mathscr{R}_{ij}$

(2) If $S(\mathscr{H}_{ij}) = S(\mathscr{R}_{ij})$, then

(i) If $A(\mathscr{H}_{ij}) > A(\mathscr{R}_{ij})$, then $\mathscr{H}_{ij} > \mathscr{R}_{ij}$

(ii) If $A(\mathscr{H}_{ij}) = A(\mathscr{R}_{ij})$, then $\mathscr{H}_{ij} = \mathscr{R}_{ij}$

Definition 5. (see [15]). Einstein sum $\bigoplus_\varepsilon$ and Einstein product $\bigotimes_\varepsilon$ are good alternatives of algebraic t-norm and t-conorm, respectively, given as follows:

$$\alpha \bigoplus_\varepsilon \beta = \frac{\alpha + \beta}{1 + (\alpha \bullet \beta)}, \alpha \bigotimes_\varepsilon \beta = \frac{\alpha \bullet \beta}{1 + (1 - \alpha) \bullet (1 - \beta)}, \quad \forall (\alpha, \beta) \in [0, 1]^2.$$

(8)

Under the Pythagorean fuzzy environment, Einstein sum $\bigoplus_\varepsilon$ and Einstein product $\bigotimes_\varepsilon$ are defined as

$$\alpha \bigoplus_\varepsilon \beta = \sqrt{\frac{\alpha^2 + \beta^2}{1 + (\alpha^2 \bullet \beta^2)}}, \alpha \bigotimes_\varepsilon \beta = \frac{\alpha \bullet \beta}{\sqrt{1 + (1 - \alpha^2)(1 - \beta^2)}}, \quad \forall (\alpha, \beta) \in [0, 1]^2,$$

(9)

where $\alpha \bigoplus_\varepsilon \beta$ and $\alpha \bigotimes_\varepsilon \beta$ are known as t-norm and t-conorm, respectively, satisfying the boundary, monotonicity, commutativity, and associativity properties.

3. Einstein-Weighted Aggregation Operators for the Pythagorean Fuzzy Soft Set

This section will construct a couple of Einstein-weighted AOs such as PFSEWA and PFSEWG operators for PFSNs with their essential properties.

3.1. Operational Laws for PFSNs

Definition 6. Let $\mathscr{H} = \langle a, b \rangle$, $\mathscr{H}_{11} = \langle a_{11}, b_{11} \rangle$, $\mathscr{H}_{12} = a_{12}$, b_{12} be PFSNs and $\partial > 0$, then based on Einstein norms, we have

(1) $\mathscr{H}_{11} \bigoplus_\varepsilon \mathscr{H}_{12} = \langle \sqrt{(1 + a_{1j}^2) - (1 - a_{1j}^2)}/ \sqrt{(1 + a_{1j}^2) + (1 - a_{1j}^2)}, \sqrt{2b_{1j}^2}/\sqrt{(2 - b_{1j}^2) + b_{1j}^2} \rangle$

(2) $\mathscr{H}_{11} \bigotimes_\varepsilon \mathscr{H}_{12} = \langle \sqrt{2a_{1j}^2}/\sqrt{(2 - a_{1j}^2) + a_{1j}^2}, \sqrt{(1 + b_{1j}^2) - (1 - b_{1j}^2)}/\sqrt{(1 + b_{1j}^2) + (1 - b_{1j}^2)} \rangle$

(3) $\partial \mathscr{H} = \langle \sqrt{(1 + a^2)^\partial - (1 - a^2)^\partial}/ \sqrt{(1 + a^2)^\partial + (1 - a^2)^\partial}, \sqrt{2(b^2)^\partial}/ \sqrt{(2 - b^2)^\partial + (b^2)^\partial} \rangle$

(4) $\mathscr{H}^\partial = \langle \sqrt{2(a^2)^\partial}/\sqrt{(2 - a^2)^\partial + (a^2)^\partial}, \sqrt{(1 + b^2)^\partial - (1 - b^2)^\partial}/\sqrt{(1 + b^2)^\partial + (1 - b^2)^\partial} \rangle$

Definition 7. Let $\mathscr{H}_{ij} = (a_{ij}, b_{ij})$ be a collection of PFSNs; then, the PFSEWA operator is defined as

$$\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \bigoplus_{j=1}^m \lambda_j \left(\bigoplus_{i=1}^n \theta_i \mathscr{H}_{ij} \right),$$

(10)

where $(i = 1, 2, \cdots, n)$, $(j = 1, 2, \cdots, m)$, and θ_i, λ_j represent the weighted vectors such that $\theta_i > 0$, $\sum_{i=1}^n \theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^n \lambda_j = 1$.

Theorem 8. *Let $\mathscr{H}_{ij} = (a_{ij}, b_{ij})$ be a collection of PFSNs; then, the aggregated value attained by equation (10) is given as*

$$\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$$
$$= \bigoplus_{j=1}^m \lambda_j \left(\bigoplus_{i=1}^n \theta_i \mathscr{H}_{ij} \right)$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n \left(1 + a_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j} - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - a_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n \left(1 + a_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j} + \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - a_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{2\prod_{j=1}^m \left(\prod_{i=1}^n \left(b_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n \left(2 - b_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j} + \prod_{j=1}^m \left(\prod_{i=1}^n \left(b_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j}}} \right\rangle,$$

(11)

where θ_i, λ_j denote the weight vectors such that $\theta_i > 0$, $\sum_{i=1}^n \theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^n \lambda_j = 1$.

Proof. We will employ mathematical induction.
For $n = 1$, we get $\theta_i = 1$.

$$\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$$
$$= \bigoplus_{j=1}^m \lambda_j \mathscr{H}_{1j}$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^m \left(1 + a_{1j}^2 \right)^{\lambda_j} - \prod_{j=1}^m \left(1 - a_{1j}^2 \right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^m \left(1 + a_{1j}^2 \right)^{\lambda_j} + \prod_{j=1}^m \left(1 - a_{1j}^2 \right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{2\prod_{j=1}^m \left(b_{1j}^2 \right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^m \left(2 - b_{1j}^2 \right)^{\lambda_j} + \prod_{j=1}^m \left(b_{1j}^2 \right)^{\lambda_j}}} \right\rangle$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^m \left(\prod_{i=1}^1 \left(1 + a_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j} - \prod_{j=1}^m \left(\prod_{i=1}^1 \left(1 - a_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^m \left(\prod_{i=1}^1 \left(1 + a_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j} + \prod_{j=1}^m \left(\prod_{i=1}^1 \left(1 - a_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{2\prod_{j=1}^m \left(\prod_{i=1}^1 \left(b_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^m \left(\prod_{i=1}^1 \left(2 - b_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j} + \prod_{j=1}^m \left(\prod_{i=1}^1 \left(b_{ij}^2 \right)^{\theta_i} \right)^{\lambda_j}}} \right\rangle.$$

(12)

For $m = 1$, we get $\lambda_j = 1$.

$$\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$$
$$= \bigoplus_{i=1}^{n} \theta_i \mathscr{H}_{i1}$$
$$= \left\langle \frac{\sqrt{\prod_{i=1}^{n}\left(1+\mathscr{a}_{i1}^2\right)^{\theta_i} - \prod_{i=1}^{n}\left(1-\mathscr{a}_{i1}^2\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(1+\mathscr{a}_{i1}^2\right)^{\theta_i} + \prod_{i=1}^{n}\left(1-\mathscr{a}_{i1}^2\right)^{\theta_i}}}, \right.$$
$$\left. \frac{\sqrt{2\prod_{i=1}^{n}\left(\mathscr{b}_{i1}^2\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(2-\mathscr{b}_{i1}^2\right)^{\theta_i} + \prod_{i=1}^{n}\left(\mathscr{b}_{i1}^2\right)^{\theta_i}}} \right\rangle$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{2\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle. \tag{13}$$

So, equation (11) true for $n = 1$ and $m = 1$.

Assume for $n = \delta_2$, $m = \delta_1 + 1$ and for $n = \delta_2 + 1$, $m = \delta_1$, the above equation holds. Then,

$$\bigoplus_{j=1}^{\delta_1+1} \lambda_j \left(\bigoplus_{i=1}^{\delta_2} \theta_i \mathscr{H}_{ij}\right)$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle, \tag{14}$$

$$\bigoplus_{j=1}^{\delta_1} \lambda_j \left(\bigoplus_{i=1}^{\delta_2+1} \theta_i \mathscr{H}_{ij}\right)$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{2\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle. \tag{14}$$

Now, for $m = \delta_1 + 1$ and $n = \delta_2 + 1$,

$$\bigoplus_{j=1}^{\delta_1+1} \lambda_j \left(\bigoplus_{i=1}^{\delta_2+1} \theta_i \mathscr{H}_{ij}\right)$$
$$= \bigoplus_{j=1}^{\delta_1+1} \lambda_j \left(\bigoplus_{i=1}^{\delta_2} \theta_i \mathscr{H}_{ij} \oplus \theta_{i+1} \mathscr{H}_{(\delta_2+1)j}\right)$$
$$= \left(\bigoplus_{j=1}^{\delta_1+1} \bigoplus_{i=1}^{\delta_2} \theta_i \lambda_j \mathscr{H}_{ij}\right) \left(\bigoplus_{j=1}^{\delta_1+1} \lambda_j \theta_{i+1} \mathscr{H}_{(\delta_2+1)j}\right)$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right.$$
$$\oplus \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+\alpha_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\left(1-\alpha_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1+\alpha_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\left(1-\alpha_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}},$$
$$\cdot \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}$$
$$\left. \oplus \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\left(\mathscr{b}_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(2-\mathscr{b}_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\left(\mathscr{b}_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}} \right\rangle$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \cdot \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$
$$= \bigoplus_{j=1}^{\delta_1+1} \lambda_j \left(\bigoplus_{i=1}^{\delta_2+1} \theta_i \mathscr{H}_{ij}\right), \tag{15}$$

So, it is true for $m = \delta_1 + 1$ and $n = \delta_2 + 1$. □

Example 9. Let $\mathscr{R} = \{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3\}$ be a set of experts with the given weight vector $\theta_i = (.1, .3, .3, .3)^T$, which want to choose a vehicle under the defined set of attributes $\mathring{A} = \{\Lambda_1$ = air conditioner, Λ_2 = airbag, Λ_3 = price, Λ_4 = comfort level , Λ_5 = design$\}$ with weight vector $\lambda_j = (.2, .2, .2, .4)^T$. The supposed rating values for all attributes in the PFSNs form $(\mathscr{H}, \mathring{A}) = (\mathscr{a}_{ij}, \mathscr{b}_{ij})_{4\times4}$ given as

$$\left(\mathscr{H}, \mathring{A}\right) = \begin{bmatrix} (0.5, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.7, 0.4) \\ (0.5, 0.6) & (0.9, 0.1) & (0.3, 0.7) & (0.4, 0.5) \\ (0.4, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.3, 0.5) \\ (0.3, 0.7) & (0.6, 0.5) & (0.5, 0.4) & (0.5, 0.7) \end{bmatrix}. \tag{16}$$

As we know that

$$\text{PFSEWA}(\mathcal{H}_{11}, \mathcal{H}_{12}, \cdots, \mathcal{H}_{nm})$$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle,$$

$$\text{PFSEWA}(\mathcal{H}_{11}, \mathcal{H}_{12}, \cdots, \mathcal{H}_{44})$$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{\begin{array}{l}\{(1.25)^{0.1}(1.25)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2} \\ \{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4} \\ - \\ \left[\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\right. \\ \left.\{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\right]\end{array}}{\begin{array}{l}\{(1.25)^{0.1}(1.25)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2} \\ \{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4} \\ + \\ \left[\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\right. \\ \left.\{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\right]\end{array}}}, \right.$$

$$\left. \sqrt{\frac{2\left[\begin{array}{l}\{(0.64)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.25)^{0.3}(0.25)^{0.3}\}^{0.2} \\ \{(0.36)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.4}\end{array}\right]}{\begin{array}{l}\{(1.36)^{0.1}(1.64)^{0.3}(1.36)^{0.3}(1.51)^{0.3}\}^{0.2}\{(1.75)^{0.1}(1.99)^{0.3}(1.75)^{0.3}(1.75)^{0.3}\}^{0.2} \\ \{(1.64)^{0.1}(1.51)^{0.3}(1.64)^{0.3}(1.84)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.75)^{0.3}(1.75)^{0.3}(1.51)^{0.3}\}^{0.4} \\ + \\ \{(0.64)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.25)^{0.3}(0.25)^{0.3}\}^{0.2} \\ \{(0.36)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.4}\end{array}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1.0324)(1.0897)(1.0309)(1.0734) - [(0.9616)(0.8350)(0.9638)(0.9105)]}{(1.0324)(1.0897)(1.0309)(1.0734) + [(0.9616)(0.8350)(0.9638)(0.9105)]}}, \frac{\sqrt{2[(0.8695)(0.6247)(0.7909)(0.6116)]}}{(1.0822)(1.1270)(1.1061)(1.2313) + (0.8695)(0.6247)(0.7909)(0.6116)}} \right\rangle$$

$$= \langle 0.5263, 0.5225 \rangle. \tag{17}$$

Lemma 10. *Let* $\mathcal{H}_{ij} = \langle \alpha_{ij}, \mathscr{b}_{ij} \rangle$, *where* $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ *and* $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$; *then,*

$$\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathcal{H}_{ij}\right)^{\theta_i}\right)^{\lambda_j} = \sum_{j=1}^{m}\lambda_j \sum_{i=1}^{n}\theta_i \mathcal{H}_{ij}. \tag{18}$$

Theorem 11. *Let* $\mathcal{H}_{ij} = (\alpha_{ij}, \mathscr{b}_{ij})$ *be a collection of PFSNs; then,*

$$\text{PFSWA}(\mathcal{H}_{11}, \mathcal{H}_{12}, \cdots, \mathcal{H}_{nm}) \geq \text{PFSEWA}(\mathcal{H}_{11}, \mathcal{H}_{12}, \cdots, \mathcal{H}_{nm}), \tag{19}$$

where θ_i, λ_j *denote the weight vectors such as* $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ *and* $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$.

Proof. As we know that

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\leq \sqrt{\sum_{j=1}^{m}\lambda_j \sum_{i=1}^{n}\theta_i\left(1+\alpha_{ij}^2\right) + \sum_{j=1}^{m}\lambda_j \sum_{i=1}^{n}\theta_i\left(1-\alpha_{ij}^2\right)}, \tag{20}$$

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{2}, \tag{21}$$

$$\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}$$

$$\leq \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}. \tag{22}$$

Again,

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\leq \sqrt{\sum_{j=1}^{m}\lambda_j \sum_{i=1}^{n}\theta_i\left(2-\mathscr{b}_{ij}^2\right) + \sum_{j=1}^{m}\lambda_j \sum_{i=1}^{n}\theta_i\left(\mathscr{b}_{ij}^2\right)}, \tag{23}$$

$$\sqrt{\sum_{j=1}^{m}\lambda_j \sum_{i=1}^{n}\theta_i\left(2-\mathscr{b}_{ij}^2\right) + \sum_{j=1}^{m}\lambda_j \sum_{i=1}^{n}\theta_i\left(\mathscr{b}_{ij}^2\right)} = \sqrt{2}, \tag{24}$$

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{2}, \tag{25}$$

$$\frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \geq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\mathscr{b}_{ij}\right)^{\theta_i}\right)^{\lambda_j}. \tag{26}$$

Let $\text{PFSWA}(\mathcal{H}_{11}, \mathcal{H}_{12}, \cdots, \mathcal{H}_{nm}) = \mathcal{H} = (\alpha_{\mathcal{H}}, \mathscr{b}_{\mathcal{H}})$ and $\text{PFSEWA}(\mathcal{H}_{11}, \mathcal{H}_{12}, \cdots \mathcal{H}_{nm}) = \mathcal{H}^{\varepsilon} = (\alpha_{\mathcal{H}^{\varepsilon}}, \mathscr{b}_{\mathcal{H}^{\varepsilon}})$. Then, (22) and (26) can be converted into the forms $\alpha_{\mathcal{H}} \geq \alpha_{\mathcal{H}^{\varepsilon}}$ and $\mathscr{b}_{\mathcal{H}} \leq \mathscr{b}_{\mathcal{H}^{\varepsilon}}$, respectively. So, $S(\mathcal{H}) = \alpha_{\mathcal{H}}^2 - \mathscr{b}_{\mathcal{H}}^2 \geq \alpha_{\mathcal{H}^{\varepsilon}}^2 - \mathscr{b}_{\mathcal{H}^{\varepsilon}}^2 = S(\mathcal{H}^{\varepsilon})$. Hence, $S(\mathcal{H}) \geq S(\mathcal{H}^{\varepsilon})$.

If $S(\mathcal{H}) > S(\mathcal{H}^{\varepsilon})$, then

$$\text{PFSWA}(\mathcal{H}_{11}, \mathcal{H}_{12}, \cdots, \mathcal{H}_{nm}) > \text{PFSEWA}(\mathcal{H}_{11}, \mathcal{H}_{12}, \cdots, \mathcal{H}_{nm}). \tag{27}$$

If $S(\mathscr{H}) = S(\mathscr{H}^{\varepsilon})$, then $S(\mathscr{H}) = a_{\mathscr{H}}^2 - b_{\mathscr{H}}^2 = a_{\mathscr{H}^{\varepsilon}}^2 - b_{\mathscr{H}^{\varepsilon}}^2 = S(\mathscr{H}^{\varepsilon})$.

So, $a_{\mathscr{H}} = a_{\mathscr{H}^{\varepsilon}}$ and $b_{\mathscr{H}} = b_{\mathscr{H}^{\varepsilon}}$; then, by accuracy function, $A(\mathscr{H}) = a_{\mathscr{H}}^2 + b_{\mathscr{H}}^2 = a_{\mathscr{H}^{\varepsilon}}^2 + b_{\mathscr{H}^{\varepsilon}}^2 = A(\mathscr{H}^{\varepsilon})$. Thus,

$$\text{PFSWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}). \tag{28}$$

From (27) and (28), we get

$$\text{PFSWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) \geq \text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}). \tag{29}$$

□

Example 12. Let $\mathscr{R} = \{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3\}$ be a set of experts with the given weight vector $\theta_i = (.1, .3, .3, .3)^T$, which want to choose a vehicle under the defined set of attributes $\mathring{A} = \{\Lambda_1 = \text{air conditioner}, \Lambda_2 = \text{airbag}, \Lambda_3 = \text{price}, \Lambda_4 = \text{comfort level}, \Lambda_5 = \text{design}\}$ with weight vector $\lambda_j = (.2, .2, .2, .4)^T$. The supposed rating values for all attributes in the PFSNs form $(\mathscr{H}, \mathring{A}) = (a_{ij}, b_{ij})_{4 \times 4}$ given as

$$\left(\mathscr{H}, \mathring{\mathscr{A}}\right) = \begin{bmatrix} (0.5, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.7, 0.4) \\ (0.5, 0.6) & (0.9, 0.1) & (0.3, 0.7) & (0.4, 0.5) \\ (0.4, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.3, 0.5) \\ (0.3, 0.7) & (0.6, 0.5) & (0.5, 0.4) & (0.5, 0.7) \end{bmatrix}. \tag{30}$$

As we know that

$$\text{PFSWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{44})$$
$$= \left\langle \sqrt{1 - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1 - a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}, \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(b_{ij}\right)^{\theta_i}\right)^{\lambda_j} \right\rangle,$$

$$\text{PFSWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{44})$$
$$= \left\langle \sqrt{1 - \begin{bmatrix} \left\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\right\}^{0.2}\left\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\right\}^{0.2} \\ \left\{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\right\}^{0.2}\left\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\right\}^{0.4} \end{bmatrix}}, \right.$$
$$\left. \begin{pmatrix} \left\{(0.8)^{0.1}(0.6)^{0.3}(0.8)^{0.3}(0.7)^{0.3}\right\}^{0.2}\left\{(0.5)^{0.1}(0.1)^{0.3}(0.5)^{0.3}(0.5)^{0.3}\right\}^{0.2} \\ \left\{(0.6)^{0.1}(0.7)^{0.3}(0.6)^{0.3}(0.4)^{0.3}\right\}^{0.2}\left\{(0.4)^{0.1}(0.5)^{0.3}(0.5)^{0.3}(0.7)^{0.3}\right\}^{0.4} \end{pmatrix} \right\rangle$$
$$= \left\langle \sqrt{1 - [(0.9616)(0.8350)(0.9638)(0.9105)]}, ((0.9324)(0.7904)(0.8893)(0.7820)) \right\rangle$$
$$= \langle 0.5404, 0.5125 \rangle. \tag{31}$$

Hence, from Examples 9 and 12, it is proven that

$$\text{PFSWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) > \text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{1\text{dis}2}, \cdots, \mathscr{H}_{nm}). \tag{32}$$

3.2. Properties of the PFSEWA Operator

3.2.1. Idempotency. If $\mathscr{H}_{ij} = \mathscr{H} = (a_{ij}, b_{ij}) \forall i, j$, then $\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}$.

Proof. As we know that

$$\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$$

$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2 - b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\left(\left(1 + a_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} - \left(\left(1 - a_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(1 + a_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} + \left(\left(1 - a_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{2\left(\left(b_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(2 - b_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} + \left(\left(b_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\left(1 + a_{ij}^2\right) - \left(1 - a_{ij}^2\right)}}{\sqrt{\left(1 + a_{ij}^2\right) + \left(1 - a_{ij}^2\right)}}, \frac{\sqrt{2b_{ij}^2}}{\sqrt{\left(2 - b_{ij}^2\right) + \left(b_{ij}^2\right)}} \right\rangle = \left\langle a_{ij}, b_{ij} \right\rangle = \mathscr{H}. \tag{33}$$

□

3.2.2. Boundedness. Let $\mathscr{H}_{ij} = (a_{ij}, b_{ij})$ be a collection PFSNs and $\mathscr{H}_{\min} = \min(\mathscr{H}_{ij})$, and $\mathscr{H}_{\max} = \max(\mathscr{H}_{ij})$. Then, $\mathscr{H}_{\min} \leq \text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) \leq \mathscr{H}_{\max}$.

Proof. Let $g(y) = \sqrt{(1 - y^2)/(1 + y^2)}$, $y \in]0, 1]$; then, $d/dy(g(y)) = -2y/(1 + y^2)^2 \sqrt{(1 + y^2)/(1 - y^2)} < 0$, which displays that $g(y)$ is a decreasing function on $]0, 1]$. So, $a_{\min} \leq a_{ij} \leq a_{\max}, \forall i, j$. Hence, $g(a_{\max}) \leq g(a_{ij}) \leq g(a_{\min}), \forall i, j$.

Let θ_i and λ_j represent the weight vectors such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$. We have

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - a_{\max}^2}{1 + a_{\max}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - a_{ij}^2}{1 + a_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$
$$\leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - a_{\min}^2}{1 + a_{\min}^2}\right)^{\theta_i}\right)^{\lambda_j}}, \tag{34}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{1 - a_{\max}^2}{1 + a_{\max}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - a_{ij}^2}{1 + a_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$
$$\leq \sqrt{\left(\left(\frac{1 - a_{\min}^2}{1 + a_{\min}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}, \tag{35}$$

$$\Leftrightarrow \sqrt{1 + \left(\frac{1 - a_{\max}^2}{1 + a_{\max}^2}\right)} \leq \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - a_{ij}^2}{1 + a_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$
$$\leq \sqrt{1 + \left(\frac{1 - a_{\min}^2}{1 + a_{\min}^2}\right)}, \tag{36}$$

$$\Leftrightarrow \sqrt{\frac{2}{1 + a_{\max}^2}} \leq \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - a_{ij}^2}{1 + a_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$
$$\leq \sqrt{\frac{2}{1 + a_{\min}^2}}, \tag{37}$$

$$\Leftrightarrow \sqrt{\frac{1 + a_{\min}^2}{2}} \leq \frac{1}{\sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1 - a_{ij}^2\right)/\left(1 + a_{ij}^2\right)\right)^{\theta_i}\right)^{\lambda_j}}}$$
$$\leq \sqrt{\frac{1 + a_{\max}^2}{2}}, \tag{38}$$

$$\Leftrightarrow \sqrt{1 + a_{\min}^2} \leq \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1 - a_{ij}^2\right)/\left(1 + a_{ij}^2\right)\right)^{\theta_i}\right)^{\lambda_j}}}$$
$$\leq \sqrt{1 + a_{\max}^2}, \tag{39}$$

$$\Leftrightarrow \sqrt{1 + a_{\min}^2 - 1} \leq \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1 - a_{ij}^2\right)/\left(1 + a_{ij}^2\right)\right)^{\theta_i}\right)^{\lambda_j}} - 1}$$
$$\leq \sqrt{1 + a_{\max}^2 - 1}, \tag{40}$$

$$\Leftrightarrow \sqrt{a_{\min}^2} \leq \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1 - a_{ij}^2\right)/\left(1 + a_{ij}^2\right)\right)^{\theta_i}\right)^{\lambda_j}} - 1}$$
$$\leq \sqrt{a_{\max}^2}, \tag{41}$$

$$\Leftrightarrow a_{\min} \leq \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1 - a_{ij}^2\right)/\left(1 + a_{ij}^2\right)\right)^{\theta_i}\right)^{\lambda_j}} - 1} \leq a_{\max}, \tag{42}$$

$$a_{\min} \leq \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - a_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq a_{\max}. \tag{43}$$

Let $f(x) = \sqrt{(2 - x^2)/x^2}, x \in\,]0, 1]$; then, $d/dx(f(x)) = -2/x^3\sqrt{x^2/(2 - x^2)} < 0$. So, $f(x)$ is a decreasing function on]

0, 1]. Since $b_{\min} \leq b_{ij} \leq b_{\max}, \forall i, j$, then, $f(b_{\max}) \leq f(b_{ij}) \leq f(b_{\min})$. So, $\sqrt{(2 - b_{\max}^2)/b_{\max}^2} \leq \sqrt{(2 - b_{ij}^2)/b_{ij}^2} \leq \sqrt{(2 - b_{\min}^2)/b_{\min}^2}$. Let θ_i and λ_j denote the weight vectors such as $\theta_i > 0, \sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$. We have

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - b_{\max}^2}{b_{\max}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - b_{ij}^2}{b_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$
$$\leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - b_{\min}^2}{b_{\min}^2}\right)^{\theta_i}\right)^{\lambda_j}}, \tag{44}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{2 - b_{\max}^2}{b_{\max}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - b_{ij}^2}{b_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$
$$\leq \sqrt{\left(\left(\frac{2 - b_{\min}^2}{b_{\min}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}, \tag{45}$$

$$\Leftrightarrow \sqrt{1 + \left(\frac{2 - b_{\max}^2}{b_{\max}^2}\right)} \leq \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - b_{ij}^2}{b_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$
$$\leq \sqrt{1 + \left(\frac{2 - b_{\min}^2}{b_{\min}^2}\right)}, \tag{46}$$

$$\Leftrightarrow \sqrt{\frac{2}{b_{\max}^2}} \leq \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - b_{ij}^2}{b_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\frac{2}{b_{\min}^2}}, \tag{47}$$

$$\Leftrightarrow \sqrt{\frac{b_{\min}^2}{2}} \leq \frac{1}{\sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(2 - b_{ij}^2\right)/b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{\frac{b_{\max}^2}{2}}, \tag{48}$$

$$\Leftrightarrow b_{\min} \leq \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(2 - b_{ij}^2\right)/b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq b_{\max}, \tag{49}$$

$$b_{\min} \leq \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2 - b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq b_{\max}. \tag{50}$$

Let PFSEWA $(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}$. Then, inequalities (43) and (50) can be written as $a_{\min} \leq a \leq a_{\max}$ and $b_{\min} \leq b \leq b_{\max}$. Thus, $S(\mathscr{H}) = a^2 - b^2 \leq a_{\max}^2 - b_{\min}^2 = S(H_{\max})$ and $S(\mathscr{H}) = a^2 - b^2 \geq a_{\min}^2 - b_{\max}^2 = S(H_{\min})$.

If $S(\mathscr{H}) < S(H_{\max})$ and $S(\mathscr{H}) > S(H_{\min})$, then

$$\mathscr{H}_{\min} < \text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) < \mathscr{H}_{\max}. \quad (51)$$

If $S(\mathscr{H}) = S(H_{\max})$, then, $a^2 = a_{\max}^2$ and $b^2 = b_{\max}^2$. Thus, $S(\mathscr{H}) = a^2 - b^2 = a_{\max}^2 - b_{\max}^2 = S(H_{\max})$. Therefore,

$$\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}_{\max}. \quad (52)$$

If $S(\mathscr{H}) = S(H_{\min})$ then, $a^2 - b^2 = a_{\min}^2 - b_{\min}^2 \Rightarrow a^2 = a_{\min}^2$ and $b^2 = b_{\min}^2$.

Thus, $A(\mathscr{H}) = a^2 + b^2 = a_{\min}^2 + b_{\min}^2 = A(H_{\min})$. So,

$$\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}_{\min},$$
$$\mathscr{H}_{\min} \leq \text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) \leq \mathscr{H}_{\max}. \quad (53)$$
$\square$

3.2.3. *Homogeneity*. Prove that PFSEWA$(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \partial$ PFSEWA $(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$ for $\partial > 0$.

Proof. Let $\mathscr{H}_{ij}$ be a PFSN and ∂ be a positive number; then,

$$\partial \mathscr{H}_{ij} = \left\langle \frac{\sqrt{(1 + a^2)^\partial - (1 - a^2)^\partial}}{\sqrt{(1 + a^2)^\partial + (1 - a^2)^\partial}}, \frac{\sqrt{2(b^2)^\partial}}{\sqrt{(2 - b^2)^\partial + (b^2)^\partial}} \right\rangle. \quad (54)$$

So,

$$\text{PFSEWA}(\partial\mathscr{H}_{11}, \partial\mathscr{H}_{12}, \cdots, \partial\mathscr{H}_{nm})$$
$$= \left\langle \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$
$$= \left\langle \frac{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^\partial - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^\partial}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^\partial + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^\partial}}, \right.$$
$$\left. \frac{\sqrt{\left(2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^\partial}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^\partial + \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^\partial}} \right\rangle$$
$$= \partial\text{PFSEWA}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) \quad (55)$$

$\square$

Definition 13. Let $\mathscr{H}_{ij} = (a_{ij}, b_{ij})$ be a collection of PFSNs; then, the PFSEWG operator is defined as

$$\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \bigotimes_{j=1}^{m}\lambda_j\left(\bigotimes_{i=1}^{n}\theta_i\mathscr{H}_{ij}\right), \quad (56)$$

where $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$ and θ_i, λ_j represent the weighted vectors such as $\theta_i > 0, \sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$.

Theorem 14. *Let $\mathscr{H}_{ij} = (a_{ij}, b_{ij})$ be a collection of PFSNs; then, the aggregated value attained by equation (56) is given as*

$$\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$$
$$= \bigotimes_{j=1}^{m}\lambda_j\left(\bigotimes_{i=1}^{n}\theta_i\mathscr{H}_{ij}\right)$$
$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle, \quad (57)$$

where θ_j, λ_i denote the weight vectors such as $\theta_j > 0$, $\sum_{i=1}^{n}\theta_j = 1$ and $\lambda_i > 0, \sum_{i=1}^{n}\lambda_i = 1$.

Proof. We will employ mathematical induction.
For $n = 1$, we get $\theta_i = 1$.

$$\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \bigotimes_{j=1}^{m}\lambda_j\mathscr{H}_{1j}$$
$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\alpha_{ij}^2\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-\alpha_{ij}^2\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\alpha_{ij}^2\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(1+b_{ij}^2\right)^{\lambda_j} - \prod_{j=1}^{m}\left(1-b_{ij}^2\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(1+b_{ij}^2\right)^{\lambda_j} + \prod_{j=1}^{m}\left(1-b_{ij}^2\right)^{\lambda_j}}} \right\rangle$$
$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(2-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1+b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1+b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1-b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle. \quad (58)$$

For $m = 1$, we get $\lambda_j = 1$.

$$\mathrm{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \bigotimes_{i=1}^{n} \theta_i \mathscr{H}_{i1}$$

$$= \left\langle \frac{\sqrt{2\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(2 - \alpha_{i1}^2\right)^{\theta_i} + \prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{i=1}^{n}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i} - \prod_{i=1}^{n}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}}}{\sqrt{\prod_{i=1}^{n}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i} + \prod_{i=1}^{n}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(2 - \alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle,$$

$$(59)$$

So, equation (57) holds for $n = 1$ and $m = 1$.

Assume for $n = \delta_2$ and $m = \delta_1 + 1$ and for $n = \delta_2 + 1$ and $m = \delta_1$, the above equation holds.

$$\bigotimes_{j=1}^{\delta_1+1} \lambda_j \left(\bigotimes_{i=1}^{\delta_2} \theta_i \mathscr{H}_{ij} \right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2 - \alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle,$$

$$\bigotimes_{j=1}^{\delta_1} \lambda_j \left(\bigotimes_{i=1}^{\delta_2+1} \theta_i \mathscr{H}_{ij} \right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(2 - \alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1}\left(\prod_{i=1}^{\delta_2+1}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle.$$

$$(60)$$

Now, for $m = \delta_1 + 1$ and $n = \delta_2 + 1$,

$$\bigotimes_{j=1}^{\delta_1+1} \lambda_j \left(\bigotimes_{i=1}^{\delta_2+1} \theta_i \mathscr{H}_{ij} \right)$$

$$= \bigotimes_{j=1}^{\delta_1+1} \lambda_j \left(\bigotimes_{i=1}^{\delta_2} \theta_i \mathscr{H}_{ij} \bigotimes \theta_{i+1} \mathscr{H}_{(\delta_2+1)j} \right)$$

$$= \left(\bigotimes_{j=1}^{\delta_1+1} \bigotimes_{i=1}^{\delta_2} \theta_i \lambda_j \mathscr{H}_{ij} \right) \left(\bigotimes_{j=1}^{\delta_1+1} \lambda_j \theta_{i+1} \mathscr{H}_{(\delta_2+1)j} \right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(2 - \alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right.$$

$$\left. \otimes \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\left(\alpha_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(2 - \alpha_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\left(\alpha_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}, \right.$$

$$\frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}$$

$$\left. \otimes \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1 + \mathscr{E}_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\left(1 - \mathscr{E}_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\left(1 + \mathscr{E}_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\left(1 - \mathscr{E}_{(\delta_2+1)j}^2\right)^{\theta_{\delta_2+1}}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(2 - \alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1 + \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{\delta_1+1}\left(\prod_{i=1}^{\delta_2+1}\left(1 - \mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \bigotimes_{j=1}^{\delta_1+1} \lambda_j \left(\bigotimes_{i=1}^{\delta_2+1} \theta_i \mathscr{H}_{ij} \right).$$

$$(61)$$

So, it is true for $m = \delta_1 + 1$ and $n = \delta_2 + 1$. □

Example 15. Let $\mathscr{R} = \{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3\}$ be a set of experts with the given weight vector $\theta_i = (.1, .3, .3, .3)^T$, which want to choose a vehicle under the defined set of attributes $\mathring{A} = \{\Lambda_1 = \text{air conditioner}, \Lambda_2 = \text{airbag}, \Lambda_3 = \text{price}, \Lambda_4 = \text{comfort level}, \Lambda_5 = \text{design}\}$ with weight vector $\lambda_j = (.2, .2, .2, .4)^T$. The supposed rating values for all attributes in the PFSNs form $(\mathscr{H}, \mathring{A}) = (\alpha_{ij}, \mathscr{E}_{ij})_{4 \times 4}$ given as

$$\left(\mathscr{H}, \mathring{\mathscr{A}}\right) = \begin{bmatrix} (0.5, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.7, 0.4) \\ (0.5, 0.6) & (0.9, 0.1) & (0.3, 0.7) & (0.4, 0.5) \\ (0.4, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.3, 0.5) \\ (0.3, 0.7) & (0.6, 0.5) & (0.5, 0.4) & (0.5, 0.7) \end{bmatrix}.$$

$$(62)$$

As we know that

$$\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle,$$

$$\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{44})$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(2-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1+\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{4}\left(\prod_{i=1}^{4}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\left[\begin{array}{c}\{(0.25)^{0.1}(0.25)^{0.3}(0.16)^{0.3}(0.09)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.81)^{0.3}(0.36)^{0.3}(0.25)^{0.3}\}^{0.2}\\ \{(0.16)^{0.1}(0.09)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.16)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\}^{0.4}\end{array}\right]}}{\begin{array}{c}\{(1.75)^{0.1}(1.75)^{0.3}(1.84)^{0.3}(1.91)^{0.3}\}^{0.2}\{(1.51)^{0.1}(1.09)^{0.3}(1.64)^{0.3}(1.75)^{0.3}\}^{0.2}\\ \{(1.84)^{0.1}(1.91)^{0.3}(1.84)^{0.3}(1.75)^{0.3}\}^{0.2}\{(1.51)^{0.1}(1.84)^{0.3}(1.91)^{0.3}(1.75)^{0.3}\}^{0.4}\\ +\\ \{(0.25)^{0.1}(0.25)^{0.3}(0.16)^{0.3}(0.09)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.81)^{0.3}(0.36)^{0.3}(0.25)^{0.3}\}^{0.2}\\ \{(0.16)^{0.1}(0.09)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.16)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\}^{0.4}\end{array}}, \right.$$

$$\left. \frac{\begin{array}{c}\{(1.64)^{0.1}(1.36)^{0.3}(1.64)^{0.3}(1.49)^{0.3}\}^{0.2}\{(1.25)^{0.1}(1.01)^{0.3}(1.25)^{0.3}(1.25)^{0.3}\}^{0.2}\\ \{(1.36)^{0.1}(1.49)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.16)^{0.1}(1.25)^{0.3}(1.49)^{0.3}(1.25)^{0.3}\}^{0.4}\\ -\\ \left[\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.51)^{0.3}\}^{0.2}\{(0.75)^{0.1}(0.99)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\}^{0.2}\right.\\ \left.\{(0.64)^{0.1}(0.51)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.84)^{0.1}(0.75)^{0.3}(0.51)^{0.3}(0.75)^{0.3}\}^{0.4}\right]\end{array}}{\begin{array}{c}\{(1.64)^{0.1}(1.36)^{0.3}(1.64)^{0.3}(1.49)^{0.3}\}^{0.2}\{(1.25)^{0.1}(1.01)^{0.3}(1.25)^{0.3}(1.25)^{0.3}\}^{0.2}\\ \{(1.36)^{0.1}(1.49)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.16)^{0.1}(1.25)^{0.3}(1.49)^{0.3}(1.25)^{0.3}\}^{0.4}\\ +\\ \left[\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.51)^{0.3}\}^{0.2}\{(0.75)^{0.1}(0.99)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\}^{0.2}\right.\\ \left.\{(0.64)^{0.1}(0.51)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.84)^{0.1}(0.75)^{0.3}(0.51)^{0.3}(0.75)^{0.3}\}^{0.4}\right]\end{array}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2[(0.4953)(0.6938)(0.5664)(0.3355)]}}{\sqrt{\begin{array}{c}(1.2346)(1.1676)(1.2208)(1.4786)+\\ (0.4953)(0.6938)(0.5664)(0.3355)\end{array}}}, \frac{\sqrt{\begin{array}{c}(1.1477)(1.0651)(1.0872)(1.1850)-\\ [(0.7841)(0.9220)(0.9035)(0.7079)]\end{array}}}{\sqrt{\begin{array}{c}(1.1477)(1.0651)(1.0872)(1.1850)+\\ [(0.7841)(0.9220)(0.9035)(0.7079)]\end{array}}} \right\rangle$$

$$= \langle 0.2211, 0.7392 \rangle.$$

$$(63)$$

Theorem 16. *Let* $\mathscr{H}_{ij} = (\alpha_{ij}, \mathscr{b}_{ij})$ *be a collection of PFSNs; then,*

$$\text{PFSWG}\left(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}\right) \leq \text{PFSEWG}\left(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}\right),$$

$$(64)$$

where θ_i, λ_j *denote the weight vectors such as* $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ *and* $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$.

Proof. As we know that

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\leq \sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(2-\alpha_{ij}^2\right) + \sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(\alpha_{ij}^2\right)}, \tag{65}$$

$$\sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(2-\alpha_{ij}^2\right) + \sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(\alpha_{ij}^2\right)} = \sqrt{2}, \tag{66}$$

$$\frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \tag{67}$$

$$\geq \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}\right)^{\theta_i}\right)^{\lambda_j}.$$

Again,

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\leq \sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1+\mathscr{b}_{ij}^2\right) + \sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1-\mathscr{b}_{ij}^2\right)}, \tag{68}$$

$$\sqrt{\sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1+\mathscr{b}_{ij}^2\right) + \sum_{j=1}^{m}\lambda_j\sum_{i=1}^{n}\theta_i\left(1-\mathscr{b}_{ij}^2\right)} = \sqrt{2}, \tag{69}$$

$$\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{2}, \tag{70}$$

$$\frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \tag{71}$$

$$\leq \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}.$$

Let $\text{PFSWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H} = (\alpha_{\mathscr{H}}, \mathscr{b}_{\mathscr{H}})$ and $\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}^{\varepsilon} = (\alpha_{\mathscr{H}^{\varepsilon}}, \mathscr{b}_{\mathscr{H}^{\varepsilon}})$.

Then, (67) and (71) can be changed into the succeeding forms $\alpha_{\mathscr{H}} \leq \alpha_{\mathscr{H}^{\varepsilon}}$ and $\mathscr{b}_{\mathscr{H}} \geq \mathscr{b}_{\mathscr{H}^{\varepsilon}}$ correspondingly.

So, $S(\mathscr{H}) = \alpha_{\mathscr{H}}^2 - \mathscr{b}_{\mathscr{H}}^2 \leq \alpha_{\mathscr{H}^{\varepsilon}}^2 - \mathscr{b}_{\mathscr{H}^{\varepsilon}}^2 = S(\mathscr{H}^{\varepsilon})$. Hence, $S(\mathscr{H}) \leq S(\mathscr{H}^{\varepsilon})$.

If $S(\mathscr{H}) < S(\mathscr{H}^{\varepsilon})$, then

$$\text{PFSWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) < \text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}).$$

$$(72)$$

If $S(\mathscr{H}) = S(\mathscr{H}^\varepsilon)$, then $a_{\mathscr{H}}^2 - b_{\mathscr{H}}^2 = a_{\mathscr{H}^\varepsilon}^2 - b_{\mathscr{H}^\varepsilon}^2$, so $a_{\mathscr{H}} = a_{\mathscr{H}^\varepsilon}$ and $b_{\mathscr{H}} = b_{\mathscr{H}^\varepsilon}$.

Then, $\quad A(\mathscr{H}) = a_{\mathscr{H}}^2 + b_{\mathscr{H}}^2 = a_{\mathscr{H}^\varepsilon}^2 + b_{\mathscr{H}^\varepsilon}^2 = A(\mathscr{H}^\varepsilon)$. Thus,

$$\text{PFSWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}). \tag{73}$$

From (72) and (73), we get

$$\text{PFSWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) \leq \text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}). \tag{74}$$

$\square$

Example 17. Let $\mathscr{R} = \{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3\}$ be a set of experts with the given weight vector $\theta_i = (.1, .3, .3, .3)^T$, which want to choose a vehicle under the defined set of attributes $\mathring{A} = \{\Lambda_1$ = air conditioner, Λ_2 = airbag, Λ_3 = price, Λ_4 = comfort level , Λ_5 = design$\}$ with weight vector $\lambda_j = (.2, .2, .2, .4)^T$. The supposed rating values for all attributes in the PFSNs form $(\mathscr{H}, \mathring{A}) = (a_{ij}, b_{ij})_{4 \times 4}$ given as

$$\left(\mathscr{H}, \mathring{A}\right) = \begin{bmatrix} (0.5, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.7, 0.4) \\ (0.5, 0.6) & (0.9, 0.1) & (0.3, 0.7) & (0.4, 0.5) \\ (0.4, 0.8) & (0.7, 0.5) & (0.4, 0.6) & (0.3, 0.5) \\ (0.3, 0.7) & (0.6, 0.5) & (0.5, 0.4) & (0.5, 0.7) \end{bmatrix}. \tag{75}$$

As we know that

$$\text{PFSWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{44})$$
$$= \left\langle \prod_{j=1}^{4} \left(\prod_{i=1}^{4} (\alpha_{ij})^{\theta_i} \right)^{\lambda_j}, \sqrt{1 - \prod_{j=1}^{4} \left(\prod_{i=1}^{4} \left(1 - b_{ij}^2\right)^{\theta_i} \right)^{\lambda_j}} \right\rangle,$$

$$\text{PFSWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{44})$$
$$= \left\langle \begin{array}{c} \left\{(0.5)^{0.1}(0.5)^{0.3}(0.4)^{0.3}(0.3)^{0.3}\right\}^{0.2}\left\{(0.7)^{0.1}(0.9)^{0.3}(0.7)^{0.3}(0.6)^{0.3}\right\}^{0.2}, \\ \left\{(0.4)^{0.1}(0.3)^{0.3}(0.4)^{0.3}(0.5)^{0.3}\right\}^{0.2}\left\{(0.7)^{0.1}(0.4)^{0.3}(0.3)^{0.3}(0.5)^{0.3}\right\}^{0.4} \end{array} \right.$$
$$\left. \sqrt{1 - \begin{bmatrix} \left\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.51)^{0.3}\right\}^{0.2}\left\{(0.75)^{0.1}(0.99)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\right\}^{0.2} \\ \left\{(0.64)^{0.1}(0.51)^{0.3}(0.64)^{0.3}(0.84)^{0.3}\right\}^{0.2}\left\{(0.84)^{0.1}(0.75)^{0.3}(0.51)^{0.3}(0.75)^{0.3}\right\}^{0.4} \end{bmatrix}} \right\rangle$$
$$= \langle((0.8330)(0.9365)(0.8293)(0.7033)), \sqrt{1 - [(0.7841)(0.9220)(0.9035)(0.7079)]}\rangle$$
$$= \langle 0.4549, 0.7332 \rangle. \tag{76}$$

Hence, from Examples 15 and 17, it has been proven that

$$\text{PFSWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) < \text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}). \tag{77}$$

3.3. Properties of the PFSEWG Operator

3.3.1. Idempotency. If $\quad \mathscr{H}_{ij} = \mathscr{H} = (a_{ij}, b_{ij}) \forall i, j, \quad$ then $\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}$.

Proof. As we know that

$$\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+b_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\left(\left(\alpha_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(2-\alpha_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} + \left(\left(\alpha_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}, \right.$$
$$\left. \frac{\sqrt{\left(\left(1+b_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} - \left(\left(1-b_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}}{\sqrt{\left(\left(1+b_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j} + \left(\left(1-b_{ij}^2\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{2\alpha_{ij}^2}}{\sqrt{\left(2-\alpha_{ij}^2\right) + \left(\alpha_{ij}^2\right)}}, \frac{\sqrt{\left(1+b_{ij}^2\right) - \left(1-b_{ij}^2\right)}}{\sqrt{\left(1+b_{ij}^2\right) + \left(1-b_{ij}^2\right)}} \right\rangle$$

$$= \langle a_{ij}, b_{ij} \rangle = \mathscr{H}. \tag{78}$$

$\square$

3.3.2. Boundedness. Let $\mathscr{H}_{ij} = (a_{ij}, b_{ij})$ be a collection PFSNs and $\mathscr{H}_{\min} = \min(\mathscr{H}_{ij})$, $\mathscr{H}_{\max} = \max(\mathscr{H}_{ij})$. Then, $\mathscr{H}_{\min} \leq \text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) \leq \mathscr{H}_{\max}$

Proof. Let $f(x) = \sqrt{(2-x^2)/x^2}, x \in \,]0, 1]$; then $d/dx(f(x)) = -2/x^3\sqrt{x^2/(2-x^2)} < 0$. So, $f(x)$ is a decreasing function on $]0, 1]$. Since $a_{\min} \leq a_{ij} \leq a_{\max}, \forall i, j$. Then, $f(a_{\max}) \leq f(a_{ij}) \leq f(a_{\min})$. So, $\quad \sqrt{(2-a_{\max}^2)/a_{\max}^2} \leq \sqrt{(2-a_{ij}^2)/a_{ij}^2} \leq \sqrt{(2-a_{\min}^2)/a_{\min}^2}$.

Let θ_i and λ_j denote the weight vectors such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$. We have

$$\Leftrightarrow \sqrt{\left(\left(\frac{2-a_{\max}^2}{a_{\max}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2-a_{ij}^2}{a_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$
$$\leq \sqrt{\left(\left(\frac{2-a_{\min}^2}{a_{\min}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}, \tag{79}$$

$$\Leftrightarrow \sqrt{\left(\frac{2 - \alpha_{\max}^2}{\alpha_{\max}^2}\right)} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - \alpha_{ij}^2}{\alpha_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \tag{80}$$
$$\leq \sqrt{\left(\frac{2 - \alpha_{\min}^2}{\alpha_{\min}^2}\right)},$$

$$\Leftrightarrow \sqrt{1 + \left(\frac{2 - \alpha_{\max}^2}{\alpha_{\max}^2}\right)} \leq \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - \alpha_{ij}^2}{\alpha_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \tag{81}$$
$$\leq \sqrt{1 + \left(\frac{2 - \alpha_{\min}^2}{\alpha_{\min}^2}\right)},$$

$$\Leftrightarrow \sqrt{\frac{2}{\alpha_{\max}^2}} \leq \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{2 - \alpha_{ij}^2}{\alpha_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\frac{2}{\alpha_{\min}^2}}, \tag{82}$$

$$\Leftrightarrow \sqrt{\frac{\alpha_{\min}^2}{2}} \leq \frac{1}{\sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(2 - \alpha_{ij}^2\right)/\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{\frac{\alpha_{\max}^2}{2}}, \tag{83}$$

$$\Leftrightarrow \alpha_{\min} \leq \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(2 - \alpha_{ij}^2\right)/\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq \alpha_{\max}, \tag{84}$$

$$\alpha_{\min} \leq \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2 - \alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq \alpha_{\max}. \tag{85}$$

Again, let $g(y) = \sqrt{(1 - y^2)/(1 + y^2)}, y \in]0, 1]$. Then, $d/dy(g(y)) = -2y/(1 + y^2)^2 \sqrt{(1 + y^2)/(1 - y^2)} < 0$. So, $g(y)$ is a decreasing function on $]0, 1]$. Thus, $\mathscr{b}_{\min} \leq \mathscr{b}_{ij} \leq \mathscr{b}_{\max}, \forall i, j$. So, $g(\mathscr{b}_{\max}) \leq g(\mathscr{b}_{ij}) \leq g(\mathscr{b}_{\min}), \forall i, j$.

$$\Rightarrow \sqrt{\frac{1 - \mathscr{b}_{\max}^2}{1 + \mathscr{b}_{\max}^2}} \leq \sqrt{\frac{1 - \mathscr{b}_{ij}^2}{1 + \mathscr{b}_{ij}^2}} \leq \sqrt{\frac{1 - \mathscr{b}_{\min}^2}{1 + \mathscr{b}_{\min}^2}}. \tag{86}$$

Let θ_i and λ_j denote the weight vectors such as $\theta_i > 0$, $\sum_{i=1}^{n}\theta_i = 1$ and $\lambda_j > 0, \sum_{j=1}^{n}\lambda_j = 1$. We have

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - \mathscr{b}_{\max}^2}{1 + \mathscr{b}_{\max}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - \mathscr{b}_{ij}^2}{1 + \mathscr{b}_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - \mathscr{b}_{\min}^2}{1 + \mathscr{b}_{\min}^2}\right)^{\theta_i}\right)^{\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\left(\frac{1 - \mathscr{b}_{\max}^2}{1 + \mathscr{b}_{\max}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - \mathscr{b}_{ij}^2}{1 + \mathscr{b}_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\left(\left(\frac{1 - \mathscr{b}_{\min}^2}{1 + \mathscr{b}_{\min}^2}\right)^{\sum_{i=1}^{n}\theta_i}\right)^{\sum_{j=1}^{m}\lambda_j}}$$

$$\Leftrightarrow \sqrt{\left(\frac{1 - \mathscr{b}_{\max}^2}{1 + \mathscr{b}_{\max}^2}\right)} \leq \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - \mathscr{b}_{ij}^2}{1 + \mathscr{b}_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\left(\frac{1 - \mathscr{b}_{\min}^2}{1 + \mathscr{b}_{\min}^2}\right)} \Leftrightarrow \sqrt{1 + \left(\frac{1 - \mathscr{b}_{\max}^2}{1 + \mathscr{b}_{\max}^2}\right)}$$

$$\leq \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - \mathscr{b}_{ij}^2}{1 + \mathscr{b}_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{1 + \left(\frac{1 - \mathscr{b}_{\min}^2}{1 + \mathscr{b}_{\min}^2}\right)} \Leftrightarrow \sqrt{\frac{2}{1 + \mathscr{b}_{\max}^2}}$$

$$\leq \sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\frac{1 - \mathscr{b}_{ij}^2}{1 + \mathscr{b}_{ij}^2}\right)^{\theta_i}\right)^{\lambda_j}} \leq \sqrt{\frac{2}{1 + \mathscr{b}_{\min}^2}} \Leftrightarrow \sqrt{\frac{1 + \mathscr{b}_{\min}^2}{2}}$$

$$\leq \frac{1}{\sqrt{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1 - \mathscr{b}_{ij}^2\right)/\left(1 + \mathscr{b}_{ij}^2\right)\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{\frac{1 + \mathscr{b}_{\max}^2}{2}} \Leftrightarrow \sqrt{1 + \mathscr{b}_{\min}^2}$$

$$\leq \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1 - \mathscr{b}_{ij}^2\right)/\left(1 + \mathscr{b}_{ij}^2\right)\right)^{\theta_i}\right)^{\lambda_j}}} \leq \sqrt{1 + \mathscr{b}_{\max}^2} \Leftrightarrow \mathscr{b}_{\min}$$

$$\leq \sqrt{\frac{2}{1 + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\left(1 - \mathscr{b}_{ij}^2\right)/\left(1 + \mathscr{b}_{ij}^2\right)\right)^{\theta_i}\right)^{\lambda_j}} - 1} \leq \mathscr{b}_{\max} \Leftrightarrow \mathscr{b}_{\min}$$

$$\leq \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + \mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 + \mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j} + \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mathscr{b}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}}} \leq \mathscr{b}_{\max}. \tag{87}$$

Let PFSEWG$(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}$; then, equations (85) and (87) can be written as $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$ and $\mathscr{b}_{\max} \leq \mathscr{b} \leq \mathscr{b}_{\min}$. Thus, $S(\mathscr{H}) = \alpha^2 - \mathscr{b}^2 \leq \alpha_{\max}^2 - \mathscr{b}_{\min}^2 = S(H_{\max})$ and $S(\mathscr{H}) = \alpha^2 - \mathscr{b}^2 \geq \alpha_{\min}^2 - \mathscr{b}_{\max}^2 = S(H_{\min})$.

If $S(\mathscr{H}) < S(H_{\max})$ and $S(\mathscr{H}) > S(H_{\min})$, then

$$\mathscr{H}_{\min} < \text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) < \mathscr{H}_{\max}. \tag{88}$$

If $S(\mathscr{H}) = S(H_{\max})$. Then, $\alpha^2 = \alpha_{\max}^2$ and $\mathscr{b}^2 = \mathscr{b}_{\max}^2$. Thus, $S(\mathscr{H}) = \alpha^2 - \mathscr{b}^2 = \alpha_{\max}^2 - \mathscr{b}_{\max}^2 = S(H_{\max})$. Therefore,

$$\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}_{\max}. \tag{89}$$

If $S(\mathscr{H}) = S(H_{\min})$. Then, $\alpha^2 - \mathscr{b}^2 = \alpha_{\min}^2 - \mathscr{b}_{\min}^2 \Rightarrow \alpha^2 = \alpha_{\min}^2$ and $\mathscr{b}^2 = \mathscr{b}_{\min}^2$. Thus, $A(\mathscr{H}) = \alpha^2 + \mathscr{b}^2 = \alpha_{\min}^2 + \mathscr{b}_{\min}^2 = A(H_{\min})$. Therefore,

$$\text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \mathscr{H}_{\min}. \tag{90}$$

So we proved that

$$\mathscr{H}_{\min} \leq \text{PFSEWG}(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) \leq \mathscr{H}_{\max}. \tag{91}$$

□

3.3.3. *Homogeneity.* Prove that PFSEWG$(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}) = \partial$ PFSEWG$(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm})$ for any $\partial > 0$.

Proof. Let $\mathscr{H}_{ij}$ be a PFSN and ∂ be a positive real number; then,

$$\partial \mathscr{H}_{ij} = \left\langle \frac{\sqrt{2(\alpha^2)^{\partial}}}{\sqrt{(2-\alpha^2)^{\partial}+(\alpha^2)^{\partial}}}, \frac{\sqrt{(1+\mathscr{E}^2)^{\partial}-(1-\mathscr{E}^2)^{\partial}}}{\sqrt{(1+\mathscr{E}^2)^{\partial}+(1-\mathscr{E}^2)^{\partial}}} \right\rangle. \quad (92)$$

So,

$$\text{PFSEWG}\left(\partial\mathscr{H}_{11}, \partial\mathscr{H}_{12}, \cdots, \partial\mathscr{H}_{nm}\right)$$

$$= \left\langle \frac{\sqrt{2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\partial\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{ij}^2\right)^{\partial\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\partial\theta_i}\right)^{\lambda_j}}}, \right.$$

$$\left. \frac{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{E}_{ij}^2\right)^{\partial\theta_i}\right)^{\lambda_j}-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{E}_{ij}^2\right)^{\partial\theta_i}\right)^{\lambda_j}}}{\sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{E}_{ij}^2\right)^{\partial\theta_i}\right)^{\lambda_j}+\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{E}_{ij}^2\right)^{\partial\theta_i}\right)^{\lambda_j}}} \right\rangle$$

$$= \left\langle \frac{\sqrt{\left(2\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(2-\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}+\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(\alpha_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}, \right. \quad (93)$$

$$\left. \frac{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}}{\sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1+\mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}+\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathscr{E}_{ij}^2\right)^{\theta_i}\right)^{\lambda_j}\right)^{\partial}}} \right\rangle$$

$$= \partial \, \text{PFSEWG}\left(\mathscr{H}_{11}, \mathscr{H}_{12}, \cdots, \mathscr{H}_{nm}\right).$$

□

4. Multiattribute Group Decision-Making Approach

This section has settled a DM methodology for resolving MAGDM complications based on the projected PFSEWA and PFSEWG operators and numerical examples.

4.1. Proposed Approach. Let $\mathfrak{H} = \{\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3, \cdots, \mathfrak{H}^s\}$ be the set of s alternatives, $O = \{O_1, O_2, O_3, \cdots, O_r\}$ be the set of r experts (decision-makers), and $\mathbb{N} = \{t_1, t_2, t_3, \cdots, t_m\}$ be the set of m attributes. Let the weighted vector of experts $O(i = 1, 2, 3, \cdots, r)$ be $\theta = (\theta_1, \theta_2, \theta_3, \cdots, \theta_n)^T$ such that $\theta_i > 0, \sum_{i=1}^{n}\theta_i = 1$. Let the weight vector of attributes $t_i(i = 1, 2, 3, \cdots, m)$ be $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n)^T$ such that $\lambda_j > 0$, $\sum_{j=1}^{n}\lambda_j = 1$. The team of experts $O_i (i = 1, 2, 3, \cdots, r)$ considers the alternatives $\mathfrak{H}^i (i = 1, 2, 3, \cdots, s)$ for attributes in the form of PFSNs such as $F = (\mathscr{H}_{ij})_{n*m} = (\alpha_{ij}, \mathscr{E}_{ij})_{n*m}$ .where $0 \le \alpha_{ij}, \mathscr{E}_{ij} \le 1$ and $0 \le \alpha_{ij}^2, b_{ij}^2 \le 1 \forall i, j$ are given in Tables 1–5.

We will apply the proposed PFSEWA and PFSEWG operators to resolve the MAGDM problem, which has the succeeding phases:

Step 1. Acquire decision matrices for each alternative $F = (\mathscr{H}_{ij})_{n*m}$ in the PFSN form.

TABLE 1: PFS decision matrix for $\mathfrak{H}^1$.

	t_1	t_2	t_3	t_4
O^1	(0.8, 0.5)	(0.7, 0.5)	(0.6, 0.4)	(0.7, 0.4)
O^2	(0.6, 0.5)	(0.9, 0.1)	(0.7, 0.3)	(0.4, 0.5)
O^3	(0.8, 0.4)	(0.7, 0.5)	(0.6, 0.4)	(0.3, 0.5)
O^4	(0.7, 0.3)	(0.6, 0.5)	(0.4, 0.5)	(0.5, 0.7)

TABLE 2: PFS decision matrix for $\mathfrak{H}^2$.

	t_1	t_2	t_3	t_4
O^1	(0.7, 0.5)	(0.8, 0.5)	(0.6, 0.4)	(0.8, 0.4)
O^2	(0.6, 0.3)	(0.9, 0.2)	(0.8, 0.3)	(0.7, 0.5)
O^3	(0.5, 0.4)	(0.6, 0.5)	(0.6, 0.3)	(0.3, 0.6)
O^4	(0.7, 0.4)	(0.6, 0.4)	(0.7, 0.5)	(0.5, 0.7)

TABLE 3: PFS decision matrix for $\mathfrak{H}^3$.

	t_1	t_2	t_3	t_4
O^1	(0.7, 0.5)	(0.7, 0.4)	(0.6, 0.4)	(0.8, 0.4)
O^2	(0.6, 0.6)	(0.9, 0.1)	(0.6, 0.3)	(0.4, 0.5)
O^3	(0.8, 0.3)	(0.7, 0.2)	(0.6, 0.5)	(0.4, 0.5)
O^4	(0.7, 0.6)	(0.3, 0.5)	(0.4, 0.5)	(0.5, 0.6)

TABLE 4: PFS decision matrix for $\mathfrak{H}^4$.

	t_1	t_2	t_3	t_4
O^1	(0.8, 0.5)	(0.7, 0.5)	(0.7, 0.4)	(0.6, 0.4)
O^2	(0.6, 0.4)	(0.8, 0.1)	(0.7, 0.3)	(0.4, 0.7)
O^3	(0.7, 0.4)	(0.7, 0.5)	(0.6, 0.4)	(0.3, 0.5)
O^4	(0.6, 0.3)	(0.6, 0.3)	(0.8, 0.5)	(0.5, 0.6)

TABLE 5: PFS decision matrix for $\mathfrak{H}^5$.

	t_1	t_2	t_3	t_4
O^1	(0.6, 0.5)	(0.6, 0.5)	(0.6, 0.4)	(0.5, 0.4)
O^2	(0.6, 0.4)	(0.8, 0.1)	(0.8, 0.3)	(0.7, 0.5)
O^3	(0.6, 0.4)	(0.7, 0.3)	(0.6, 0.4)	(0.6, 0.5)
O^4	(0.7, 0.4)	(0.7, 0.5)	(0.4, 0.5)	(0.5, 0.8)

Step 2. Normalize the decision matrix to convert the rating value of cost-type parameters into benefit-type parameters by using the normalization formula.

$$M_{ij} = \begin{cases} \mathscr{H}_{ij}^c = (\mathscr{E}_{ij}, \alpha_{ij}) & \text{cost-type parameter}, \\ \mathscr{H}_{ij} = (\alpha_{ij}, \mathscr{E}_{ij}) & \text{benefit-type parameter}. \end{cases} \quad (94)$$
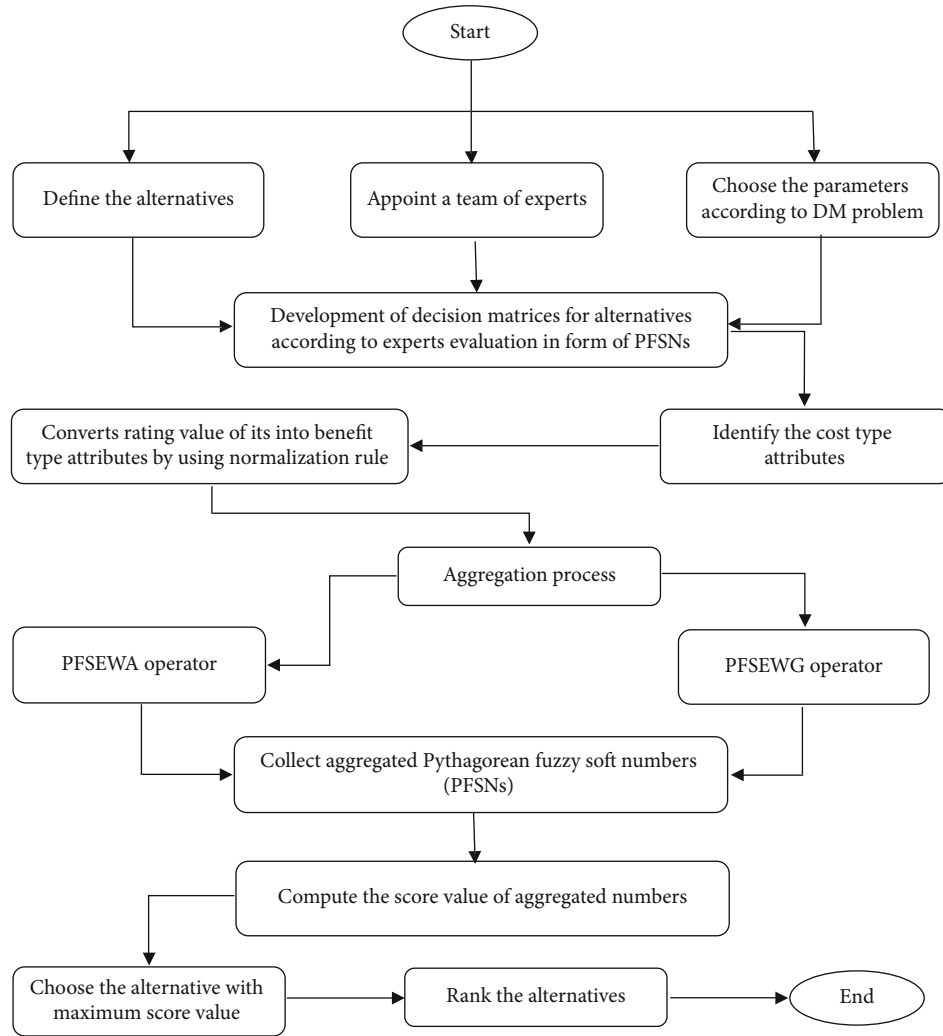
FIGURE 1: Graphical model of the proposed PFSEWA and PFSEWG operators.

Step 3. Use the developed PFSEWA and PFSEWG operators to aggregate the PFSNs $\mathscr{H}_{ij}$ for each alternative $\mathfrak{H} = \{\mathfrak{H}^1, \mathfrak{H}^2, \mathfrak{H}^3, \cdots, \mathfrak{H}^s\}$.

Step 4. Compute the score values for each alternative using equation (6).

Step 5. Choose the most feasible alternative with the maximum score value.

The graphical representation of the proposed model is given in Figure 1.

4.2. Numerical Example. Suppose a businessman desires to invest money, and he has five alternatives such as $\mathfrak{H}^1$; a restaurant, $\mathfrak{H}^2$; a filling station, $\mathfrak{H}^3$; a pharmacy, $\mathfrak{H}^4$; a leather factory, and $\mathfrak{H}^5$; a supermart. There are four considered attributes, according to which people in business must have to take decision such as t_1; socioeconomic impact, t_2; environment, t_3; risk of loss, and t_4; growth rate, with the

weight vector $\lambda = (0.2, 0.2, 0.2, 0.4)^T$. Here, t_1, t_3 are cost-type parameters and t_2, t_4 are benefit-type parameters. People in business hire a team of four experts O_r $(r = 1, 2, 3, 4)$ for decision-making with the weight vector $\theta = (0.1, 0.3, 0.3, 0.3)^T$.

4.2.1. By the PFSEWA Operator

Step 1. Decision-maker's opinions in the PFSN form for each alternative are prearranged in Tables 1–5.

Step 2. The normalization rule developed the normalized decision matrices for each alternative. Because t_1 and t_3 are cost-type parameters, the normalized PFS decision matrices are given in Tables 6–10.

Step 3. Using the PFSEWA operator acquired the aggregated values of each alternative in the form of PFSN such as

Table 6: Normalized PFS decision matrix for $\mathfrak{H}^1$.

	t_1	t_2	t_3	t_4
O^1	(0.5, 0.8)	(0.7, 0.5)	(0.4, 0.6)	(0.7, 0.4)
O^2	(0.5, 0.6)	(0.9, 0.1)	(0.3, 0.7)	(0.4, 0.5)
O^3	(0.4, 0.8)	(0.7, 0.5)	(0.4, 0.6)	(0.3, 0.5)
O^4	(0.3, 0.7)	(0.6, 0.5)	(0.5, 0.4)	(0.5, 0.7)

Table 7: Normalized PFS decision matrix for $\mathfrak{H}^2$.

	t_1	t_2	t_3	t_4
O^1	(0.5, 0.7)	(0.8, 0.5)	(0.4, 0.6)	(0.8, 0.4)
O^2	(0.3, 0.6)	(0.9, 0.2)	(0.3, 0.8)	(0.7, 0.5)
O^3	(0.4, 0.5)	(0.6, 0.5)	(0.3, 0.6)	(0.3, 0.6)
O^4	(0.4, 0.7)	(0.6, 0.4)	(0.5, 0.7)	(0.5, 0.7)

Table 8: Normalized PFS decision matrix for $\mathfrak{H}^3$.

	t_1	t_2	t_3	t_4
O^1	(0.5, 0.7)	(0.7, 0.4)	(0.4, 0.6)	(0.8, 0.4)
O^2	(0.6, 0.6)	(0.9, 0.1)	(0.3, 0.6)	(0.4, 0.5)
O^3	(0.3, 0.8)	(0.7, 0.2)	(0.5, 0.6)	(0.4, 0.5)
O^4	(0.6, 0.7)	(0.3, 0.5)	(0.5, 0.4)	(0.5, 0.6)

Table 9: Normalized PFS decision matrix for $\mathfrak{H}^4$.

	t_1	t_2	t_3	t_4
O^1	(0.5, 0.8)	(0.7, 0.5)	(0.4, 0.7)	(0.6, 0.4)
O^2	(0.4, 0.6)	(0.8, 0.1)	(0.3, 0.7)	(0.4, 0.7)
O^3	(0.4, 0.7)	(0.7, 0.5)	(0.4, 0.6)	(0.3, 0.5)
O^4	(0.3, 0.6)	(0.6, 0.3)	(0.5, 0.8)	(0.5, 0.6)

Table 10: Normalized PFS decision matrix for $\mathfrak{H}^5$.

	t_1	t_2	t_3	t_4
O^1	(0.5, 0.6)	(0.6, 0.5)	(0.4, 0.6)	(0.5, 0.4)
O^2	(0.4, 0.6)	(0.8, 0.1)	(0.3, 0.8)	(0.7, 0.5)
O^3	(0.4, 0.6)	(0.7, 0.3)	(0.4, 0.6)	(0.6, 0.5)
O^4	(0.4, 0.7)	(0.7, 0.5)	(0.5, 0.4)	(0.5, 0.8)

$$\mathscr{H}_1 = \left\langle \sqrt{\frac{\begin{bmatrix} \{(1.25)^{0.1}(1.25)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2} \\ \{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4} \\ - \\ \begin{bmatrix}\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\end{bmatrix} \\ \{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{bmatrix}}{\begin{bmatrix}\{(1.25)^{0.1}(1.25)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2} \\ \{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4} \\ + \\ \begin{bmatrix}\{(0.75)^{0.1}(0.75)^{0.3}(0.84)^{0.3}(0.91)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\end{bmatrix} \\ \{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.51)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{bmatrix}}}, \right.$$

$$\left. \sqrt{\frac{2\begin{bmatrix}\{(0.64)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.25)^{0.3}(0.25)^{0.3}\}^{0.2} \\ \{(0.36)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.4}\end{bmatrix}}{\begin{bmatrix}\{(1.36)^{0.1}(1.64)^{0.3}(1.36)^{0.3}(1.51)^{0.3}\}^{0.2}\{(1.75)^{0.1}(1.99)^{0.3}(1.75)^{0.3}(1.75)^{0.3}\}^{0.2} \\ \{(1.64)^{0.1}(1.51)^{0.3}(1.64)^{0.3}(1.84)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.75)^{0.3}(1.75)^{0.3}(1.51)^{0.3}\}^{0.4} \\ + \\ \{(0.64)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.25)^{0.3}(0.25)^{0.3}\}^{0.2} \\ \{(0.36)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.4}\end{bmatrix}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1.0324)(1.0897)(1.0309)(1.0734) - [(0.9616)(0.8350)(0.9638)(0.9105)]}{(1.0324)(1.0897)(1.0309)(1.0734) + [(0.9616)(0.8350)(0.9638)(0.9105)]}}, \sqrt{\frac{2[(0.8695)(0.6247)(0.7909)(0.6116)]}{(1.0822)(1.1270)(1.1061)(1.2313) + (0.8695)(0.6247)(0.7909)(0.6116)}} \right\rangle$$

$$\mathscr{H}_1 = \langle 0.5263, 0.5225 \rangle,$$

$$\mathscr{H}_2 = \left\langle \sqrt{\frac{\begin{bmatrix}\{(1.25)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.16)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.81)^{0.3}(1.36)^{0.3}(1.36)^{0.3}\}^{0.2} \\ \{(1.16)^{0.1}(1.09)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.49)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4} \\ - \\ \begin{bmatrix}\{(0.75)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.84)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.19)^{0.3}(0.64)^{0.3}(0.64)^{0.3}\}^{0.2}\end{bmatrix} \\ \{(0.84)^{0.1}(0.91)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.51)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{bmatrix}}{\begin{bmatrix}\{(1.25)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.16)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.81)^{0.3}(1.36)^{0.3}(1.36)^{0.3}\}^{0.2} \\ \{(1.16)^{0.1}(1.09)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.49)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4} \\ + \\ \begin{bmatrix}\{(0.75)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.84)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.19)^{0.3}(0.64)^{0.3}(0.64)^{0.3}\}^{0.2}\end{bmatrix} \\ \{(0.84)^{0.1}(0.91)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.51)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{bmatrix}}}, \right.$$

$$\left. \sqrt{\frac{2\begin{bmatrix}\{(0.49)^{0.1}(0.36)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.49)^{0.3}(0.25)^{0.3}(0.16)^{0.3}\}^{0.2} \\ \{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.36)^{0.3}(0.49)^{0.3}\}^{0.4}\end{bmatrix}}{\begin{bmatrix}\{(1.51)^{0.1}(1.64)^{0.3}(1.75)^{0.3}(1.51)^{0.3}\}^{0.2}\{(1.75)^{0.1}(1.51)^{0.3}(1.75)^{0.3}(1.84)^{0.3}\}^{0.2} \\ \{(1.64)^{0.1}(1.36)^{0.3}(1.64)^{0.3}(1.51)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.75)^{0.3}(1.64)^{0.3}(1.51)^{0.3}\}^{0.4} \\ + \\ \{(0.49)^{0.1}(0.36)^{0.3}(0.25)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.41)^{0.3}(0.25)^{0.3}(0.16)^{0.3}\}^{0.2} \\ \{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.36)^{0.3}(0.49)^{0.3}\}^{0.4}\end{bmatrix}}} \right\rangle$$

$$= \left\langle \sqrt{\frac{(1.0278)(1.0858)(1.0270)(1.1104) - [(0.9681)(0.8406)(0.9684)(0.8457)]}{(1.0278)(1.0858)(1.0270)(1.1104) + [(0.9681)(0.8406)(0.9684)(0.8457)]}}, \sqrt{\frac{2[(0.8174)(0.7682)(0.8595)(0.6389)]}{(1.1010)(1.1113)(1.0862)(1.2218) + (0.8174)(0.7682)(0.8595)(0.6389)}} \right\rangle$$

$$\mathscr{H}_2 = \langle 0.5591, 0.5918 \rangle,$$

$$\mathcal{H}_3 = \left\langle \sqrt{\dfrac{\begin{array}{l}\{(1.25)^{0.1}(1.36)^{0.3}(1.09)^{0.3}(1.36)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.09)^{0.3}\}^{0.2}\\\{(1.16)^{0.1}(1.09)^{0.3}(1.25)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.04)^{0.3}(1.04)^{0.3}(1.25)^{0.3}\}^{0.4}\\-\\[(0.75)^{0.1}(0.64)^{0.3}(0.91)^{0.3}(0.64)^{0.3}]^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.91)^{0.3}\}^{0.2}\\\{(0.84)^{0.1}(0.91)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.96)^{0.3}(0.96)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}{\begin{array}{l}\{(1.25)^{0.1}(1.36)^{0.3}(1.09)^{0.3}(1.36)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.81)^{0.3}(1.49)^{0.3}(1.09)^{0.3}\}^{0.2}\\\{(1.16)^{0.1}(1.09)^{0.3}(1.25)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.04)^{0.3}(1.04)^{0.3}(1.25)^{0.3}\}^{0.4}\\+\\[(0.75)^{0.1}(0.64)^{0.3}(0.91)^{0.3}(0.64)^{0.3}]^{0.2}\{(0.51)^{0.1}(0.19)^{0.3}(0.51)^{0.3}(0.91)^{0.3}\}^{0.2}\\\{(0.84)^{0.1}(0.91)^{0.3}(0.75)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.96)^{0.3}(0.96)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}} \; , \right.$$

$$\left. \sqrt{\dfrac{2\left[\begin{array}{l}\{(0.49)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.01)^{0.3}(0.04)^{0.3}(0.25)^{0.3}\}^{0.2}\\\{(0.36)^{0.1}(0.36)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.36)^{0.3}\}^{0.4}\end{array}\right]}{\begin{array}{l}\{(1.51)^{0.1}(1.64)^{0.3}(1.36)^{0.3}(1.51)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.99)^{0.3}(1.96)^{0.3}(1.75)^{0.3}\}^{0.2}\\\{(1.64)^{0.1}(1.64)^{0.3}(1.64)^{0.3}(1.84)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.75)^{0.3}(1.75)^{0.3}(1.64)^{0.3}\}^{0.4}\\+\\\{(0.49)^{0.1}(0.36)^{0.3}(0.64)^{0.3}(0.49)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.01)^{0.3}(0.04)^{0.3}(0.25)^{0.3}\}^{0.2}\\\{(0.36)^{0.1}(0.36)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.36)^{0.3}\}^{0.4}\end{array}}} \right\rangle$$

$$= \left\langle \sqrt{\dfrac{(1.0476)(1.0753)(1.0355)(1.0575) - [(0.9370)(0.8528)(0.9572)(0.9183)]}{(1.0476)(1.0753)(1.0355)(1.0575) + [(0.9370)(0.8528)(0.9572)(0.9183)]}} \; , \; \sqrt{\dfrac{2[(0.8762)(0.5547)(0.7764)(0.5894)]}{(1.0844)(1.1359)(1.1116)(1.2436) + (0.8762)(0.5547)(0.7764)(0.5894)}} \right\rangle$$

$$\mathcal{H}_3 = \langle 0.5238, 0.4806 \rangle,$$

$$\mathcal{H}_4 = \left\langle \sqrt{\dfrac{\begin{array}{l}\{(1.25)^{0.1}(1.16)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.64)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2}\\\{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.36)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4}\\-\\[(0.75)^{0.1}(0.84)^{0.3}(0.84)^{0.3}(0.91)^{0.3}]^{0.2}\{(0.51)^{0.1}(0.36)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\\\{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.64)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}{\begin{array}{l}\{(1.25)^{0.1}(1.16)^{0.3}(1.16)^{0.3}(1.09)^{0.3}\}^{0.2}\{(1.49)^{0.1}(1.64)^{0.3}(1.49)^{0.3}(1.36)^{0.3}\}^{0.2}\\\{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.36)^{0.1}(1.16)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.4}\\+\\[(0.75)^{0.1}(0.84)^{0.3}(0.84)^{0.3}(0.91)^{0.3}]^{0.2}\{(0.51)^{0.1}(0.36)^{0.3}(0.51)^{0.3}(0.64)^{0.3}\}^{0.2}\\\{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.64)^{0.1}(0.84)^{0.3}(0.91)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}} \; , \right.$$

$$\left. \sqrt{\dfrac{2\left[\begin{array}{l}\{(0.64)^{0.1}(0.36)^{0.3}(0.49)^{0.3}(0.36)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\}^{0.2}\\\{(0.49)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.49)^{0.3}(0.25)^{0.3}(0.36)^{0.3}\}^{0.4}\end{array}\right]}{\begin{array}{l}\{(1.36)^{0.1}(1.64)^{0.3}(1.51)^{0.3}(1.64)^{0.3}\}^{0.2}\{(1.75)^{0.1}(1.99)^{0.3}(1.91)^{0.3}(1.75)^{0.3}\}^{0.2}\\\{(1.51)^{0.1}(1.51)^{0.3}(1.64)^{0.3}(1.84)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.51)^{0.3}(1.75)^{0.3}(1.64)^{0.3}\}^{0.4}\\+\\\{(0.64)^{0.1}(0.36)^{0.3}(0.49)^{0.3}(0.36)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\}^{0.2}\\\{(0.49)^{0.1}(0.49)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.49)^{0.3}(0.25)^{0.3}(0.36)^{0.3}\}^{0.4}\end{array}}} \right\rangle$$

$$= \left\langle \sqrt{\dfrac{(1.0278)(1.0833)(1.0309)(1.0695) - [(0.9681)(0.8676)(0.9638)(0.9188)]}{(1.0278)(1.0833)(1.0309)(1.0695) + [(0.9681)(0.8676)(0.9638)(0.9188)]}} \; , \; \sqrt{\dfrac{2[(0.8400)(0.5976)(0.7958)(0.6389)]}{(1.0944)(1.1330)(1.1043)(1.2292) + (0.8400)(0.5976)(0.7958)(0.6389)}} \right\rangle$$

$$\mathcal{H}_4 = \langle 0.4953, 0.5131 \rangle,$$

$$\mathcal{H}_5 = \left\langle \sqrt{\dfrac{\begin{array}{l}\{(1.25)^{0.1}(1.16)^{0.3}(1.16)^{0.3}(1.16)^{0.3}\}^{0.2}\{(1.36)^{0.1}(1.64)^{0.3}(1.49)^{0.3}(1.49)^{0.3}\}^{0.2}\\\{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.25)^{0.1}(1.49)^{0.3}(1.36)^{0.3}(1.25)^{0.3}\}^{0.4}\\-\\[(0.75)^{0.1}(0.84)^{0.3}(0.84)^{0.3}(0.84)^{0.3}]^{0.2}\{(0.64)^{0.1}(0.36)^{0.3}(0.51)^{0.3}(0.51)^{0.3}\}^{0.2}\\\{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.75)^{0.1}(0.51)^{0.3}(0.64)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}{\begin{array}{l}\{(1.25)^{0.1}(1.16)^{0.3}(1.16)^{0.3}(1.16)^{0.3}\}^{0.2}\{(1.36)^{0.1}(1.64)^{0.3}(1.49)^{0.3}(1.49)^{0.3}\}^{0.2}\\\{(1.16)^{0.1}(1.09)^{0.3}(1.16)^{0.3}(1.25)^{0.3}\}^{0.2}\{(1.25)^{0.1}(1.49)^{0.3}(1.36)^{0.3}(1.25)^{0.3}\}^{0.4}\\+\\[(0.75)^{0.1}(0.84)^{0.3}(0.84)^{0.3}(0.84)^{0.3}]^{0.2}\{(0.64)^{0.1}(0.36)^{0.3}(0.51)^{0.3}(0.51)^{0.3}\}^{0.2}\\\{(0.84)^{0.1}(0.91)^{0.3}(0.84)^{0.3}(0.75)^{0.3}\}^{0.2}\{(0.75)^{0.1}(0.51)^{0.3}(0.64)^{0.3}(0.75)^{0.3}\}^{0.4}\end{array}}} \; , \right.$$

$$\left. \sqrt{\dfrac{2\left[\begin{array}{l}\{(0.36)^{0.1}(0.36)^{0.3}(0.49)^{0.3}(0.36)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\}^{0.2}\\\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.64)^{0.3}\}^{0.4}\end{array}\right]}{\begin{array}{l}\{(1.64)^{0.1}(1.64)^{0.3}(1.51)^{0.3}(1.64)^{0.3}\}^{0.2}\{(1.75)^{0.1}(1.99)^{0.3}(1.91)^{0.3}(1.75)^{0.3}\}^{0.2}\\\{(1.64)^{0.1}(1.36)^{0.3}(1.64)^{0.3}(1.84)^{0.3}\}^{0.2}\{(1.84)^{0.1}(1.75)^{0.3}(1.75)^{0.3}(1.36)^{0.3}\}^{0.4}\\+\\\{(0.36)^{0.1}(0.36)^{0.3}(0.49)^{0.3}(0.36)^{0.3}\}^{0.2}\{(0.25)^{0.1}(0.01)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\}^{0.2}\\\{(0.36)^{0.1}(0.64)^{0.3}(0.36)^{0.3}(0.16)^{0.3}\}^{0.2}\{(0.16)^{0.1}(0.25)^{0.3}(0.25)^{0.3}(0.64)^{0.3}\}^{0.4}\end{array}}} \right\rangle$$

$$= \left\langle \sqrt{\dfrac{(1.0316)(1.0872)(1.0309)(1.1280) - [(0.9635)(0.8598)(0.9638)(0.8349)]}{(1.0316)(1.0872)(1.0309)(1.1280) + [(0.9635)(0.8598)(0.9638)(0.8349)]}} \; , \; \sqrt{\dfrac{2[(0.8301)(0.5876)(0.8037)(0.6315)]}{(1.0985)(1.1330)(1.0992)(1.2160) + (0.8301)(0.5876)(0.8037)(0.6315)}} \right\rangle$$

$$\mathcal{H}_5 = \langle 0.5687, 0.5089 \rangle. \tag{95}$$

Step 4. Compute the score values using equation (6); $S = \alpha_{ij}^2 - \ell_{ij}^2$.

$$S(\mathcal{H}_1) = 0.0039, \; S(\mathcal{H}_2) = -0.0376, \; S(\mathcal{H}_3) = 0.0433,$$

$$S(\mathcal{H}_4) = -0.0179, \; S(\mathcal{H}_5) = 0.0644. \tag{96}$$

Step 5. Compute the ranking of the alternatives $S(\mathcal{H}_5) > S(\mathcal{H}_3) > S(\mathcal{H}_1) > S(\mathcal{H}_4) > S(\mathcal{H}_2)$. So, $\mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1 > \mathfrak{H}^4 > \mathfrak{H}^2$.

So, $\mathfrak{H}^5$ is the most suitable alternative.

4.2.2. By the PFSEWG Operator

Step 1. Obtain the Pythagorean fuzzy soft decision matrices (Tables 1–5).

Step 2. Use the normalization formula to normalize the obtained Pythagorean fuzzy soft decision matrices (Tables 6–10).

Step 3. Using the PFSEWG operator acquired the aggregated values of each alternative in the form of PFSN such as

$$\mathcal{H}_1 = \left\langle \frac{\sqrt{2\left[\{(0.25)^{0.1}(0.25)^{0.3}(0.16)^{0.3}(0.09)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.81)^{0.3}(0.36)^{0.3}(0.25)^{0.3}\}^{0.2}\atop \{(0.16)^{0.1}(0.09)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.16)^{0.3}(0.09)^{0.3}(0.25)^{0.3}\}^{0.4}\right]}}{\cdots}, \cdots \right\rangle$$

$$\mathcal{H}_1 = \left\langle \frac{\sqrt{2[(0.4953)(0.6938)(0.5664)(0.3355)]}}{\sqrt{(1.2346)(1.1676)(1.2208)(1.4786) + (0.4953)(0.6938)(0.5664)(0.3355)}}, \frac{\sqrt{(1.1477)(1.0651)(1.0872)(1.1850) - [(0.7841)(0.9220)(0.9035)(0.7079)]}}{\sqrt{(1.1477)(1.0651)(1.0872)(1.1850) + [(0.7841)(0.9220)(0.9035)(0.7079)]}} \right\rangle$$

$$\mathcal{H}_1 = \langle 0.2211, 0.7392 \rangle,$$

$$\mathcal{H}_3 = \left\langle \frac{\sqrt{2[(0.4466)(0.8071)(0.7064)(0.5357)]}}{\sqrt{(1.1155)(1.0857)(1.12254)(1.2533) + (074466)(0.8071)(0.7064)(0.5357)}}, \frac{\sqrt{(1.0833)(1.0194)(1.0533)(1.1011) - [(0.8676)(0.9764)(0.8981)(0.8784)]}}{\sqrt{(1.08338)(1.0194)(1.0533)(1.1011) + [(0.8676)(0.9764)(0.8981)(0.8784)]}} \right\rangle$$

$$\mathcal{H}_3 = \langle 0.3696, 0.5605 \rangle,$$

$$\mathcal{H}_2 = \left\langle \frac{\sqrt{2[(0.6756)(0.8657)(0.6644)(0.5719)]}}{\sqrt{(1.1310)(1.0789)(1.1313)(1.2294) + (0.6756)(0.8657)(0.6644)(0.5719)}}, \frac{\sqrt{(1.0658)(1.0295)(1.0813)(1.1246) - [(0.9067)(0.9647)(0.8716)(0.8387)]}}{\sqrt{(1.0658)(1.0295)(1.0813)(1.1246) + [(0.9067)(0.9647)(0.8716)(0.8387)]}} \right\rangle$$

$$\mathcal{H}_2 = \langle 0.4811, 0.5932 \rangle,$$

$$\mathcal{H}_4 = \left\langle \frac{\sqrt{2[(0.5848)(0.8756)(0.6877)(0.5787)]}}{\sqrt{(1.1128)(1.0809)(1.1228)(1.2193) + [(0.5848)(0.8756)(0.6877)(0.5787)]}}, \frac{\sqrt{(1.0692)(1.0239)(1.0652)(1.1262) - [(0.9022)(0.9711)(0.8981)(0.8198)]}}{\sqrt{(1.0692)(1.0239)(1.0652)(1.1262) + [(0.9022)(0.9711)(0.8981)(0.8198)]}} \right\rangle$$

$$\mathcal{H}_4 = \langle 0.4691, 0.5841 \rangle,$$

TABLE 11: Comparison of projected operators with some prevailing operators.

Approach	$\mathfrak{H}^1$	$\mathfrak{H}^2$	$\mathfrak{H}^3$	$\mathfrak{H}^4$	$\mathfrak{H}^5$	Alternative ranking
Proposed PFSEWA operator	0.0039	−0.0376	0.0433	−0.0179	0.0644	$\mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1 > \mathfrak{H}^4 > \mathfrak{H}^2$
Proposed PFSEWG operator	−0.4975	−0.1204	−0.1775	−0.1211	−0.0778	$\mathfrak{H}^5 > \mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$
PFSWA operator	0.0293	0.0369	0.0783	−0.0938	0.0858	$\mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^2 > \mathfrak{H}^1 > \mathfrak{H}^4$
PFSWG operator	−0.3306	−0.1383	−0.1092	−0.1661	−0.5957	$\mathfrak{H}^3 > \mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^1 > \mathfrak{H}^5$
PFEWA operator	0.0039	−0.0376	0.0433	−0.0179	0.0644	$\mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1 > \mathfrak{H}^4 > \mathfrak{H}^2$
PFEWG operator	−0.4975	−0.1204	−0.1775	−0.0778	−0.1211	$\mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^5 > \mathfrak{H}^3 > \mathfrak{H}^1$

$$
\mathcal{H}_5 = \left\langle \begin{array}{c}
\sqrt{\dfrac{2\left[\begin{array}{c}\{(0.25)^{0.1}(0.16)^{0.3}(0.16)^{0.3}(0.09)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.64)^{0.3}(0.49)^{0.3}(0.36)^{0.3}\}^{0.2}\\ \{(0.16)^{0.1}(0.09)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.16)^{0.3}(0.96)^{0.3}(0.25)^{0.3}\}^{0.4}\end{array}\right]}{\begin{array}{c}\{(1.75)^{0.1}(1.84)^{0.3}(1.84)^{0.3}(1.91)^{0.3}\}^{0.2}\{(1.51)^{0.1}(1.36)^{0.3}(1.51)^{0.3}(1.64)^{0.3}\}^{0.2}\\ \{(1.84)^{0.1}(1.91)^{0.3}(1.84)^{0.3}(1.75)^{0.3}\}^{0.2}\{(1.64)^{0.1}(1.84)^{0.3}(1.04)^{0.3}(1.75)^{0.3}\}^{0.4}\\ +\\ \{(0.25)^{0.1}(0.16)^{0.3}(0.16)^{0.3}(0.09)^{0.3}\}^{0.2}\{(0.49)^{0.1}(0.64)^{0.3}(0.49)^{0.3}(0.36)^{0.3}\}^{0.2}\\ \{(0.16)^{0.1}(0.09)^{0.3}(0.16)^{0.3}(0.25)^{0.3}\}^{0.2}\{(0.36)^{0.1}(0.16)^{0.3}(0.96)^{0.3}(0.25)^{0.3}\}^{0.4}\end{array}}}, \\[40pt]
\sqrt{\dfrac{\left[\begin{array}{c}\{(0.36)^{0.1}(0.64)^{0.3}(0.64)^{0.3}(0.51)^{0.3}\}^{0.2}\{(0.75)^{0.1}(0.99)^{0.3}(0.51)^{0.3}(0.75)^{0.3}\}^{0.2}\\ \{(0.51)^{0.1}(0.51)^{0.3}(0.64)^{0.3}(0.36)^{0.3}\}^{0.2}\{(0.84)^{0.1}(0.75)^{0.3}(0.75)^{0.3}(0.64)^{0.3}\}^{0.4}\end{array}\right]}{\begin{array}{c}\{(1.64)^{0.1}(1.36)^{0.3}(1.36)^{0.3}(1.49)^{0.3}\}^{0.2}\{(1.25)^{0.1}(1.01)^{0.3}(1.09)^{0.3}(1.25)^{0.3}\}^{0.2}\\ \{(1.49)^{0.1}(1.49)^{0.3}(1.36)^{0.3}(1.64)^{0.3}\}^{0.2}\{(1.16)^{0.1}(1.49)^{0.3}(1.25)^{0.3}(1.36)^{0.3}\}^{0.4}\\ +\\ \left[(0.36)^{0.1}(0.64)^{0.3}(0.64)^{0.3}(0.51)^{0.3}\right]^{0.2}\{(0.75)^{0.1}(0.99)^{0.3}(0.51)^{0.3}(0.75)^{0.3}\}^{0.2}\\ \{(0.51)^{0.1}(0.51)^{0.3}(0.64)^{0.3}(0.36)^{0.3}\}^{0.2}\{(0.84)^{0.1}(0.75)^{0.3}(0.75)^{0.3}(0.64)^{0.3}\}^{0.4}\end{array}}}
\end{array}\right\rangle
$$

$$
= \left\langle \sqrt{\dfrac{2[(0.6756)(0.8648)(0.6877)(0.6491)]}{\begin{array}{c}(1.1310)(1.0844)(1.1228)(1.1791) +\\ (0.6756)(0.8648)(0.6877)(0.6491)\end{array}}}, \sqrt{\dfrac{\begin{array}{c}(1.0732)(1.0239)(1.0833)(1.1246) -\\ {[(0.8919)(0.9379)(0.8676)(0.8788)]}\end{array}}{\begin{array}{c}(1.0732)(1.0239)(1.0833)(1.1246) +\\ {[(0.8919)(0.9379)(0.8676)(0.8788)]}\end{array}}} \right\rangle
$$

$$
= \langle 0.5260, 0.5954 \rangle .
$$

(97)

Step 4. Compute the score values using equation (6); $S = \alpha_{ij}^2 - \mathscr{E}_{ij}^2$.

$$
S(\mathcal{H}_1) = -0.4975, S(\mathcal{H}_2) = -0.1204,
$$

$$
S(\mathcal{H}_3) = -0.1775, S(\mathcal{H}_4) = -0.1211, S(\mathcal{H}_5) = -0.0778.
$$

(98)

Step 5. Compute the ranking of the alternatives $S(\mathcal{H}_5) > S(\mathcal{H}_2) > S(\mathcal{H}_4) > S(\mathcal{H}_3) > S(\mathcal{H}_1)$. So, $\mathfrak{H}^5 > \mathfrak{H}^2 > \mathfrak{H}^4 > \mathfrak{H}^3 > \mathfrak{H}^1$.

So, $\mathfrak{H}^5$ is the most suitable alternative.

5. Comparative Studies

To demonstrate the efficiency of the projected technique, a comparison with some prevailing methods under the PFS and PFSS environments is presented.

5.1. Comparative Analysis. To authenticate the usefulness of the anticipated technique, we compare the obtained results with some existing techniques under the environment of PFS and PFSS. A summary of all outcomes is given in Table 11. Firstly, we present a comparison with methods proposed by Zulqarnain et al. [37]. Their proposed aggregation operators are based on algebraic norms, while the proposed operators in this work are based on the Einstein norms. Secondly, we compare PFEWA and PFEWG operators proposed by Garg [15]. He developed the DM technique for PFNs by utilizing the Einstein norms that cannot accommodate the parametrized values of the alternatives. On the other hand, our established approach capably contracts with parametrized values of the alternatives and supplies superior facts comparative to existing methods. In this work, two aggregation operators, such as PFSEWA and PFSEWG operators, are planned to fuse the assessment data and then use the score function to estimate the ranking of the alternatives. Moreover, PFSS theory condenses to PFS if only one parameter is presumed. Therefore, the PFSS concept is the most general Pythagorean fuzzy set (PFS). So, the operators planned in this work are influential, more consistent, and more effective based on the above facts.

6. Conclusion

The core objective of this research is to formulate some operational laws using the Einstein norms. Then, we developed two new operators such as PFSEWA and PFSEWG, using our established operational laws. Some fundamental properties have been discussed for developed Einstein AOs, such as idempotency, homogeneity, and boundedness. Moreover, a MAGDM approach has been established to solve real-life complications based on developed Einstein AOs. To confirm the strength of a specified method, we deliver an inclusive numerical illustration for selecting the best business for investors. A comparative analysis with some prevailing approaches is offered, showing the constructed approach's practicality. Finally, based on the attained outcomes, it is determined that the technique

anticipated in this study is the utmost reasonable and adequate to solve the MAGDM difficulties. Further research will use many other operators under PFSS to present DM techniques. In addition, several other structures can be proven and projected, such as topological structures, algebraic structures, and sequential structures. This study will open new avenues for researchers in this field.

Data Availability

No such type of data is used in this manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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