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Article

Estimates for a Rough Fractional Integral Operator and Its Commutators on p-Adic Central Morrey Spaces

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Abstract: In the current paper, we obtain the boundedness of a rough p-adic fractional integral operator on p-adic central Morrey spaces. Moreover, we establish the λ-central bounded mean oscillations estimate for commutators of a rough p-adic fractional integral operator on p-adic central Morrey spaces.

Keywords: central Morrey spaces; commutators; rough p-adic fractional integral operator; λ-central boundedned mean oscillations

MSC: 42B20; 42B25

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1. Introduction

In this day and age, fractional calculus is a key area because of its heaps of applications in engineering science and technology, see for instance [1,2]. Moreover, fractional integral operators are major part of the mathematical analysis. These operators have been used to formulate and construct new results in the theory of inequalities. Many of the familiar inequalities and relevant results are generalized and extended via fractional integral operators [3,4]. Fractional integral operator of order β is defined by

$$T_\beta f(x) = \frac{1}{\zeta_n(\beta)} \int_{\mathbb{R}^n} \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-\beta}} d\mathbf{y}, \tag{1}$$

where $\zeta_n(\beta) = \frac{\pi^{n/2} 2^\beta \Gamma(\beta/2)}{\Gamma((n-\beta)/2)}$. A fractional integral operator is a smooth operator and has been applied in several branches such as partial differential equations, harmonic analysis, non-linear control theory, and potential analysis, see for example [5,6] and references therein. Over the years, the boundedness properties of T_β has put many researchers in the spotlight [7–9].

In the last few years. the field of p-adic numbers $\mathbb{Q}_p$ is wildly used in harmonic analysis [10–12] and mathematical physics [13,14]. Let p be a prime number. The field of the p-adic absolute value $|y|_p$ is defined by setting $|0|_p = 0$,

$$|y|_p = p^{-\gamma} \quad if \quad y = p^\gamma \frac{s}{t},$$

where $\gamma, s, t \in \mathbb{Z}$, and p, s and t are coprime. $|\cdot|_p$ undergoes many axioms of a real norm with the below ultrametric inequality

$$|y+z|_p \le \max\{|y|_p, |z|_p\}. \tag{2}$$

In [14], we see that any $y \neq 0 \in \mathbb{Q}_p$ can be uniquely represented as:

$$y = p^{\gamma} \sum_{i=0}^{\infty} \alpha_i p^i, \tag{3}$$

where $\alpha_i, \gamma \in \mathbb{Z}, \alpha_i \in \frac{\mathbb{Z}}{p\mathbb{Z}_p}, \alpha_0 \neq 0$. The convergent of the series (3) is from $|p^{\gamma}\alpha_i p^i|_p = p^{-\gamma-i}$.

The space $\mathbb{Q}_p^n$ consists all n-tuples of $\mathbb{Q}_p$ with the following norm

$$|\mathbf{y}|_p = \max_{1 \leq k \leq n} |y_k|_p. \tag{4}$$

Now, let

$$B_{\gamma}(\mathbf{a}) = \{\mathbf{y} \in \mathbb{Q}_p^n : |\mathbf{y} - \mathbf{a}|_p \leq p^{\gamma}\},\ S_{\gamma}(\mathbf{a}) = \{\mathbf{y} \in \mathbb{Q}_p^n : |\mathbf{y} - \mathbf{a}|_p = p^{\gamma}\}$$

be, respectively, the ball and sphere with radius p^{γ} and the center at $\mathbf{a}$.

It is a familiar fact that $\mathbb{Q}_p^n$ is a locally compact commutative group under addition; denote by $d\mathbf{y}$, the Haar measure on $\mathbb{Q}_p^n$ normalized by $\int_{B_0(\mathbf{0})} d\mathbf{y} = 1$. Additionally, $\int_{B_{\gamma}(\mathbf{a})} d\mathbf{y} = p^{n\gamma}$ and $\int_{S_{\gamma}(\mathbf{a})} d\mathbf{y} = p^{n\gamma}(1 - p^{-n})$, for any $\mathbf{a} \in \mathbb{Q}_p^n$.

Suppose $L^r(\mathbb{Q}_p^n)$ $(1 \leq r < \infty)$ is the space of all complex-valued functions f on $\mathbb{Q}_p^n$ such that

$$\|f\|_{L^r(\mathbb{Q}_p^n)} = \left(\int_{\mathbb{Q}_p^n} |f(\mathbf{y})|^r d\mathbf{y} \right)^{1/r} < \infty.$$

In [15], author introduced the fractional integral operator on $\mathbb{Q}_p^n$ as

$$T_{\beta}^{p} f(\mathbf{x}) = \frac{1}{\Gamma_n(\beta)} \int_{\mathbb{Q}_p^n} \frac{f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|_p^{n-\beta}}, \quad 0 < \beta < n,$$

where $\Gamma_n(\beta) = \frac{1-p^{\beta-n}}{1-p^{-\beta}}$.

The explicit formula of the above operator on the p-adic field is acquired in [16,17]. The fundamental properties of the fractional integral operator on local fields are given in [15]. Moreover, λ central bounded mean oscillations estimate for commutators of fractional integral operator on p-adic Morrey spaces are reported in [18]. Recently, the boundedness of the fractional integral operator on Morrey spaces is shown in [12,19]. The current paper deals with the roughness of an operator which is a key topic in analysis in this day and age; see for instance [20,21] and the references therein. Motivated by [21], we define the rough fractional integral operator. Suppose $b\colon \mathbb{Q}_p^n \to \mathbb{R}$, $f\colon \mathbb{Q}_p^n \to \mathbb{R}$ and $\Omega\colon S_0 \to \mathbb{R}$ are measurable mappings, then

$$T_{\beta,\Omega}^{p} f(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{\Omega(|\mathbf{y}|_p \mathbf{y}) f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|_p^{n-\beta}} d\mathbf{y} \tag{5}$$

and

$$T_{\beta,\Omega}^{p,b} f(\mathbf{x}) = \int_{\mathbb{Q}_p^n} \frac{(b(\mathbf{x}) - b(\mathbf{y}))\Omega(|\mathbf{y}|_p \mathbf{y}) f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|_p^{n-\beta}} d\mathbf{y}, \tag{6}$$

respectively, whenever

$$\int_{\mathbb{Q}_p^n} \left| \frac{\Omega(|\mathbf{y}|_p \mathbf{y}) f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|_p^{n-\beta}} \right| d\mathbf{y} < \infty \tag{7}$$

and

$$\int_{\mathbb{Q}_p^n} \left| \frac{b(\mathbf{y})\Omega(|\mathbf{y}|_p \mathbf{y}) f(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|_p^{n-\beta}} \right| d\mathbf{y} < \infty. \tag{8}$$

In this article, we consider the rough fractional integral operator $T^{p}_{\beta,\Omega}$ along with its commutator $T^{p,b}_{\beta,\Omega}$ and acquire the boundedness on p-adic central Morrey spaces. In the latter case, the symbol function is from the λ-*central* bounded mean oscillations$(C\dot{M}O^{s,\lambda})(\mathbb{Q}_p^n)$. The results of the paper can also be implied in locally compact Vilenkin groups and Heisenberg groups. From here on, the letter C means a constant with a different values at separate occurrence.

Definition 1 ([22]). *Suppose* $1 < s < \infty$. *and* $\lambda \in \mathbb{R}$. *The space* $\dot{B}^{s,\lambda}(\mathbb{Q}_p^n)$ *is defined as*

$$\|f\|_{\dot{B}^{s,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda s}} \int_{B_\gamma} |f(\mathbf{x})|^s d\mathbf{x} \right)^{1/s} < \infty,$$

where $B_\gamma = B_\gamma(0)$. *Moreover,* $\dot{B}^{s,\lambda}(\mathbb{Q}_p^n)$ *reduces to* $\{0\}$ *for* $\lambda < -1/s$.

Definition 2 ([22]). *Suppose* $1 < s < \infty$ *and* $\lambda < 1/n$. *The space* $C\dot{M}O^{s,\lambda}(\mathbb{Q}_p^n)$ *is as follows*

$$\|f\|_{C\dot{M}O^{s,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{1}{|B_\gamma|_H^{1+\lambda s}} \int_{B_\gamma} |f(\mathbf{x}) - f_{B_\gamma}|^s d\mathbf{x} \right)^{1/s} < \infty, \tag{9}$$

where $f_{B_\gamma} = \frac{1}{|B_\gamma|_H} \int_{B_\gamma} f(\mathbf{x}) d\mathbf{x}$.

Remark 1. $C\dot{M}O^{s,\lambda}(\mathbb{Q}_p^n)$ *is a mere* $CMO^s(\mathbb{Q}_p^n)$ *for* $\lambda = 0$, *(see [23]).*

Definition 3 ([18]). *Suppose* $1 < s < \infty$ *and* $\lambda \in \mathbb{R}$. *The space* $W\dot{B}^{s,\lambda}(\mathbb{Q}_p^n)$ *is as follows*

$$\|f\|_{W\dot{B}^{s,\lambda}(\mathbb{Q}_p^n)} = \sup_{\gamma\in\mathbb{Z}} \left(\frac{\sup_{\sigma>0} \sigma^s |\{\mathbf{x} \in B_\gamma : |f(\mathbf{x})| > \sigma|_H}{|B_\gamma|_H^{1+\lambda s}} \right)^{1/s} < \infty,$$

where $B_\gamma = B_\gamma(0)$.

It is noteworthy to illustrate the importance of our main results before stating them. The following example will do a world of good in this context.

Example 1. *The solution* $u(\mathbf{y}, t)$ *of the homogeneous Cauchy problem of linear evolutionary pseudo-differential equation*

$$\begin{cases} \frac{\partial u}{\partial t}(\mathbf{y}, t) + T^p_\beta u(\mathbf{y}, t) = 0, & (\mathbf{y}, t) \in \mathbb{Q}_p^n \times \mathbb{R}^+, \\ u(\mathbf{y}, 0) = u^0(\mathbf{y}) \end{cases}$$

is given by $u(\mathbf{y}, t) = (T^p_\beta u^0)(\mathbf{y})$. *For the regularity of the solution, we consider two function spaces* X *and* Y. *Since* T^p_β *is linear, then we have*

$$\|T^p_\beta(u^0 - v^0)\|_Y = \|T^p_\beta(u^0) - T^p_\beta(v^0)\|_Y \le C\|u^0 - v^0\|_X.$$

Here, we came across the boundedness inequality

$$\|T^p_\beta f\|_Y \le C\|f\|_X.$$

It is imperative to mention here that our operator is very helpful in finding the regularity of Cauchy problem of Schrödinger equation.

2. Boundedness of Rough p-Adic Fractional Integral Operator on Central Morrey Spaces

The current section deals the boundednesss of $T^p_{\beta,\Omega}$ on central Morrey spaces. However, in order to do this, we need a lemma which can be proved in the same way as [15].

Lemma 1. *Suppose* $1 \leq q < r < \infty, 0 < \beta < n, \beta/n + 1/r = 1/q$, *and* $\Omega \in L^{q'}(S_{\mathbf{0}}(\mathbf{0}))$.

(i) *If* $f \in L^q(\mathbb{Q}_p^n)$, $q > 1$, *then*

$$\|T^p_{\beta,\Omega} f\|_{L^r(\mathbb{Q}_p^n)} \leq C\|f\|_{L^q(\mathbb{Q}_p^n)}.$$

(ii) *If* $f \in L^1(\mathbb{Q}_p^n)$, $\sigma > 0$, *then*

$$\left|\left\{\mathbf{x} \in \mathbb{Q}_p^n : |T^p_{\beta,\Omega} f(\mathbf{x})| > \sigma\right\}\right|_H \leq C\left(\frac{\|f\|_{L^1(\mathbb{Q}_p^n)}}{\sigma}\right)^r.$$

Now, we turn towards our key result of the section.

Theorem 1. *Suppose* $0 < \beta < n, 1 \leq q < n/\beta, \mu = \beta/n + \lambda, 1/q - 1/r = \beta/n, \lambda < -\beta/n$, *and* $\Omega \in L^{q'}(S_{\mathbf{0}}(\mathbf{0}))$.

(i) *For* $q > 1$, $T^p_{\beta,\Omega}$ *satisfies the following inequality:*

$$\|T^p_{\beta,\Omega} f\|_{\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \leq C\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.$$

(ii) *For* $q = 1$, $T^p_{\beta,\Omega}$ *satisfies the following inequality*

$$\|T^p_{\beta,\Omega} f\|_{W\dot{B}^{r,\mu}(\mathbb{Q}_p^n)} \leq C\|f\|_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}.$$

Proof. (i) Suppose $f \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$. Now for fixed $\gamma \in \mathbb{Z}$, representing $B_\gamma(\mathbf{0})$ by B_γ, we begin as

$$\begin{aligned}
&\left(\frac{1}{|B_\gamma|_H^{1+\mu r}}\int_{B_\gamma}|T^p_{\beta,\Omega} f(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} \\
&\leq \left(\frac{1}{|B_\gamma|_H^{1+\mu r}}\int_{B_\gamma}|T^p_{\beta,\Omega}(f\chi_{B_\gamma})(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} \\
&\quad + \left(\frac{1}{|B_\gamma|_H^{1+\mu r}}\int_{B_\gamma}|T^p_{\beta,\Omega}(f\chi_{B_\gamma^c})(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} \\
&= I + II.
\end{aligned}$$

For I, we make use of Lemma 1 together with $\beta/n = 1/q - 1/r$ and $\mu = \lambda + \beta/n$.

$$\begin{aligned}
I =& \left(\frac{1}{|B_\gamma|_H^{1+\mu r}}\int_{B_\gamma}|T^p_{\beta,\Omega}(f\chi_{B_\gamma})(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} \\
&\leq |B_\gamma|_H^{-1/r-\mu}\left(\int_{B_\gamma}|f\chi_{B_\gamma(\mathbf{x})}|^q d\mathbf{x}\right)^q \\
&\leq \|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{10}$$

For II, first we have

$$\int_{S_k}|\Omega(p^k\mathbf{y})|^{q'}d\mathbf{y} = \int_{|\mathbf{x}|_p=1}|\Omega(\mathbf{x})|^{q'}p^{kn}d\mathbf{x} = Cp^{kn} \tag{11}$$

By the application of Hölder's inequality, equality (11) and $\lambda < -\beta/n$, we proceed as

$$\begin{aligned}
\left|T^p_{\beta,\Omega}(f\chi_{B^c_\gamma})(\mathbf{x})\right| &= \left|\int_{B^c_\gamma} \frac{\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\right| \\
&\le \int_{B^c_\gamma} \frac{|\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y} \\
&= \sum_{k=\gamma+1}^{\infty} p^{-k(n-\beta)} \int_{S_k} |\Omega(p^k\mathbf{y})f(\mathbf{y})|d\mathbf{y} \\
&\le \sum_{k=\gamma+1}^{\infty} p^{-k(n-\beta)} \left(\int_{S_k} |\Omega(p^k\mathbf{y})|^{q'}d\mathbf{y}\right)^{1/q'} \left(\int_{S_k} |f(\mathbf{y})|^q d\mathbf{y}\right)^{1/q} \\
&\le \sum_{k=\gamma+1}^{\infty} p^{-k(n-\beta)} \left(\int_{S_k} |\Omega(p^k\mathbf{y})|^{q'}d\mathbf{y}\right)^{1/q'} \left(\int_{B_k} |f(\mathbf{y})|^q d\mathbf{y}\right)^{1/q} \\
&\le C\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}^n_p)} \sum_{k=\gamma+1}^{\infty} p^{-k(n-\beta)}|B_k|_H^{1+\lambda} \\
&\le C|B_\gamma|_H^{\mu}\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}^n_p)}.
\end{aligned} \tag{12}$$

Consequently,

$$\begin{aligned}
II &= \left(\frac{1}{|B_\gamma|_H^{1+\mu r}} \int_{B_\gamma} |T^p_{\beta,\Omega}(f\chi_{B^c_\gamma})(\mathbf{x})|^r d\mathbf{x}\right)^{1/r} \\
&\le C\|f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}^n_p)}.
\end{aligned} \tag{13}$$

From (10) and (13), we have the desired result.
(ii) For $q = 1$ we set $f_2 = f - f_1$ and $f_1 = f\chi_{B_\gamma}$. Then by Lemma 1, we have

$$\begin{aligned}
&|\{\mathbf{x} \in B_\gamma : |T^p_{\beta,\Omega}f_1(\mathbf{x})| > \sigma|_H \\
&\quad\le C\left(\frac{\|f_1\|_{L^1(\mathbb{Q}^n_p)}}{\sigma}\right)^r \\
&\quad= C\sigma^{-r}\left(\int_{B_\gamma} |f(\mathbf{x})|d\mathbf{x}\right)^r \\
&\quad\le C\sigma^{-r}|B_\gamma|_H^{(1+\lambda)r}\|f\|^r_{\dot{B}^{1,\lambda}(\mathbb{Q}^n_p)} \\
&\quad= C\sigma^{-r}|B_\gamma|_H^{1+\mu r}\|f\|^r_{\dot{B}^{1,\lambda}(\mathbb{Q}^n_p)}.
\end{aligned}$$

Now, from the similar estimate as in (12), we have

$$|T^p_{\beta,\Omega}f_2(\mathbf{x})| \le C|B_\gamma|_H^{\mu}\|f_2\|^r_{\dot{B}^{1,\lambda}(\mathbb{Q}^n_p)}.$$

Making use of Chebyshev's inequality, we obtain

$$\begin{aligned}
&|\{\mathbf{x} \in B_\gamma : |T^p_{\beta,\Omega}f_2(\mathbf{x})| > \sigma|_H \\
&\quad\le \sigma^{-r}\int_{B_\gamma} |T^p_{\beta,\Omega}f_2(\mathbf{x})|^r d\mathbf{x} \\
&\quad\le C\sigma^{-r}|B_\gamma|_H^{1+\mu r}\|f_2\|^r_{\dot{B}^{1,\lambda}(\mathbb{Q}^n_p)} \\
&\quad\le C\sigma^{-r}|B_\gamma|_H^{1+\mu r}\|f\|^r_{\dot{B}^{1,\lambda}(\mathbb{Q}^n_p)}.
\end{aligned}$$

Since

$$|T^{p}_{\beta,\Omega}f(\mathbf{x})| \leq |T^{p}_{\beta,\Omega}f_1(\mathbf{x})| + |T^{p}_{\beta,\Omega}f_2(\mathbf{x})|,$$

we obtain

$$\begin{aligned}|\{\mathbf{x} \in B_\gamma : |T^{p}_{\beta,\Omega}f(\mathbf{x})| > \sigma|_H &\leq |\{\mathbf{x} \in B_\gamma : |T^{p}_{\beta,\Omega}f_1(\mathbf{x})| > \sigma/2|_H \\ &\quad + |\{\mathbf{x} \in B_\gamma : |T^{p}_{\beta,\Omega}f_2(\mathbf{x})| > \sigma/2|_H \\ &\leq C\sigma^{-r}|B_\gamma|_H^{1+\mu r}\|f\|^r_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}. \end{aligned} \tag{14}$$

Ultimately,

$$\left(\frac{\sigma^r|\{\mathbf{x} \in B_\gamma : |T^{p}_{\beta,\Omega}f(\mathbf{x})| > \sigma|_H}{|B_\gamma|_H^{1+\mu r}}\right)^{1/r} \leq C|B_\gamma|_H^{1+\mu r}\|f\|^r_{\dot{B}^{1,\lambda}(\mathbb{Q}_p^n)}, \tag{15}$$

for some $\gamma \in \mathbb{Z}$ and $\sigma > 0$. Hence proof is completed. □

3. λ-Central Bounded Mean Oscillation Estimates of $T^{p,b}_{\beta,\Omega}$ on Central Morrey Spaces

The following section discusses the λ-central bounded mean oscillation estimates of $T^{p,b}_{\beta,\Omega}$ on p-adic central Morrey spaces. We need an important result before proving this.

Lemma 2 ([22])**.** *Suppose $b \in C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)$, $\lambda \geq 0$ and $i, j \in \mathbb{Z}$. Then*

$$|b_{B_i} - b_{B_j}| \leq p^n|i-j|\|b\|_{C\dot{M}O^{r,\lambda}(\mathbb{Q}_p^n)} \max\{|B_i|_H^\lambda, |B_j|_H^\lambda\}.$$

Now are are firmly in a position to prove our key result.

Theorem 2. *Suppose $\beta \in \mathbb{R}$, $0 < \beta < n$, $1 < q_1 < n/\beta$, $q_1' < q_2 < \infty$, $\beta/n = 1/q_1 + 1/q_2 - 1/q$. Let also $0 \leq \lambda_2 < 1/n$, $\lambda_1 + \beta/n < \lambda_2 \leq 0$, $\lambda = \lambda_1 + \lambda_2 + \beta/n$, and $\Omega \in L^{q_1'}(S_{\mathbf{0}}(\mathbf{0}))$. Then $T^{p,b}_{\beta,\Omega}$ is bounded from $\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$ and satisfies*

$$\|T^{p,b}_{\beta,\Omega}f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq C\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.$$

Proof. Suppose $f \in \dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$. Now for fixed $\gamma \in \mathbb{Z}$, represent $B_\gamma(\mathbf{0})$ with B_γ,

$$\begin{aligned}&\left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}|T^{p,b}_{\beta,\Omega}f(\mathbf{x})|^q d\mathbf{x}\right)^{1/q} \\ &\leq \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|\Big(b(\mathbf{x}) - b_{B_\gamma}\Big)\Big(T^{p}_{\beta,\Omega}f\chi_{B_\gamma}\Big)(\mathbf{x})\right|^q d\mathbf{x}\right)^{1/q} \\ &\quad + \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|\Big(b(\mathbf{x}) - b_{B_\gamma}\Big)\Big(T^{p}_{\beta,\Omega}f\chi_{B^c_\gamma}\Big)(\mathbf{x})\right|^q d\mathbf{x}\right)^{1/q} \\ &\quad + \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|T^{p}_{\beta,\Omega}\Big(\Big(b(\mathbf{x}) - b_{B_\gamma}\Big)f\chi_{B_\gamma}\Big)(\mathbf{x})\right|^q d\mathbf{x}\right)^{1/q} \\ &\quad + \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|T^{p}_{\beta,\Omega}\Big(\Big(b(\mathbf{x}) - b_{B_\gamma}\Big)f\chi_{B^c_\gamma}\Big)(\mathbf{x})\right|^q d\mathbf{x}\right)^{1/q} \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

In order to evaluate I_1, we set $1/r = \beta/n - 1/q_1$, $1/q = 1/q_2 + 1/r$, then by Lemma 1 along with Hölder's inequality to have

$$
\begin{aligned}
I_1 &= \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|\left(b(\mathbf{x})-b_{B_\gamma}\right)\left(T^p_{\beta,\Omega}f\chi_{B_\gamma}\right)(\mathbf{x})\right|^q d\mathbf{x}\right)^{1/q} \\
&\leq |B_\gamma|_H^{-1/q}\left(\int_{B_\gamma}\left|b(\mathbf{x})-b_{B_\gamma}\right|^{q_2}d\mathbf{x}\right)^{1/q_2} \\
&\quad\cdot\left(\int_{B_\gamma}\left|T^p_{\beta,\Omega}(f\chi_{B_\gamma})(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r} \\
&\leq C|B_\gamma|_H^{-1/r+\lambda_1}\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)} \\
&\quad\cdot\left(\int_{B_\gamma}\left|f\chi_{B_\gamma}(\mathbf{x})\right|^{q_1}d\mathbf{x}\right)^{1/q_1} \\
&\leq C|B_\gamma|_H^{\lambda}\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{16}
$$

In a similar fashion, we estimate I_3, for this represent $1/q_1+1/q_2=1/r, \beta/n=1/r-1/q$, with Lemma 1; together with Hölder's inequality, we are down to

$$
\begin{aligned}
I_3 &= \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|T^p_{\beta,\Omega}\left(\left(b(\mathbf{x})-b_{B_\gamma}\right)f\chi_{B_\gamma}\right)(\mathbf{x})\right|^q d\mathbf{x}\right)^{1/q} \\
&\leq C|B_\gamma|_H^{-1/q}\left(\int_{B_\gamma}\left|\left(b(\mathbf{x})-b_{B_\gamma}\right)f(\mathbf{x})\right|^r d\mathbf{x}\right)^{1/r} \\
&\leq C|B_\gamma|_H^{-1/q}\left(\int_{B_\gamma}\left|b(\mathbf{x})-b_{B_\gamma}\right|^{q_2}d\mathbf{x}\right)^{1/q_2} \\
&\quad\cdot\left(\int_{B_\gamma}|f(\mathbf{x})|^{q_1}d\mathbf{x}\right)^{1/q_1} \\
&\leq C|B_\gamma|_H^{\lambda}\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{17}
$$

To evaluate I_2, we use Hölder's inequality, equality (11) and $\lambda_1+\beta/n<-\lambda_2\leq 0$, we obtain

$$
\begin{aligned}
\left|T^p_{\beta,\Omega}(f\chi_{B^c_\gamma})(\mathbf{x})\right| &= \left|\int_{B^c_\gamma}\frac{\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\right| \\
&\leq \int_{B^c_\gamma}\frac{|\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y} \\
&= \sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\int_{S_k}|\Omega(p^k\mathbf{y})f(\mathbf{y})|d\mathbf{y} \\
&\leq \sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\left(\int_{S_k}|\Omega(p^k\mathbf{y})|^{q_1'}d\mathbf{y}\right)^{1/q_1'}\left(\int_{S_k}|f(\mathbf{y})|^{q_1}d\mathbf{y}\right)^{1/q_1} \\
&\leq \sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\left(\int_{S_k}|\Omega(p^k\mathbf{y})|^{q_1'}d\mathbf{y}\right)^{1/q_1'}\left(\int_{B_k}|f(\mathbf{y})|^{q_1}d\mathbf{y}\right)^{1/q_1} \\
&\leq C\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}|B_k|_h^{1+\lambda_1} \\
&= C\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{kn(\lambda_1+\beta/n)} \\
&= C|B_\gamma|_H^{\lambda_1+\beta/n}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned} \tag{18}
$$

Now, we are well and truly in a position to estimate I_2. From (18) and Hölder's inequality, we acquire

$$
\begin{aligned}
I_2 &= \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|\left(b(\mathbf{x})-b_{B_\gamma}\right)\left(T^p_{\beta,\Omega}f\chi_{B^c_\gamma}\right)(\mathbf{x})\right|^q d\mathbf{x}\right)^{1/q}\\
&\leq C|B_\gamma|_H^{\lambda_1+\beta/n-1/q}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\\
&\quad\cdot\left(\int_{B_\gamma}\left|b(\mathbf{x})-b_{B_\gamma}\right|^q d\mathbf{x}\right)^{1/q}\\
&\leq C|B_\gamma|_H^{\lambda_1+\beta/n-1/q_2}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\\
&\quad\cdot\left(\int_{B_\gamma}\left|b(\mathbf{x})-b_{B_\gamma}\right|^{q_2} d\mathbf{x}\right)^{1/q_2}\\
&\leq C|B_\gamma|_H^{\lambda}\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.
\end{aligned}
\tag{19}
$$

Finally, we turn our attention towards estimating I_4. For this we need to give the following estimates. Making use of Hlder's inequality, equality (11), inequality (18), Lemma 2 and of the fact that $\gamma+1\leq k$, we have

$$
\begin{aligned}
&\left|T^p_{\beta,\Omega}\left(\left(b(\mathbf{x})-b_{B_\gamma}\right)f\chi_{B^c_\gamma}\right)(\mathbf{x})\right|\\
&=\left|\int_{B^c_\gamma}\frac{\left(b(\mathbf{y})-b_{B_\gamma}\right)\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\right|\\
&\leq\int_{B^c_\gamma}\frac{|b(\mathbf{y})-b_{B_\gamma}||\Omega(|\mathbf{y}|_p\mathbf{y})f(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|_p^{n-\beta}}d\mathbf{y}\\
&=\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\int_{S_k}|b(\mathbf{y})-b_{B_\gamma}||\Omega(p^k\mathbf{y})f(\mathbf{y})|d\mathbf{y}\\
&\leq\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\left(\int_{S_k}|\Omega(p^k\mathbf{y})|^{q_1'}d\mathbf{y}\right)^{1/q_1'}\\
&\quad\cdot\left(\int_{S_k}|f(\mathbf{y})|^{q_1}d\mathbf{y}\right)^{1/q_1}\left(\int_{S_k}|b(\mathbf{y}-b_{B_\gamma})|^{q_2}d\mathbf{y}\right)^{1/q_2}\\
&\leq\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}\left(\int_{S_k}|\Omega(p^k\mathbf{y})|^{q_1'}d\mathbf{y}\right)^{1/q_1'}\\
&\quad\cdot\left(\int_{B_k}|f(\mathbf{y})|^{q_1}d\mathbf{y}\right)^{1/q_1}\left(\int_{B_k}|b(\mathbf{y}-b_{B_\gamma})|^{q_2}d\mathbf{y}\right)^{1/q_2}\\
&\leq C\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}|B_k|_H^{1-1/q_2+\lambda_1}\\
&\quad\cdot\left(\int_{B_k}|b(\mathbf{y}-b_{B_\gamma})|^{q_2}d\mathbf{y}\right)^{1/q_2}\\
&\leq C\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}|B_k|_H^{1-1/q_2+\lambda_1}\\
&\quad\cdot\left[\left(\int_{B_k}|b(\mathbf{y}-b_{B_k})|^{q_2}d\mathbf{y}\right)^{1/q_2}+\left(\int_{B_k}|b_{B_k}-b_{B_\gamma})|^{q_2}d\mathbf{y}\right)^{1/q_2}\right]\\
&\leq C\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}p^{-k(n-\beta)}|B_k|_H^{1-1/q_2+\lambda_1}\\
&\quad\cdot\left[|B_k|_H^{1/q_2+\lambda_2}+p^n(k-\gamma)|B_k|_H^{1/q_2+\lambda_2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)} \\
&\quad \cdot \sum_{k=\gamma+1}^{\infty}(k-\gamma)p^{-k(n-\beta)}|B_k|_H^{1+\lambda_1+\lambda_2} \\
&\leq C\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}\sum_{k=\gamma+1}^{\infty}(k-\gamma)p^{kn\lambda} \\
&= C|B_\gamma|_H^{\lambda}\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}. \qquad (20)
\end{aligned}
$$

Now, it follows from (20) that

$$
\begin{aligned}
I_4 &= \left(\frac{1}{|B_\gamma|_H}\int_{B_\gamma}\left|T_{\beta,\Omega}^{p}\left(\left(b(\mathbf{x})-b_{B_\gamma}\right)f\chi_{B_\gamma^c}\right)(\mathbf{x})\right|^q d\mathbf{x}\right)^{1/q} \\
&\leq C|B_\gamma|_H^{\lambda}\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}. \qquad (21)
\end{aligned}
$$

From (16), (17), (19) and (21), we obtain

$$
\|T_{\beta,\Omega}^{p,b}f\|_{\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)} \leq C\|b\|_{C\dot{M}O^{q_2,\lambda_2}(\mathbb{Q}_p^n)}\|f\|_{\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)}.
$$

Hence $T_{\beta,\Omega}^{p,b}$ is bounded from $\dot{B}^{q_1,\lambda_1}(\mathbb{Q}_p^n)$ to $\dot{B}^{q,\lambda}(\mathbb{Q}_p^n)$. This completes the proof. □

4. Conclusions

The boundedness of rough p-adic fractional integral operator on central Morrey spaces and weak central Morrey spaces in the p-adic field is studied. In addition, the boundedness for commutators of rough p-adic fractional integral operator on central Morrey spaces is also obtained when the symbol function is from λ-central bounded mean oscillations. It is noteworthy here that rough p-adic fractional integral operator and its commutator can be further considered in locally compact Vilenkin groups, Heisenberg groups, and variable exponent in the p-adic field, which will appear elsewhere.

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