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Interpolative Kannan- Meir-Keeler type contraction

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Abstract

In this short manuscript, we revisit the renowned contraction's of Meir-Keeler by involving the interpolation theory in the context of complete metric space. We provide a simple example to illustrate the validity of the observed result.

Keywords: Kannan Type Contraction, Interpolation, fixed point, Meir-Keeler contraction
2010 MSC: 47H 10, 49T99, 54H25

1. Introduction

There is no need to emphasize it again, as it is very well known how effectively fixed point theory is used in nonlinear functional analysis, topology, and applied mathematics. Instead, it makes more sense to emphasize the effective use of fixed point theory in all branches of qualitative science. In this paper, we combine three exciting ideas and trends in the metric fixed point theory. After the famous fixed point theorem of Banach [1], one of the most significant advances in metric fixed point theory was given by Kannan [3, 4]. It was later understood that the Banach contraction and Kannan contractions are independent [1]. On the other hand, in another aspect, Meir-Keeler [6] proposed an interesting contraction inequality that can be called a uniform contraction. Another interesting contraction concept that was recently published is interpolative contraction [5].

This paper aims to revisit the slightly modified version of the Meir-Keeler type contraction that is observed by combining the Kannan type contraction and interpolative contraction. We first recall the basic definitions and results.

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Definition 1.1. *Let* (X, d) *be a complete metric space. A mapping* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *is said to be a* Meir-Keeler contraction *on* X *(on short,* MK*-contraction), if for every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\varepsilon \leq d(\mathsf{x}, \mathsf{y}) < \varepsilon + \delta \ \Rightarrow \ d(\mathsf{Tx}, \mathsf{Ty}) < \varepsilon, \tag{1.1}$$

for every $\mathsf{x}, \mathsf{y} \in \mathsf{X}$.

Theorem 1.2. *[6] On a complete metric space* (X, d)*, any* MK*-contraction* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *has a unique fixed point.*

Definition 1.3. *Let* (X, d) *be a complete metric space. A mapping* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *is said to be an* interpolative Kannan type contraction *on* X *(on short,* MK*-contraction), if there exist* $\kappa \in [0, 1)$ *and* $\gamma \in (0, 1)$ *such that*

$$d(\mathsf{Tx}, \mathsf{Ty}) \leq \kappa [d(\mathsf{x}, \mathsf{Tx})]^{\gamma} [d(\mathsf{y}, \mathsf{Ty})]^{1-\gamma}, \tag{1.2}$$

for every $\mathsf{x}, \mathsf{y} \in \mathsf{X} \setminus \mathrm{Fix}(\mathsf{T})$*, where* $\mathrm{Fix}(\mathsf{T}) = \{\mathsf{x} \in \mathsf{X} | \mathsf{Tx} = \mathsf{x}\}$.

Theorem 1.4. *[5] On a complete metric space* (X, d)*, any interpolative Kannan-contraction* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *has a fixed point.*

2. Main Results

We start this section with the definition of *interpolative Kannan-Meir-Keeler type contraction.*

Definition 2.1. *Let* (X, d) *be a complete metric space. A mapping* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *is said to be an* interpolative Kannan-Meir-Keeler type contraction *on* X *(on short,* KMK*-contraction), if there exists* $\gamma \in (0, 1)$ *such that for every* $\mathsf{x}, \mathsf{y} \in \mathsf{X} \setminus \mathrm{Fix}(\mathsf{T})$ *we have*

(1) given $\varepsilon > 0$*, there exists* $\delta > 0$ *so that*

$$\varepsilon < [d(\mathsf{x}, \mathsf{Tx})]^{\gamma} [d(\mathsf{y}, \mathsf{Ty})]^{1-\gamma} < \varepsilon + \delta \implies d(\mathsf{Tx}, \mathsf{Ty}) \leq \varepsilon, \tag{2.1}$$

(2)

$$d(\mathsf{Tx}, \mathsf{Ty}) < d(\mathsf{x}, \mathsf{Tx})]^{\gamma} [d(\mathsf{y}, \mathsf{Ty})]^{1-\gamma}. \tag{2.2}$$

Theorem 2.2. *On a complete metric space* (X, d)*, any interpolative* KMK*-contraction* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *has a fixed point.*

Proof. Starting with a point $\mathsf{x}_0 \in \mathsf{X}$, we build the sequence $\{\mathsf{x}_m\}$, by the following rule:

$$\mathsf{x}_m = \mathsf{T}\mathsf{x}_{m-1} = \mathsf{T}^m \mathsf{x}_0,$$

for all $m \in \mathbb{N}$. Thus, by the assumption (2), we have

$$\begin{aligned} d(\mathsf{x}_m, \mathsf{x}_{m+1}) &= d(\mathsf{T}\mathsf{x}_{m-1}, \mathsf{T}\mathsf{x}_m) < [d(\mathsf{x}_{m-1}, \mathsf{T}\mathsf{x}_{m-1})]^{\gamma} [d(\mathsf{x}_m, \mathsf{T}\mathsf{x}_m)]^{1-\gamma} \\ &= [d(\mathsf{x}_{m-1}, \mathsf{x}_m)]^{\gamma} [d(\mathsf{x}_m, \mathsf{x}_{m+1})]^{1-\gamma}, \end{aligned}$$

and then, equivalent,

$$[d(\mathsf{x}_m, \mathsf{x}_{m+1})]^{\gamma} < [d(\mathsf{x}_{m-1}, \mathsf{x}_m)]^{\gamma}.$$

Then, the sequence $\{d(\mathsf{x}_m, \mathsf{x}_{m+1})\}$ is strictly decreasing and since $d(\mathsf{x}_m, \mathsf{x}_{m+1}) > 0$, for every $m \in \mathbb{N} \cup \{0\}$, it follows that the sequence $\{d(\mathsf{x}_m, \mathsf{x}_{m+1})\}$ tends to a point $\omega \geq 0$. We claim that $\omega = 0$. Indeed, if we suppose that $\omega > 0$, we can find $N \in \mathbb{N}$, such that

$$\omega < d(\mathsf{x}_m, \mathsf{x}_{m+1}) < \omega + \delta(\omega),$$

for any $m \geq N$. Then, since $\omega < d(\mathsf{x}_m, \mathsf{x}_{m+1}) < [d(\mathsf{x}_{m-1}, \mathsf{x}_m)]^\gamma [d(\mathsf{x}_m, \mathsf{x}_{m+1})]^{1-\gamma}$, keeping in mind (2.1), it follows that $d(\mathsf{x}_m, \mathsf{x}_{m+1}) \leq \omega$, for any $m \geq N$. This is a contradiction, and that's why we get $\omega = 0$.
In order to show that $\{\mathsf{x}_m\}$ is a Cauchy sequence, let $\varepsilon > 0$ be fixed and we can consider that $\delta(\varepsilon)$ can be choose such that $\delta(\varepsilon) < \varepsilon$. Since $\lim_{m\to\infty} d(\mathsf{x}_m, \mathsf{x}_{m+1}) = 0$, we can find $l \in \mathbb{N}$ such that $d(\mathsf{x}_m, \mathsf{x}_{m+1}) < \frac{\varepsilon}{2}$, for $m \geq l$, and we claim that

$$d(\mathsf{x}_m, \mathsf{x}_{m+p}) < \varepsilon, \tag{2.3}$$

, for any $p \in \mathbb{N}$. Of course, the above inequality holds for $p = 1$. Supposing that for some p, (2.3) holds, we will prove it for $p + 1$. Indeed, using the triangle inequality, together with (2.2) we have

$$\begin{aligned} d(\mathsf{x}_m, \mathsf{x}_{m+p+1}) &\leq d(\mathsf{x}_m, \mathsf{x}_m + 1) + d(\mathsf{x}_{m+1}, \mathsf{x}_{m+p+1}) \\ &= d(\mathsf{x}_m, \mathsf{x}_{m+1}) + d(\mathsf{T}\mathsf{x}_m, \mathsf{T}\mathsf{x}_{m+p}) \\ &< d(\mathsf{x}_m, \mathsf{x}_{m+1}) + [d(\mathsf{x}_m, \mathsf{x}_{m+1})]^\gamma [d(\mathsf{x}_{m+p}, \mathsf{x}_{m+p+1})]^{1-\gamma} \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, the sequence $\{\mathsf{x}_m\}$ is Cauchy and by the completeness of the space X it follows that there exists $\mathsf{x}_* \in \mathsf{X}$ such that

$$\lim_{m\to\infty} \mathsf{x}_m = \mathsf{x}_*. \tag{2.4}$$

We shall show that $\mathsf{x}_* = \mathsf{T}\mathsf{x}_*$. Supposing on the contrary, that $\mathsf{x}_* \neq \mathsf{T}\mathsf{x}_*$, by (2.2) we have

$$\begin{aligned} 0 < d(\mathsf{x}_*, \mathsf{T}\mathsf{x}_*) \leq \; & d(\mathsf{x}_*, \mathsf{x}_{m+1}) + d(\mathsf{x}_{m+1}, \mathsf{T}\mathsf{x}_*) = d(\mathsf{x}_*, \mathsf{x}_{m+1}) + d(\mathsf{T}\mathsf{x}_m, \mathsf{T}\mathsf{x}_*) \\ & < [d(\mathsf{x}_m, \mathsf{T}\mathsf{x}_m)]^\gamma [d(\mathsf{x}_*, \mathsf{T}\mathsf{x}_*)]^{1-\gamma} \\ & = [d(\mathsf{x}_m, \mathsf{x}_{m+1})]^\gamma [d(\mathsf{x}_*, \mathsf{T}\mathsf{x}_*)]^{1-\gamma} \to 0 \text{ as } m \to \infty. \end{aligned}$$

Therefore, $d(\mathsf{x}_*, \mathsf{T}\mathsf{x}_*) = 0$, that is, x_* is a fixed point of the mapping T. □

Example 2.3. *Let* $\mathsf{X} = \mathbb{R}^2$ *and* $\mathcal{A} = \{A, B, C, D\}$, *where* $A = (1, -1)$, $B = (-1, 0)$, $C = (2, -1)$, $D = (2, 0)$. *Let* $d : \mathsf{X} \times \mathsf{X} \to [0, \infty)$ *be defined as* $d(P, Q) = \sqrt{(\mathsf{x}_1 - \mathsf{y}_1)^2 + (\mathsf{x}_2 - \mathsf{y}_2)^2}$ *for any* $P, Q \in \mathsf{X}$, $P = (\mathsf{x}_1, \mathsf{x}_2)$, $Q = (\mathsf{y}_1, \mathsf{y}_2)$, *with* $\mathsf{x}_1, \mathsf{x}_2, \mathsf{y}_1, \mathsf{y}_2 \in \mathbb{R}$. *Define the mapping* $\mathsf{T} : \mathsf{X} \to \mathsf{X}$ *as follows*

$$\mathsf{T}A = \mathsf{T}C = \mathsf{T}D = C, \mathsf{T}B = D, \text{ and } \mathsf{T}P = P \text{ for any } P \in \mathsf{X} \backslash \mathcal{A}.$$

We choose $\gamma = \frac{1}{2}$.
Thus, we claim that T *satisfies the conditions of Theorem 2.2.*
Indeed, for $\varepsilon < 1$, *with* $\delta = \sqrt{2} - \varepsilon$,

$$\varepsilon < 1 \;\; = \sqrt{d(A, \mathsf{T}A) d(D, \mathsf{T}D)} = \sqrt{d(A, C) d(D, C)} < \sqrt{2} = \varepsilon + \delta \Rightarrow$$

$$d(\mathsf{T}A, \mathsf{T}D) = d(C, C) = 0 < \varepsilon,$$

and also

$$d(\mathsf{T}A, \mathsf{T}D) < \sqrt{d(A, C) d(D, C)}.$$

For $\varepsilon \geq 1$, *choosing* $\delta = 1$, *we get*

$$\varepsilon \;\; < \sqrt{d(A, \mathsf{T}A) d(B, \mathsf{T}B)} = \sqrt{d(A, C) d(B, D)} = \sqrt{3} < \varepsilon + \delta \Rightarrow$$

$$d(\mathsf{T}A, \mathsf{T}B) = d(C, D) = 1 < \varepsilon,$$

and

$$d(\mathsf{T}A, \mathsf{T}B) = 1 < \sqrt{3} = \sqrt{d(A, C) d(B, D)}.$$

Similar,

$$\varepsilon \;\; < \sqrt{d(D, \mathsf{T}D) d(B, \mathsf{T}B)} = \sqrt{d(D, C) d(B, D)} = \sqrt{3} < \varepsilon + \delta \Rightarrow$$

$$d(\mathsf{T}D, \mathsf{T}B) = d(C, D) = 1 < \varepsilon,$$

and

$$\mathscr{d}(\mathsf{T}D, \mathsf{T}B) = 1 < \sqrt{3} = \sqrt{\mathscr{d}(A, C)\mathscr{d}(B, D)}.$$

Consequently, by Theorem 2.2, the mappings T *has fixed points; these are* $C = (2, -1)$ *and all* $P \in \mathsf{X} \backslash \mathcal{A}$*.*

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