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Existence of fixed point and best proximity point of p-cyclic orbital ϕ-contraction map

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Abstract. In this manuscript, p-cyclic orbital ϕ-contraction map over closed, nonempty, convex subsets of a uniformly convex Banach space X possesses a unique best proximity point if the auxiliary function ϕ is strictly increasing. The given result unifies and extend some existing results in the related literature. We provide an illustrative example to indicate the validity of the observed result.

Keywords: p-cyclic map, fixed point, best proximity point, p-cyclic orbital nonexpansive map.

1 Introduction

Fixed point theory appeared first in the solution of the certain differential equations, see, e.g., Liouville [15] and Picard [18]. Banach [2] successfully derived the successive approximation method from the proofs of Picard [18], and he initiated the first fixed point theorem: For every contraction T on a complete metric space (X, d), by starting from an arbitrary point $x \in X$ one can construct a recursive sequence $\{x_n := T^{n-1}x\}$ such that $d(Tx_n, x_n) \to d(Tx^*, x^*)$, that is, x^* is a fixed point, and it is unique. It should be noted that in this proof the continuity of the mapping is used efficiently, although it is not assumed so. Indeed, the continuity of the operators is a necessary consequence of the "fulfilling contraction" condition. Roughly speaking, "finding the unique fixed point for a given operator" is equivalent to the existence and uniqueness of the solution of the corresponding differential equations. After Banach, a huge number of papers reported to improve, extend, and generalize the metric fixed point theory, which implicitly improved the differential equations theory but not only that. Metric fixed point theory has a wide application potential in almost all quantitative sciences, in particular, theoretical computer science, economics, and engineering.

Besides this improvement in fixed point theory, there are some operators that do not admit a fixed point. In other words, in any point in its domain, we have $d(x, Tx) > 0$. Accordingly, we could not find a solution for the considered differential equations or some other equations that are fulfilled by the given operator T. Roughly speaking, we could not find an exact solution for the given problem. At this handicap, optimization brings an approximate solution via "best proximity point."

Let (X, d) be a metric space and A, B be nonempty subsets of it. Suppose, for a mapping $T : A \to B$, that the corresponding functional equation $Tx = x$ ($x \in A$) does not necessarily have a solution. Regarding that $d(A, B)$ is a lower bound for $d(x, Tx)$, an approximate solution $z^* \in A$ to the corresponding functional equation $Tx = x$ yields the least possible error when $d(z^*, Tz^*) = \operatorname{dist}(A, B)$, where $\operatorname{dist}(A, B) = \inf\{d(a, b)\colon a \in A,\ b \in B\}$. Here the approximate solution $z*$ is called a best proximity point of the considered nonself mapping $T : A \to B$. Note that a best proximity point yields the global minimum of the nonlinear programming problem $\min_{x \in A} d(x, Tx)$ since $d(x, Tx) \geqslant d(A, B)$ for all $x \in A$.

As it is emphasized above, the continuity of the given mapping has a crucial role in obtaining the existence and uniqueness of the fixed point. On the other hand, the continuity is a heavy condition for the given mappings. Consequently, the following natural question appears: is it possible to find a fixed point of a given mapping that is not necessarily continuous? An interesting affirmative answer was given by Kirk, Srinivasan, and Veeramani [14] (see also, for example, [7, 9, 16]).

Theorem 1. *Suppose that (X, d) is complete metric space and the letters A, B reserved to denote nonempty closed subsets of it. If, for a nonself mapping $T : A \cup B \to A \cup B$ with $T(A) \subset B$ and $T(B) \subset A$ there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leqslant kd(x, y)$ for all $x \in A$ and $y \in B$, then T possesses a unique fixed point in $A \cap B$.*

In Theorem 1, T is called cyclic map. In [10], the concept of "cyclic map" was extended as p-cyclic map as follows.

Definition 1. (See [10, Defs. 3.1, 3.2].) Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}$, $p \geqslant 2$) be nonempty subsets of a metric space (X, d).

(i) A map $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is called a *p-cyclic map* if $T(A_i) \subseteq A_{i+1}$ for all $i \in \{1, 2, \dots, p\}$, where $A_{p+i} = A_i$. If $p = 2$, the map T is called cyclic.

(ii) A point $x \in A_i$ is said to be a *best proximity point* of T in A_i if $d(x, Tx) = \operatorname{dist}(A_i, A_{i+1})$, where $\operatorname{dist}(A_i, A_{i+1}) := \inf\{d(x, y)\colon x \in A_i,\ y \in A_{i+1}\}$.

For p-cyclic maps, the distances between the adjacent sets play an important role in the existence of a best proximity point. In [17, 21] and [12], the authors investigated the problem of finding a best proximity point for a p-cyclic map in which the distances between the adjacent sets need not be equal.

In [3], the following lemma is proved, and it is used to prove the main results.

Lemma 1. (*See* [3, Lemma 3].) *Let A and B be nonempty closed subsets of a uniformly convex Banach space $(X, \|\cdot\|)$ such that A is convex. Let $\{x_n\}$,$\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B satisfying*:

(i) $\|z_n - y_n\| \to \operatorname{dist}(A, B)$;
(ii) *For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|x_m - y_n\| \leqslant \operatorname{dist}(A, B) + \epsilon$ for all $m > n \geqslant N$.*

Then for every $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $m > n \geqslant N_1$, $\|x_m - z_n\| \leqslant \epsilon$.

In [11], the following notion of cyclic orbital contraction is introduced in which the contraction condition need not be satisfied for all the points.

Definition 2. Let A and B be nonempty subsets of a metric space X. A cyclic map $T : A \cup B \to A \cup B$ is said to be *cyclic orbital contraction* if for some $x \in A$, there exists a $k_x \in (0, 1)$ such that

$$d\bigl(T^{2n}x, Ty\bigr) \leqslant k_x d\bigl(T^{2n-1}x, y\bigr) \quad \forall y \in A,\ n \in \mathbb{N}.$$

In [13], the following notion of p-cyclic orbital nonexpansive map is introduced.

Definition 3. (See [13, DEf. 6].) Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}, p \geqslant 2$) be nonempty subsets of a metric space X. A p-cyclic map $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is said to be *p-cyclic orbital non expansive* if for some $x \in A_i$ ($1 \leqslant i \leqslant p$) and for each $k = 0, 1, 2, \dots, (p-1)$, the following inequality holds:

$$d\bigl(T^{pn+k}x, T^{k+1}y\bigr) \leqslant d\bigl(T^{pn+k-1}x, T^{k}y\bigr) \quad \forall y \in A_i,\ n \in \mathbb{N}.$$

In [5, 13] and [20], the authors investigated the existence of fixed points and best proximity points for various types of cyclic orbital contractions. Cyclic orbital contractions can be compared with the notion of "contractive iterate at a point" introduced in [19], later generalized in [4] and [6].

In [1], the following notion of cyclic ϕ-contraction is introduced.

Definition 4. (See [1, Def. 1].) Let A and B be nonempty subsets of a metric space X and $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. A cyclic map $T : A \cup B \to A \cup B$ is said to be *cyclic ϕ-contraction* if

$$d(Tx, Ty) \leqslant d(x, y) - \phi\bigl(d(x, y)\bigr) + \phi\bigl(\operatorname{dist}(A, B)\bigr) \quad \forall x \in A,\ y \in B.$$

Proposition 1. (*See* [8, Prop. 1].) *Let X be a strictly convex normed linear space. Let $A_1, A_2, \dots, A_p$ be nonempty and convex subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map. Then T has at most one best proximity point in A_i, $1 \leqslant i \leqslant p$.*

2 Main results

We introduce a notion called p-cyclic orbital ϕ-contraction, which is defined as follows.

Definition 5. Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}$, $p \geqslant 2$) be nonempty subsets of a metric space X and $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing map such that $\phi(o) = 0$. We say that a p-cyclic map $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is *p-cyclic orbital ϕ-contraction* if for each $k = 0, 1, 2, \dots, (p-1)$ and for some $x \in A_i$ ($1 \leqslant i \leqslant p$), the following inequality holds:

$$\begin{aligned} d\big(T^{pn+k}x, T^{k+1}y\big) &\leqslant d\big(T^{pn+k-1}x, T^k y\big) - \phi\big(d\big(T^{pn+k-1}x, T^k y\big)\big) \\ &\quad + \phi\big(d(A_{i+k-1}, A_{i+k})\big) \quad \forall y \in A_i,\ n \in \mathbb{N}. \end{aligned} \tag{1}$$

Proposition 2. *Every p-cyclic orbital ϕ-contraction map is p-cyclic orbital nonexpansive.*

Proof. Let T be a p-cyclic orbital ϕ-contraction map satisfying (1) for some $x \in A_i$ ($1 \leqslant i \leqslant p$). Since $d(A_{i+k-1}, A_{i+k}) \leqslant d(T^{pn+k-1}x, T^k y)$ and ϕ is a strictly increasing map, we have

$$\phi\big(d(A_{i+k-1}, A_{i+k})\big) \leqslant \phi\big(d\big(T^{pn+k-1}x, T^k y\big)\big). \tag{2}$$

Substituting equation (2) in equation (1), we get

$$\begin{aligned} d\big(T^{pn+k}x, T^{k+1}y\big) &\leqslant d\big(T^{pn+k-1}x, T^k y\big) - \phi\big(d\big(T^{pn+k-1}x, T^k y\big)\big) \\ &\quad + \phi\big(d\big(T^{pn+k-1}x, T^k y\big)\big) \\ &= d\big(T^{pn+k-1}x, T^k y\big). \end{aligned}$$

This completes the proof. $\square$

Proposition 3. *Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}, p \geqslant 2$) be nonempty subsets of a metric space X. Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. If $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a p-cyclic orbital ϕ-contraction map such that equation* (1) *holds for some $x \in A_i$, ($1 \leqslant i \leqslant p$), then*

$$\lim_{n\to\infty} d\big(T^{pn+k}x, T^{pn+k+1}y\big) = \operatorname{dist}(A_{i+k}, A_{i+k+1}) \\ \forall y \in A_i,\ k \in \{0, 1, 2, \dots, p\}.$$

Proof. Let $y \in A_i$ ($1 \leqslant i \leqslant p$) be arbitrary. Let $d = \operatorname{dist}(A_{i+k}, A_{i+k+1})$ and $d_n = d(T^{pn+k}x, T^{pn+k+1}y)$ for $n \in \mathbb{N}$. Then $\{d_n\}_{n=1}^\infty$ is a nonincreasing sequence of nonnegative real numbers and bounded below by $\operatorname{dist}(A_{i+k}, A_{i+k-1})$. Therefore, $\{d_n\}_{n=1}^\infty$ converges to r (say). This implies that $r = \inf_{n\geqslant 1} d_n$ and $r \geqslant d$. Since T is p-cyclic orbital nonexpansive, we have

$$\begin{aligned} d_{n+1} &\leqslant d\big(T^{pn+k+1}x, T^{pn+k+2}y\big) \\ &\leqslant d\big(T^{pn+k}x, T^{pn+k+1}y\big) - \phi\big(d(T^{pn+k}x, T^{pn+k+1}y)\big) \\ &\quad + \phi\big(d(A_{i+k}, A_{i+k+1})\big). \end{aligned}$$

This implies that

$$\phi(d_n) \leqslant d_n - d_{n+1} + \phi(d). \tag{3}$$

As ϕ is strictly increasing and $d \leqslant r \leqslant d_n$ for all $n \in \mathbb{N}$, we have

$$\phi(d) \leqslant \phi(r) \leqslant \phi(d_n) \quad \forall n \in \mathbb{N}. \tag{4}$$

From (3) and (4) we get

$$\phi(d) \leqslant \phi(d_n) \leqslant d_n - d_{n+1} + \phi(d).$$

Hence,

$$\lim_{n\to\infty} \phi(d_n) = \phi(d). \tag{5}$$

By combining (4) and (5) we have $\phi(r) = \phi(d)$. This gives $r = d$ as ϕ is a strictly increasing map. Hence the proof. □

Proposition 4. *Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}, p \geqslant 2$) be nonempty subsets of a metric space X. Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. If $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a p-cyclic orbital ϕ-contraction map satisfying* (1) *for some $x \in A_i$ ($1 \leqslant i \leqslant p$), then the following hold*:

(i) $\lim_{n\to\infty} d(T^{pn+k-1}x, T^{pn+k}y) = \operatorname{dist}(A_{i+k-1}, A_{i+k})$ *for every $y \in A_i$ and $k \in \{0, 1, 2, \dots, p\}$*;
(ii) $\lim_{n\to\infty} d(T^{pn+p}x, T^{pn+1}x) = \operatorname{dist}(A_i, A_{i+1})$;
(iii) $\lim_{n\to\infty} d(T^{pn-p}x, T^{pn+1}x) = \operatorname{dist}(A_i, A_{i+1})$;
(iv) $\lim_{n\to\infty} d(T^{pn}x, T^{pn+p+1}x) = \operatorname{dist}(A_i, A_{i+1})$.

Proof. By using similar argument as in Proposition 3(i)–(iv) can be proved. □

The following proposition is useful to prove the main result whose proof follows from Lemma 1, Propositions 3 and 4.

Proposition 5. *Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}$, $p \geqslant 2$) be nonempty closed convex subsets of a uniformly convex Banach space X. Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing map. If $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a p-cyclic orbital ϕ-contraction map satisfying* (1) *for some $x \in A_i$ ($1 \leqslant i \leqslant p$), then the following hold*:

(i) $\lim_{n\to\infty} \|T^{pn}x - T^{pn+p}x\| = 0$;
(ii) $\lim_{n\to\infty} \|T^{pn}x - T^{pn-p}x\| = 0$;
(iii) $\lim_{n\to\infty} \|T^{pn+1}x - T^{pn+p+1}x\| = 0$.

Proposition 6. *Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}$, $p \geqslant 2$) be nonempty subsets of a metric space X. Let $\phi : [0,\infty) \to [0,\infty)$ be a strictly increasing map and $\phi(0) = 0$. If $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a p-cyclic orbital ϕ-contraction map satisfying* (1) *for some $x \in A_i$ ($1 \leqslant i \leqslant p$), then*

(i) $\operatorname{dist}(A_1, A_2) = \operatorname{dist}(A_2, A_3) = \dots = \operatorname{dist}(A_{p-1}, A_p) = \operatorname{dist}(A_p, A_1)$;
(ii) *If $\{T^{pn}x\}$ converges to some $\xi \in A_i$, then ξ is a best proximity point of T in A_i.*

Proof. (i) Let $k \in \{0, 1, 2, \dots, (p-1)\}$ be arbitrary. As T is p-cyclic orbital nonexpansive, we have

$$d\big(T^{pn+k}x, T^{pn+k+1}x\big) \leqslant d\big(T^{pn+k-1}x, T^{pn+k}x\big).$$

Also,

$$\operatorname{dist}(A_{i+k}, A_{i+k+1}) \leqslant d\big(T^{pn+k}x, T^{pn+k+1}x\big).$$

Thus,

$$\operatorname{dist}(A_{i+k}, A_{i+k+1}) \leqslant d\big(T^{pn+k-1}x, T^{pn+k}x\big).$$

Now by taking limit on both sides and using Proposition 4(i) we get

$$\begin{aligned}\operatorname{dist}(A_{i+k}, A_{i+k+1}) &\leqslant \lim_{n\to\infty} d\big(T^{pn+k-1}x, T^{pn+k}x\big) \\ &= \operatorname{dist}(A_{i+k-1}, A_{i+k}).\end{aligned}$$

From the above inequality we get the following chain of inequalities:

$$\begin{aligned}\operatorname{dist}(A_{i+1}, A_{i+p}) &= \operatorname{dist}(A_{i+p}, A_{i+1}) = \operatorname{dist}(A_{i+p}, A_{i+p+1}) \\ &\leqslant \operatorname{dist}(A_{i+p-1}, A_{i+p}) \leqslant \operatorname{dist}(A_{i+p-2}, A_{i+p-1}) \\ &\leqslant \cdots \leqslant \operatorname{dist}(A_{i+k}, A_{i+k+1}) \leqslant \cdots \leqslant \operatorname{dist}(A_{i+1}, A_{i+2}) \\ &\leqslant \operatorname{dist}(A_i, A_{i+1}) = \operatorname{dist}(A_{i+p}, A_{i+1}).\end{aligned}$$

Hence,

$$\operatorname{dist}(A_1, A_2) = \operatorname{dist}(A_2, A_3) = \cdots = \operatorname{dist}(A_{p-1}, A_p) = \operatorname{dist}(A_p, A_1).$$

(ii) Let $d(x, y)$ be the metric induced by the norm $\|x - y\|$, $x, y \in X$. Now

$$\begin{aligned}\operatorname{dist}(A_i, A_{i+1}) \leqslant d(\xi, T\xi) &= \lim_{n\to\infty} d\big(T^{pn}x, T\xi\big) \\ &\leqslant \lim_{n\to\infty} d\big(T^{pn-1}x, \xi\big) = \lim_{n\to\infty} d\big(T^{pn-1}x, T^{pn}x\big) \\ &= \operatorname{dist}(A_{i-1}, A_i) = \operatorname{dist}(A_i, A_{i+1}).\end{aligned}$$

Thus, $\operatorname{dist}(A_i, A_{i+1}) = d(\xi, T\xi)$ and ξ is a best proximity point of T in A_i. $\square$

Theorem 2. *Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}, p \geqslant 2$) be nonempty closed subsets of a complete metric space X. Let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing map and $\phi(0) = 0$. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a p-cyclic orbital ϕ-contraction map of type one satisfying* (1) *for some $x \in A_i$ ($1 \leqslant i \leqslant p$). Then there exists a fixed point of T, say, $\xi \in \bigcap_{i=1}^p A_i$, such that for any $z \in A_i$ ($1 \leqslant i \leqslant p$) satisfying* (1)*, the sequence $\{T^{pn}z\}$ converges to ξ.*

Proof. Let $x \in A_i$ ($1 \leqslant i \leqslant p$) satisfy equation (1). Let us prove that given $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$d\big(T^{pn}x, T^{pm}x\big) < \epsilon \quad \forall n, m \geqslant n_0$$

by induction on m. Let $\epsilon > 0$ be given. Now

$$d\big(T^{pn}x, T^{pm}x\big) \leqslant d\big(T^{pn}x, T^{pm+1}x\big) + d\big(T^{pm+1}x, T^{pm}x\big).$$

From Proposition 4(i), for $k = 1$, we have $\lim_{m\to\infty} d(T^{pm}x, T^{pm+1}x) = 0$. Hence, there exists an $n_0 \in \mathbb{N}$ such that

$$d\big(T^{pm}x, T^{pm+1}x\big) < \left(\frac{\delta}{p}\right), \quad 0 < \delta < \left(\frac{\epsilon}{2}\right), \; m \geqslant n_0. \tag{6}$$

Hence, it is enough to show that

$$d\big(T^{pn}x, T^{pm+1}x\big) < \left(\frac{\epsilon}{2}\right), \quad m, n \geqslant n_0. \tag{7}$$

Fix $n \geqslant n_0$ such that (6) holds. Now (7) is true for $m = n$. Assume that (7) is true for some $m \geqslant n_0$. We will prove that (7) is true for $m + 1$ in place of m. Now

$$\begin{aligned} d\big(T^{pn}x, T^{p(m+1)+1}x\big) &\leqslant d\big(T^{pn}x, T^{pm+1}x\big) + d\big(T^{pm+1}x, T^{pm+2}x\big) \\ &\quad + \cdots + d\big(T^{pm+p}x, T^{pm+p+1}x\big) \\ &< \left(\frac{\epsilon}{2}\right) + \left(\frac{\delta}{p}\right)p < \epsilon. \end{aligned}$$

Hence, $\{T^{pn}x\}$ is a Cauchy sequence, and it converges to a limit, say, $\xi \in A_i$. For $k = 0$ in Proposition 4(i), we get $\lim_{n\to\infty} d(T^{pn-1}x, T^{pn}x) = 0$. Now

$$\begin{aligned} d(\xi, T\xi) &= \lim_{n\to\infty} d\big(T^{pn}x, T\xi\big) \leqslant \lim_{n\to\infty} d\big(T^{pn-1}x, \xi\big) \\ &= \lim_{n\to\infty} d\big(T^{pn-1}x, T^{pn}x\big) = 0. \end{aligned}$$

This implies that $\xi = T\xi$, and therefore, ξ is a fixed point in A_i. Since T is p-cyclic, $\xi \in \bigcap_{i=1}^{p} A_i$. To prove that ξ is unique, suppose $\eta \in A_i$ such that $\eta = T\eta$. Now from Proposition 4(i)

$$d(\xi, \eta) = \lim_{n\to\infty} d\big(T^{pn}x, T^{pn+1}\eta\big) = 0$$

for $k = 1$. Thus, we have $\xi = \eta$. $\square$

Theorem 3. *Let $A_1, A_2, \dots, A_p$ ($p \in \mathbb{N}$, $p \geqslant 2$) be nonempty closed and convex subsets of a uniformly convex Banach space $(X, \|\cdot\|)$. Let $d(x, y) = \|x - y\|$, $x, y \in X$, be the metric induced by the norm. Let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing map. If $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is a p-cyclic orbital ϕ-contraction map of type two, then for every $x \in A_i$ ($1 \leqslant i \leqslant p$) satisfying equation* (1)*, the sequence $\{T^{pn}x\}$ converges to η, which is a unique best proximity point of T in A_i.*

Proof. If for every $k = 0, 1, \dots (p-1)$, $\operatorname{dist}(A_{i+k}, A_{i+k+1}) = 0$, then T has a unique fixed point in $\bigcap_{i=1}^{p} A_i$ by Theorem 2. Let us assume that $\operatorname{dist}(A_i, A_{i+1}) > 0$. We claim that for every $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$,

$$\big\|T^{pm}x - T^{pn+1}x\big\| \leqslant \operatorname{dist}(A_i, A_{i+1}) + \epsilon.$$

Suppose not. Then there exists an $\epsilon_0 > 0$ such that for all $k \in \mathbb{N}$, there exists $m_k > n_k > k$ for which

$$\left\|T^{pm_k}x - T^{pn_k+1}x\right\| > \operatorname{dist}(A_i, A_{i+1}) + \epsilon_0. \tag{8}$$

By choosing m_k to be the least integer greater than n_k, to satisfy the above inequality, we have

$$\left\|T^{p(m_k-1)}x - T^{pn_k+1}x\right\| \leqslant \operatorname{dist}(A_i, A_{i+1}) + \epsilon_0. \tag{9}$$

Now by (9), for each k,

$$\begin{aligned}
\operatorname{dist}(A_i, A_{i+1}) + \epsilon_0 &< \left\|T^{pm_k}x - T^{pn_k+1}x\right\| \\
&\leqslant \left\|T^{pm_k}x - T^{pm_k-p}x\right\| + \left\|T^{pm_k-p}x - T^{pn_k+1}x\right\| \\
&< \left\|T^{pm_k}x - T^{pm_k-p}x\right\| + \operatorname{dist}(A_i, A_{i+1}) + \epsilon_0.
\end{aligned}$$

By taking limit on both sides of above inequality as $k \to \infty$ and by using Proposition 5(ii) we have

$$\operatorname{dist}(A_i, A_{i+1}) + \epsilon_0 \leqslant \lim_{k\to\infty} \left\|T^{pm_k}x - T^{pn_k+1}x\right\| \leqslant \operatorname{dist}(A_i, A_{i+1}) + \epsilon_0.$$

That is,

$$\lim_{k\to\infty} \left\|T^{pm_k}x - T^{pn_k+1}x\right\| = \operatorname{dist}(A_i, A_{i+1}) + \epsilon_0.$$

Now

$$\begin{aligned}
\left\|T^{pm_k}x - T^{pn_k+1}x\right\| &\leqslant \left\|T^{pm_k}x - T^{pm_k+p}x\right\| + \left\|T^{pm_k+p}x - T^{pn_k+p+1}x\right\| \\
&\quad + \left\|T^{pn_k+p+1}x - T^{pn_k+1}x\right\|.
\end{aligned} \tag{10}$$

Now by using $p-1$ times p-cyclic orbital nonexpansiveness of T to $\|T^{pm_k+p}x - T^{pn_k+p+1}x\|$, we get

$$\begin{aligned}
\left\|T^{pm_k+p}x - T^{pn_k+p+1}x\right\| &\leqslant \left\|T^{pm_k+1}x - T^{pn_k+2}x\right\| \\
&\leqslant \left\|T^{pm_k}x - T^{pn_k+1}x\right\| - \phi\bigl(\left\|T^{pm_k}x - T^{pn_k+1}x\right\|\bigr) \\
&\quad + \phi\bigl(\operatorname{dist}(A_i, A_{i+1})\bigr).
\end{aligned}$$

Let $\operatorname{dist}(A_i, A_{i+1}) = d$ and $\|T^{pm_k}x - T^{pn_k+1}x\| = \mu_k$. Then the above inequality becomes

$$\left\|T^{pm_k+p}x - T^{pn_k+p+1}x\right\| \leqslant \mu_k - \phi(\mu_k) + \phi(d). \tag{11}$$

By using (11) in (10) we get

$$\mu_k \leqslant \left\|T^{pm_k}x - T^{pm_k+p}x\right\| + \mu_k - \phi(\mu_k) + \phi(d) + \left\|T^{pn_k+p+1}x - T^{pn_k+1}x\right\|.$$

That is,

$$\begin{aligned}
\mu_k - \mu_k + \phi(\mu_k) - \phi(d) &\leqslant \left\|T^{pm_k}x - T^{pm_k+p}x\right\| + \left\|T^{pn_k+p+1}x - T^{pn_k+1}x\right\|, \\
\phi(\mu_k) - \phi(d) &\leqslant \left\|T^{pm_k}x - T^{pm_k+p}x\right\| + \left\|T^{pn_k+p+1}x - T^{pn_k+1}x\right\|.
\end{aligned} \tag{12}$$

By taking limit on both sides of (12) as $k \to \infty$ and using Proposition 5(i) and (iii) in (12) we have

$$\lim_{k\to\infty} \phi(\mu_k) \leqslant \phi(d).$$

Since for each k, $d \leqslant \mu_k$, we have $\phi(d) \leqslant \lim_{k\to\infty} \phi(\mu_k)$. Hence,

$$\lim_{k\to\infty} \phi(\mu_k) = \phi(d).$$

Since $\mu_k > d + \epsilon_0$, by (8) we have $\phi(\mu_k) > \phi(d + \epsilon_0)$. Thus, $\lim_{k\to\infty} \phi(\mu_k) \geqslant \phi(d + \epsilon_0)$. That is, $\phi(d) \geqslant \phi(d + \epsilon_0)$. This is a contradiction to the fact that ϕ is strictly increasing and $d < d + \epsilon_0$. Hence the claim.

Now by Proposition 4(i) for $k = 1$, $\|T^{pn}x - T^{pn+1}x\| \to \mathrm{dist}(A_i, A_{i+1})$. Combining this with the claim, by Lemma 1 we have the following: for every $\epsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$\bigl\|T^{pm}x - T^{pn}x\bigr\| \leqslant \epsilon, \quad m > n > n_1.$$

Therefore, $\{T^{pn}x\}$ is a Cauchy sequence in A_i, and it converges to a point $\xi \in A_i$. By Proposition 6(ii) and Proposition 1, ξ is the unique best proximity point of T in A_i. $\square$

Example. Consider $X = \mathbb{R}^2$ endowed with the Euclidean metric. Let A_1, A_2, A_3, A_4 be the subsets of X defined as follows:

$$\begin{aligned}
A_1 &= \bigl\{(x_1, x_2)\colon -2 \leqslant x_1 \leqslant -1,\ x_2 = 0\bigr\},\\
A_2 &= \bigl\{(x_1, x_2)\colon x_1 = 0,\ 1 \leqslant x_2 \leqslant 2\bigr\},\\
A_3 &= \bigl\{(x_1, x_2)\colon 1 \leqslant x_1 \leqslant 2,\ x_2 = 0\bigr\},\\
A_4 &= \bigl\{(x_1, x_2)\colon x_1 = 0,\ -2 \leqslant x_2 \leqslant -1\bigr\}.
\end{aligned}$$

Define $T : \bigcup_{i=1}^{4} A_i \to \bigcup_{i=1}^{4} A_i$ as follows:

$$T\bigl((x_1, x_2)\bigr) = \begin{cases}
(0, 1.5) & \text{for } -2 \leqslant x_1 \leqslant -1.5,\ x_2 = 0,\\
(0, 1) & \text{for } -1.5 < x_1 \leqslant -1,\ x_2 = 0,\\
(1.5, 0) & \text{for } x_1 = 0,\ 1 < x_2 \leqslant 2,\\
(0, -1.5) & \text{for } 1 < x_1 \leqslant 2,\ x_2 = 0,\\
(-1.5, 0) & \text{for } x_1 = 0,\ -2 \leqslant x_2 < -1;
\end{cases}$$

$$T\bigl((0,1)\bigr) = (1,0), \quad T\bigl((1,0)\bigr) = (0,-1), \quad \text{and} \quad T\bigl((0,-1)\bigr) = (-1,0).$$

Define $\phi : [0,\infty) \to [0,\infty)$ as $\phi(t) = t^2/(1+t)$, $t \geqslant 0$. It is easy to see that T is a 4-cyclic map and ϕ is a strictly increasing map. We note that $\mathrm{dist}(A_i, A_{i+1}) = \sqrt{2}$, $i = 1, 2, 3, 4$. T is 4-cyclic orbital ϕ-contraction for all points in the set $S = \{\{(x_1, x_2) \in A_1\colon -1 \leqslant x_1 < -1.5,\ x_2 = 0\},\ (0,1) \in A_2,\ (1,0) \in A_3,\ \text{and}\ (0,-1) \in A_4\}$. The unique best proximity point of A_1 is $(-1,0)$, A_2 is $(0,1)$, A_3 is $(1,0)$, and A_4 is $(0,-1)$. We see that for all $x \in S$, the sequence $\{T^{4n}x\}$ converges to the unique best proximity point of T in the respective set. Further, $y \in X \setminus S$ do not satisfy condition (1), and $\{T^{4n}y\}$ do not converge to the best proximity point.

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