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Finite-time stability results for fractional damped dynamical systems with time delays*

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Abstract. This paper is explored with the stability procedure for linear nonautonomous multiterm fractional damped systems involving time delay. Finite-time stability (FTS) criteria have been developed based on the extended form of Gronwall inequality. Also, the result is deduced to a linear autonomous case. Two examples of applications of stability analysis in numerical formulation are described showing the expertise of theoretical prediction.

Keywords: damped system, fractional order, finite-time stability, time delay.

1 Introduction

Fractional differential equations provide the outstanding device for account of remembrance and heritable characteristics of numerous complex systems. Fractional derivatives like Caputo derivative, Riemann–Liouville derivative have their individual advantages and disadvantages. The Riemann–Liouville derivative cannot be used in the situation when the particular function is differentiable. In that case, the Caputo derivative can be used to solve the differential equation. The research related to fractional-order derivatives is well established and absolutely adequate in many different applications [1, 11, 15, 16, 26, 28]. Time delay occurs in the system subject to different causes such as communication delay, energy conversation, etc. The appearance of time delay in system state, measurement or control input is an unavoidable one in several practical systems [6, 7, 35, 36]. It is the main cause for instability of the system. Time delay is one of the most analysed

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phenomena of control systems and in control process, which may cause the degradation of the controller performance. Consequently, much attention has been paid on stability research of dynamical systems involving time delay [13, 14, 19, 30].

Stability criteria are classified into several types such as asymptotic stability, exponential stability, globally exponential stability, FTS and so on. In FTS, the considered system's state tends to zero in a finite time. But in other types of stability, the convergence time is sufficiently large. In the literature, numerous reports have been established on the asymptotic stability, which concerns the behaviours of state variable over an infinite time interval [25]. The main disadvantage of the asymptotic stability behaviour is that the large value of state variable may exist for the duration of transient period. The occurrence of large values should not go beyond its limit in several practical systems. Therefore, FTS concept has been introduced and concentrating on the behaviour of state variables for the duration of momentary time, which must not beyond the definite value, while the initial condition's upper bound is specified [2]. So, the researchers developed the FTS criteria [3, 10, 22, 33] and established several results by using Gronwall inequalities, Holder inequalities and inequality scaling skills [17, 32, 34].

Gronwall-type inequalities play an essential role in the analysis of behaviour of solution of differential equations as well as integral equations. Also, these types of inequalities used to model the engineering and applied science problems. The Gronwall inequality is also known as Gronwall–Bellman inequality, which bounds the solution of given fractional system. Due to this application, many researchers followed this inequality to analyze the existence of solution, stability related problems, oscillation and also to check boundedness property of the given system. So, recently this inequality gets much attention of many researchers [23, 27].

Sufficient condition of FTS analysis for a class of fractional system with time delay has been derived by utilizing the Bellman–Gronwall's approach in [17]. FTS of fractional delay systems has been investigated in [8, 9, 20]. The FTS concept for the fractional-order delay system with two parameter Mittag-Leffler matrix function is presented in [21]. In [31], the authors examined FTS of considered fractional system involving discrete time delay. FTS of discrete fractional delay system examined by utilizing Gronwall's inequality approach in [32]. In [29], existence results for fractional-order damped systems are studied by using Holder and Gronwall inequalities.

Many authors have investigated the controllability of damped system [4] and the controllability of fractional-order damped system [5, 12], but not yet studied the stability analysis for multiterm fractional-order damped system. FTS of multiterm fractional-order damped dynamical system involving time delay has been studied. The key notions can be highlighted below:

- Analyzing FTS concept, some difficulties have been occurred to bound the solution of the considered system. To overcome this difficulty, we use the extended form of generalized Gronwall inequality.
- Many of the previous results on fractional systems are reported without damping effect. It is more essential to study the FTS of fractional system with damping behaviour.

- So, it is crucial to pay attention to check the FTS of linear nonautonomous multiterm fractional damped time delay system with order $0 < \alpha_2 \leqslant 1 < \alpha_1 \leqslant 2$, which is examined by using extended form of generalized Gronwall's inequality.
- Further, we deduce the results to linear autonomous systems.
- The formulated stability conditions can be easily validated through two numerical examples.

The structure of the paper is outlined as follows. Problem description with necessary facts and lemmas are given in the following section. Section 3 provides the stability criteria for considered linear nonautonomous system and also some deduction from derived results. Section 4 contains two numerical examples, which shows the validity of obtained results. Conclusion is drawn in Section 5.

2 Problem formulation

Consider the linear nonautonomous multiterm fractional damped dynamical system

$$\begin{aligned} &{}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}\,{}_0^C D_t^{\alpha_2} y(t) = \mathcal{B} y(t) + \mathcal{C} y(t-\rho) + \mathcal{D} u(t), \\ &\qquad t \in L = [t_0, t_0 + T], \\ &y(t) = \phi(t), \quad y'(t) = \phi'(t), \quad -\rho \leqslant t \leqslant 0, \end{aligned} \tag{1}$$

with $0 < \alpha_2 \leqslant 1 < \alpha_1 \leqslant 2$. Here state vector $y(t)$ is in $\mathbb{R}^n$. ${}_0^C D_t^{\alpha_1}$ and ${}_0^C D_t^{\alpha_2}$ denote the Caputo fractional derivative with orders α_1 and α_2, respectively. The matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are in $\mathbb{R}^{n\times n}$, and matrix $\mathcal{D}$ in $\mathbb{R}^{n\times m}$. $u(t) \in \mathbb{R}^m$ denoted as control vector. ρ denotes the pure time delay. Also, ρ is a constant, and it should be greater than zero. T is either positive or $+\infty$. $\|\cdot\|$ denotes the maximum norm. The following results are well known, and this provides some hints to reach our main result.

Definition 1. (See [1].) Mittag-Leffler function (MLF) for one parameter:

$$E_{\alpha_1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + 1)} \equiv E_{\alpha_1}(z);$$

MLF for two parameters:

$$E_{\alpha_1,\alpha_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \alpha_2)}, \quad \alpha_1 > 0,\ \alpha_2 > 0.$$

Definition 2. (See [1].) Fractional derivative for $y(t)$ in terms of Caputo with $\alpha_1 \in \mathbb{R}^+$ is given by

$${}_0^C D_{t_0,t}^{\alpha_1} y(t) = \frac{1}{\Gamma(n-\alpha_1)} \int_{t_0}^{t} (t-\theta)^{n-\alpha_1-1} y^{(n)}(\theta)\,\mathrm{d}\theta$$

with $n - 1 < \alpha_1 < n \in \mathbb{Z}^+$.

Definition 3. (See [18, Def. 2.2], [24, Def. 2.4].) System (1) is finite-time stable w.r.t $\{t_0, L, \delta, \epsilon, \alpha_{1u}, \rho\}$ iff $\kappa < \delta$ and for all $t \in L$, $\|u(t)\| < \alpha_{1u}$ implies $\|y(t)\| < \epsilon$ for all $t \in L$, where $\kappa = \max\{\|\phi\|, \|\phi'\|\}$ represents the initial time of observation of system, and δ, α_{1u}, ϵ are positive constants.

Definition 4. (See [18, 24].) System (1) is finite-time stable w.r.t $\{t_0, L, \delta, \epsilon, \rho\}$ at $(u(t) \equiv 0\ \forall t)$ iff $\kappa < \delta$ for all $t \in L$ implies $\|y(t)\| < \epsilon$ for all $t \in L$, where $\kappa = \max\{\|\phi\|, \|\phi'\|\}$ represents the initial time of observation of system, and δ, ϵ are positive constants.

Lemma 1 [Generalized Gronwall inequality (GGI)]. (*See* [34].) *Assume $y(t) > 0$, $v(t) > 0$ be locally integrable and the continuous function $r(t) > 0$ is nondecreasing on $t \in [0, T)$. Now $r(t) \leqslant M$, $\alpha_1 > 0$ with*

$$y(t) \leqslant v(t) + r(t) \int_0^t (t-\theta)^{\alpha_1 - 1} y(\theta)\, \mathrm{d}\theta, \quad 0 \leqslant t < T.$$

Then

$$y(t) \leqslant v(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(r(t)\Gamma(\alpha_1))^n}{\Gamma(n\alpha_1)} (t-\theta)^{n\alpha_1 - 1} v(\theta) \right] \mathrm{d}\theta, \quad 0 \leqslant t < T.$$

Corollary 1. (*See* [34].) *From the assumption of above Lemma* 1 *and on* $[0, T)$, $v(t)$ *is a nondecreasing function. Then* $y(t) \leqslant v(t) E_{\alpha_1}(r(t)\Gamma(\alpha_1) t^{\alpha_1})$.

Lemma 2 [Extended form of Gronwall inequality]. (*See* [29].) *If both fractional orders α_1 and α_2 are nonzero and positive, $v(t) > 0$ is locally integrable, the continuous functions $r_1(t) > 0$ and $r_2(t) > 0$ are nondecreasing on $[0, T)$, $r_1(t) \leqslant M_1$, $r_2(t) \leqslant M_2$. Assume $y(t) > 0$ is locally integrable on $[0, T)$ and*

$$y(t) \leqslant v(t) + r_1(t) \int_0^t (t-\theta)^{\alpha_1 - 1} y(\theta)\, \mathrm{d}\theta + r_2(t) \int_0^t (t-\theta)^{\alpha_2 - 1} y(\theta)\, \mathrm{d}\theta.$$

Then for $t \in [0, T)$,

$$y(t) \leqslant v(t) + \int_0^t \sum_{n=1}^{\infty} \left[r(t)\right]^n \sum_{k=0}^{n} \frac{c_n^k [\Gamma(\alpha_1)]^{n-k} [\Gamma(\alpha_2)]^k}{\Gamma((n-k)\alpha_1 + k\alpha_2)} (t-\theta)^{(n-k)\alpha_1 + k\alpha_2 - 1} v(\theta)\, \mathrm{d}\theta,$$

where $r(t) = r_1(t) + r_2(t)$ *and* $c_n^k = n(n-1)(n-2)\cdots(n-k+1)/k!$.

Corollary 2. (*See* [29].) *From the assumption of above Lemma* 2 *and on the interval* $[0, T)$, $v(t)$ *is a nondecreasing function. Then*

$$y(t) \leqslant v(t) E_\gamma \left[r(t) \left(\Gamma(\alpha_1) t^{\alpha_1} + \Gamma(\alpha_2) t^{\alpha_2} \right) \right],$$

where $\gamma = \min\{\alpha_1, \alpha_2\}$.

3 Main results

Theorem 1. *Assume that $t_0 = 0$. The linear nonautonomous fractional damped system* (1) *is finite-time stable w.r.t $\{\delta, \epsilon, L_0, \alpha_{1u}\}$, $\delta < \epsilon$ if it satisfies the following*:

$$\left\{1 + |t| + \frac{\|\mathcal{A}\||t|^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\} E_\gamma\big(r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\big)\big) + \frac{\eta_{u0}}{\Gamma(\alpha_1+1)} t^{\alpha_1} \leqslant \frac{\epsilon}{\delta}, \quad t \in L_0 = [0, T], \tag{2}$$

where $\eta_{u0} = \alpha_{1u} d_0/\delta$, and $\sigma_{\max}(\cdot)$ is largest singular value of matrix $(\cdot)$. Here $\sigma_{\max}(\Lambda) = \sigma_{\max}(\mathcal{B}) + \sigma_{\max}(\mathcal{C})$.

Proof. One can obtain the solution of the damped system with delay given by (1) in terms of equivalent form of Volterra integral equation:

$$\begin{aligned} y(t) &= y(0) + ty'(0) \\ &\quad - \frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)} y(0) + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} y(\theta)\,\mathrm{d}\theta \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} \big[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + \mathcal{D}u(\theta)\big]\,\mathrm{d}\theta. \end{aligned}$$

Now by taking norm on both sides we get the following:

$$\begin{aligned} \big\|y(t)\big\| &\leqslant \|\phi\| + |t|\|\phi'\| \\ &\quad + \frac{\|\mathcal{A}\||t|^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \int_0^t |t-\theta|^{\alpha_1-\alpha_2-1} \big\|y(\theta)\big\|\,\mathrm{d}\theta \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_0^t |t-\theta|^{\alpha_1-1} \big\|\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + \mathcal{D}u(\theta)\big\|\,\mathrm{d}\theta. \end{aligned} \tag{3}$$

Also, we can write

$$\big\|\mathcal{B}y(t) + \mathcal{C}y(t-\rho) + \mathcal{D}u(t)\big\| \leqslant \|\mathcal{B}\|\big\|y(t)\big\| + \|\mathcal{C}\|\big\|y(t-\rho)\big\| + \|\mathcal{D}\|\big\|u(t)\big\|. \tag{4}$$

Here $\|\mathcal{B}\|$ indicates induced norm of $\mathcal{B}$. Substituting (4) into (3), we get

$$\begin{aligned} \big\|y(t)\big\| &\leqslant \|\phi\| + t\|\phi'\| \\ &\quad + \frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)} \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} \big\|y(\theta)\big\|\,\mathrm{d}\theta \\ &\quad + \frac{1}{\Gamma(\alpha_1)} \int_0^t (t-\theta)^{\alpha_1-1} \big[\|\mathcal{B}\|\big\|y(\theta)\big\| + \|\mathcal{C}\|\big\|y(\theta-\rho)\big\| + \|\mathcal{D}\|\big\|u(\theta)\big\|\big]\,\mathrm{d}\theta. \end{aligned}$$

Now let

$$z(t)=\sup_{\eta\in[-\rho,t]}\big\|y(\eta)\big\|,\quad t\in L,$$
$$\big\|y(\theta)\big\|\leqslant z(\theta),\quad \big\|y(\theta-\rho)\big\|\leqslant z(\theta),\quad \theta\in[0,t].$$

$$\begin{aligned}\big\|y(t)\big\|\leqslant{}&\|\phi\|+t\|\phi'\|\\&+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\theta)^{\alpha_1-\alpha_2-1}z(\theta)\,\mathrm{d}\theta\\&+\frac{1}{\Gamma(\alpha_1)}\int_0^t(t-\theta)^{\alpha_1-1}\big[\sigma_{\max}(\mathcal{B})z(\theta)+\sigma_{\max}(\mathcal{C})z(\theta)+\|\mathcal{D}\|\big\|u(\theta)\big\|\big]\,\mathrm{d}\theta,\end{aligned}$$

$$\begin{aligned}\big\|y(t)\big\|\leqslant{}&\|\phi\|+t\|\phi'\|\\&+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t(t-\theta)^{\alpha_1-\alpha_2-1}z(\theta)\,\mathrm{d}\theta\\&+\frac{\sigma_{\max}(\varLambda)}{\Gamma(\alpha_1)}\int_0^t(t-\theta)^{\alpha_1-1}z(\theta)\,\mathrm{d}\theta+\frac{\alpha_{1u}d_0}{\Gamma(\alpha_1+1)}t^{\alpha_1}\\={}&\|\phi\|+t\|\phi'\|\\&+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t\theta^{\alpha_1-\alpha_2-1}z(t-\theta)\,\mathrm{d}\theta\\&+\frac{\sigma_{\max}(\varLambda)}{\Gamma(\alpha_1)}\int_0^t\theta^{\alpha_1-1}z(t-\theta)\,\mathrm{d}\theta+\frac{\alpha_{1u}d_0}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned}$$

Here $\|u(\theta)\|\leqslant\alpha_{1u}$, and $\sigma_{\max}(\mathcal{B})+\sigma_{\max}(\mathcal{C})$ is notated as $\sigma_{\max}(\varLambda)$.

Note that for all $\eta\in[0,t]$, we have

$$\begin{aligned}\big\|y(\eta)\big\|\leqslant{}&\|\phi\|+t\|\phi'\|\\&+\frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\|+\frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^\eta\theta^{\alpha_1-\alpha_2-1}z(\eta-\theta)\,\mathrm{d}\theta\\&+\frac{\sigma_{\max}(\varLambda)}{\Gamma(\alpha_1)}\int_0^\eta\theta^{\alpha_1-1}z(\eta-\theta)\,\mathrm{d}\theta+\frac{\alpha_{1u}d_0}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned}$$

Since the functions $\int_0^t(\theta)^{\alpha_1-\alpha_2-1}z(t-\theta)\,\mathrm{d}\theta$ and $\int_0^t(\theta)^{\alpha_1-1}z(t-\theta)\,\mathrm{d}\theta$ are increasing w.r.t $t \geqslant 0$, because of increasing of the nonnegative function $z(t)$, we get

$$\int_0^\eta (\theta)^{\alpha_1-\alpha_2-1}z(\eta-\theta)\,\mathrm{d}\theta \leqslant \int_0^t (\theta)^{\alpha_1-\alpha_2-1}z(t-\theta)\,\mathrm{d}\theta,$$
$$\int_0^\eta (\theta)^{\alpha_1-1}z(\eta-\theta)\,\mathrm{d}\theta \leqslant \int_0^t (\theta)^{\alpha_1-1}z(t-\theta)\,\mathrm{d}\theta.$$

Hence,

$$\begin{aligned}\big\|y(\eta)\big\| &\leqslant \|\phi\| + t\|\phi'\| \\ &\quad + \frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t \theta^{\alpha_1-\alpha_2-1}z(t-\theta)\,\mathrm{d}\theta \\ &\quad + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^t \theta^{\alpha_1-1}z(t-\theta)\,\mathrm{d}\theta + \frac{\alpha_{1u}d_0}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned}$$

Now we have

$$\begin{aligned}z(t) &= \sup_{\eta\in[-\rho,t]}\big\|y(\eta)\big\| \leqslant \max\Big\{\sup_{\eta\in[-\rho,0]}\big\|y(\eta)\big\|, \sup_{\eta\in[0,t]}\big\|y(\eta)\big\|\Big\} \\ &\leqslant \max\Bigg\{\|\phi\|,\ \|\phi\| + t\|\phi'\| + \frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\| \\ &\qquad + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t \theta^{\alpha_1-\alpha_2-1}z(t-\theta)\,\mathrm{d}\theta \\ &\qquad + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^t \theta^{\alpha_1-1}z(t-\theta)\,\mathrm{d}\theta + \frac{\alpha_{1u}d_0}{\Gamma(\alpha_1+1)}t^{\alpha_1}\Bigg\}, \\ &= \|\phi\| + t\|\phi'\| \\ &\quad + \frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\| + \frac{\|\mathcal{A}\|}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}z(\theta)\,\mathrm{d}\theta \\ &\quad + \frac{\sigma_{\max}(\Lambda)}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}z(\theta)\,\mathrm{d}\theta + \frac{\alpha_{1u}d_0}{\Gamma(\alpha_1+1)}t^{\alpha_1}.\end{aligned}$$

Now we present the nondecreasing function

$$v(t) = \|\phi\| + t\|\phi'\| + \frac{\|\mathcal{A}\|t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\|\phi\|,$$

and we let $r_1(t) = \|\mathcal{A}\|/\Gamma(\alpha_1 - \alpha_2)$ and $r_2(t) = \sigma_{\max}(\Lambda)/\Gamma(\alpha_1)$. Therefore, from the above equation we get

$$z(t) \leqslant v(t) + r_1(t) \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1} z(\theta)\,\mathrm{d}\theta + r_2(t) \int_0^t (t-\theta)^{\alpha_1-1} z(\theta)\,\mathrm{d}\theta + \frac{d_0 \alpha_{1u}}{\Gamma(\alpha_1+1)} t^{\alpha_1}.$$

Now, to apply the GGI, we obtain

$$\|y(t)\| \leqslant z(t) \leqslant v(t) E_\gamma\bigl(r(t)\bigl(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\bigr)\bigr),$$

where $r(t) = r_1(t) + r_2(t)$, $r_1(t) = \|\mathcal{A}\|/\Gamma(\alpha_1-\alpha_2)$, $r_2(t) = \sigma_{\max}(\Lambda)/\Gamma(\alpha_1)$, $\gamma = \min\{\alpha_1, \alpha_1 - \alpha_2\}$ and

$$\|y(t)\| \leqslant \delta\biggl\{1 + t + \frac{\|\mathcal{A}\| t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\biggr\} E_\gamma\bigl(r(t)\bigl(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\bigr)\bigr) + \frac{\alpha_{1u} d_0}{\Gamma(\alpha_1+1)} t^{\alpha_1}.$$

Now using condition (2), we can attain the required finite-time stability condition

$$\|y(t)\| \leqslant \epsilon, \quad t \in L_0.$$

This is our required result. □

Corollary 3. *Suppose* α_1 *and* α_2 *are integers, i.e., take* $\alpha_1 = 2$ *and* $\alpha_2 = 1$. *The nonautonomous system with integer order defined by*

$$\begin{aligned} &\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} - \mathcal{A}\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \mathcal{B}y(t) + \mathcal{C}y(t-\rho) + \mathcal{D}u(t), \quad t \in L, \\ &y(t) = \phi(t), \quad y'(t) = \phi'(t), \quad -\rho \leqslant t \leqslant 0, \end{aligned} \tag{5}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ *and* $\mathcal{D}$ *are as in* (1). *Then the FTS condition of* (5) *is*

$$\biggl\{1 + |t| + \frac{\sigma_{\max}(\mathcal{A})|t|^1}{1}\biggr\} \mathrm{e}^{r(t)(t+t^2)} + \eta_{u_0}\frac{t^2}{2} \leqslant \frac{\epsilon}{\delta},$$

where

$$\eta_{u0} = \frac{\alpha_{1u} d_0}{\delta}, \qquad r(t) = \frac{\sigma_{\max}(\mathcal{A})}{1} + \frac{\sigma_{\max}(\Lambda)}{1}$$

and $\Gamma(2) = 1$, $E_{\gamma=1}(z) = \mathrm{e}^z$.

Proof. Using the method of converting the differential equation with initial condition to Volterra integral equation, we can get the solution of system (5) following in the equivalent form of integral equation

$$y(t) = y(0) + ty'(0) - \mathcal{A}ty(0) + \mathcal{A}\int_0^t y(\theta)\,\mathrm{d}\theta + \int_0^t (t-\theta)\big[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho) + \mathcal{D}u(\theta)\big]\,\mathrm{d}\theta.$$

Now proceeding the same technique as in Theorem 1, we get the required proof of this corollary. □

Theorem 2. *Consider the linear autonomous fractional-order damped dynamical system involving time delay*

$$\begin{aligned} &{}^C_0D_t^{\alpha_1}y(t) - \mathcal{A}{}^C_0D_t^{\alpha_2}y(t) = \mathcal{B}y(t) + \mathcal{C}y(t-\rho), \quad t \in L,\\ &y(t) = \phi(t), \quad y'(t) = \phi'(t), \quad -\rho \leqslant t \leqslant 0, \end{aligned} \tag{6}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are defined as in (1). *System* (6) *is finite-time stable w.r.t* $\{\delta, \epsilon, L_0\}$, $\delta < \epsilon$, *if*

$$\begin{aligned} &\left\{1 + |t| + \frac{\|\mathcal{A}\||t|^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}\right\}E_\gamma\big(r(t)\big(\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2} + \Gamma(\alpha_1)t^{\alpha_1}\big)\big)\\ &\quad \leqslant \frac{\epsilon}{\delta}, \quad t \in L_0 = [0, T], \end{aligned} \tag{7}$$

where $r(t) = r_1(t) + r_2(t)$, $r_1(t) = \|\mathcal{A}\|/\Gamma(\alpha_1-\alpha_2)$ *and* $r_2(t) = \sigma_{\max}(\Lambda)/\Gamma(\alpha_1)$.

Proof. The following $y(t)$ is the solution of (6):

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \frac{\mathcal{A}t^{\alpha_1-\alpha_2}}{\Gamma(\alpha_1-\alpha_2+1)}y(0)\\ &\quad + \frac{\mathcal{A}}{\Gamma(\alpha_1-\alpha_2)}\int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}y(\theta)\,\mathrm{d}\theta\\ &\quad + \frac{1}{\Gamma(\alpha_1)}\int_0^t (t-\theta)^{\alpha_1-1}\big[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho)\big]\,\mathrm{d}\theta. \end{aligned} \tag{8}$$

Following similar procedure of proof of Theorem 1, we get immediate proof of this theorem by using (7) and (8). □

Corollary 4. *Suppose that α_1 and α_2 are integers, i.e., take $\alpha_1 = 2$ and $\alpha_2 = 1$. Define the autonomous system with integer order*

$$\begin{aligned} &\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2} - \mathcal{A}\frac{\mathrm{d}y(t)}{\mathrm{d}t} = \mathcal{B}y(t) + \mathcal{C}y(t-\rho), \quad t \in L,\\ &y(t) = \phi(t), \quad y'(t) = \phi'(t), \quad -\rho \leqslant t \leqslant 0, \end{aligned} \tag{9}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are as in (1)*. Then FTS condition of system* (9) *is*

$$\left\{1 + |t| + \frac{\sigma_{\max}(\mathcal{A})|t|^1}{1}\right\}\mathrm{e}^{r(t)(t+t^2)} \leqslant \frac{\epsilon}{\delta},$$

where

$$r(t) = \frac{\sigma_{\max}(\mathcal{A})}{1} + \frac{\sigma_{\max}(\Lambda)}{1}$$

and $\Gamma(2) = 1$, $E_{\gamma=1}(z) = \mathrm{e}^z$.

Proof. The following $y(t)$ is the solution of (9):

$$\begin{aligned} y(t) &= y(0) + ty'(0) - \mathcal{A}ty(0) \\ &\quad + \mathcal{A}\int_0^t y(\theta)\,\mathrm{d}\theta + \int_0^t (t-\theta)\big[\mathcal{B}y(\theta) + \mathcal{C}y(\theta-\rho)\big]\,\mathrm{d}\theta. \end{aligned}$$

Now proceeding the same steps as in Theorem 1, we get the required result. □

4 Numerical examples

Example 1. Consider the multiterm fractional damped system

$$\begin{aligned} &{}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}{}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}y(t) + \mathcal{C}y(t-\rho) + \mathcal{D}u(t), \\ &y(t) = 0, \quad y'(t) = 0, \quad -\rho \leqslant t \leqslant 0. \end{aligned}$$

The parameters are taken explicitly as $\alpha_2 = 0.75$, $\alpha_1 = 1.25$, $y(t) = (y_1, y_2)^{\mathrm{T}}$ and

$$\mathcal{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}, \qquad \mathcal{D} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now, to check the FTS condition w.r.t $\delta = 0.05$, $\epsilon = 2$, $t_0 = 0$, $\alpha_{1u} = 1$,

$$\gamma = \min\{\alpha_1, \alpha_1 - \alpha_2\} = 0.5 \quad \text{and} \quad \rho = 0.1.$$

Then $\|\mathcal{A}\| = 1$, $\sigma_{\max}(\mathcal{B}) = 1$ and $\sigma_{\max}(\mathcal{C}) = 0.5$ implies $\sigma_{\max}(\Lambda) = 1.5$. Hence, from this we can calculate $r(t) = 2.2191$ and $\eta_{u0} = 20$. Applying these values to the condition given in Theorem 1, we can get the estimated time $T \approx 0.13$ of finite-time stability.

Example 2. Consider the multiterm fractional-order damped system

$$\begin{aligned} &{}_0^C D_t^{\alpha_1} y(t) - \mathcal{A}{}_0^C D_t^{\alpha_2} y(t) = \mathcal{B}y(t) + \mathcal{C}y(t-\rho), \\ &y(t) = 0, \quad y'(t) = 0, \quad -\rho \leqslant t \leqslant 0. \end{aligned}$$

The parameters are taken explicitly as $\alpha_2 = 0.75$, $\alpha_1 = 1.25$, $y(t) = (y_1, y_2, y_3)^{\mathrm{T}}$. Also,

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.04 & 0.04 & 0 \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 3 & 0 & 0 \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, to check the FTS condition w.r.t $\delta = 0.01$, $\epsilon = 1$, $t_0 = 0$,

$$\gamma = \min\{\alpha_1, \alpha_1 - \alpha_2\} = 0.5 \quad \text{and} \quad \rho = 0.1.$$

Then $\|\mathcal{A}\| = 0$, $\sigma_{\max}(\mathcal{B}) = 3$ and $\sigma_{\max}(\mathcal{C}) = 1$ implies $\sigma_{\max}(\Lambda) = 4$. From this we can calculate $r(t) = 4.4131$. Applying these values to the condition given in Theorem 2, we can get the estimated time $T \approx 1.19$ of finite-time stability.

5 Conclusion

The analysis related to stability is studied for many fractional systems using the Lyapunov method and Gronwall inequality over the finite and infinite interval of time. So, it is important to discuss the FTS for fractional systems with damping behaviour. This work concerned with the FTS of multiterm time-delayed fractional-order system with $0 < \alpha_2 \leqslant 1 < \alpha_1 \leqslant 2$. For this, we obtained some inequalities with the help of Gronwall's inequality and its extended form, which proved our FTS results. At last, the obtained results are verified through examples. Moreover, the results derived in this work can be also extended to stochastic cases with various behaviours like impulses, delay in multi states and so on.

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