mathematics

Review

Fixed-Point Results for Meir–Keeler Type Contractions in Partial Metric Spaces: A Survey

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Abstract: In this paper, we aim to review Meir–Keeler contraction mappings results on various abstract spaces, in particular, on partial metric spaces, dislocated (metric-like) spaces, and M-metric spaces. We collect all significant results in this direction by involving interesting examples. One of the main reasons for this work is to help young researchers by giving a framework for Meir Keeler's contraction.

Keywords: Meir–Keeler contraction; uniform contraction mapping; fixed-point; partial metric spaces; metric-like spaces; M-metric spaces

MSC: 46T99; 47H10; 54H25

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Citation: Karapınar, E.; Agarwal, R.P.; Yeşilkaya, S.S.; Wang, C. Fixed-Point Results for Meir–Keeler Type Contractions in Partial Metric Spaces: A Survey. *Mathematics* **2022**, *10*, 3109. https://doi.org/10.3390/math10173109

Academic Editor: Christopher Goodrich

Received: 3 August 2022
Accepted: 18 August 2022
Published: 30 August 2022

1. Introduction

Fixed-point theory is a powerful tool for solving pure mathematics and other applied science problems such as computer science, engineering, and so on. Following Banach's pioneering fixed-point result (also known as the Banach contraction mapping principle) in 1922, many authors have published numerous research papers. The famous Banach fixed-point theorem has been generalized by many researchers from different directions in various structures.

In 1969, one of the essential such generalizations was reported by Meir and Keeler in the article "*A theorem on contraction mappings,* [1]". More precisely, in this article, the authors introduced the notion of weakly uniformly strict contraction, later called the Meir–Keeler contraction. It is worth mentioning that it is one of the most cited studies in the fixed-point theory literature.

In this study, we conduct a literature review in which we systematically collect studies on Meir–Keeler contraction by starting with the original result of Meir–Keeler [1]. On the other hand, since there are too many articles and results on this subject, we had to limit ourselves to focus on the Meir–Keeler contraction studies in the framework of the partial metric spaces (and extensions of these spaces; M- metric and metric-like). This paper will present only the primary studies on this subject.

This paper consists of five sections. In the first section, we shall recollect some fundamental contraction results, including the Meir–Keeler contraction theorem. In Section 2, we first give the definition of standard partial metric spaces, some critical properties, and examples. Next, we examine Meir–Keeler contraction studies on partial metric space. In addition, we examine Meir–Keeler contraction studies on metric-like space in Section 3.

The Meir–Keeler results on M-metric space are collected in Section 4. The last section is reserved for the conclusion.

1.1. Some Definitions and Fundamental Results

In this section, we shall fix some notations and recall some basic definitions and well-known results. Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers, that is, $\mathbb{N} = \{1, 2, \ldots\}$. We denote by $\mathbb{Z}$ the set of integers, that is, $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$ and $\mathbb{Z}^+ := \mathbb{N}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbols $\mathbb{R}$ the set of all real numbers, $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers, that is, $\mathbb{R}_0^+ := [0, \infty)$ and $\mathbb{R}^+ := (0, \infty)$. Throughout the paper, all considered sets will be presumed nonempty.

Definition 1. *Let X denotes a complete metric space with distance function d, and T a function mapping X into itself.*

(i) *(Banach [2]) There exists a number $\lambda, 0 \leqslant \lambda < 1$, such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leqslant \lambda d(x, y).$$

(ii) *(Rakotch [3]) There exists a monotone decreasing function $\lambda : (0, \infty) \to [0, 1)$ such that, for each $x, y \in X, x \neq y$,*

$$d(Tx, Ty) \leqslant \lambda(d(x, y)).d(x, y).$$

(iii) *(Edelstein [4]) For each $x, y \in X, x \neq y$,*

$$d(Tx, Ty) < d(x, y).$$

(iv) *(Kannan [5]) There exists a number $\lambda, 0 < \lambda < \frac{1}{2}$, such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leqslant \lambda[d(x, Tx) + d(y, Ty)].$$

(v) *(Reich [6]) There exist nonnegative numbers λ, β, γ satisfying $\lambda + \beta + \gamma < 1$ such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leqslant \lambda d(x, Tx) + \beta d(y, Ty) + \gamma d(x, y).$$

(vi) *(Chatterjea [7])*

(a) *There exists a number $\lambda, 0 < \lambda < \frac{1}{2}$, such that, for each $x, y \in X$*

$$d(Tx, Ty) \leqslant \lambda\{d(x, Ty) + d(y, Tx)\}.$$

(b) *There exists a number $\lambda, 0 \leqslant \lambda < 1$, such that, for each $x, y \in X$,*

$$d(Tx, Ty) \leqslant \lambda \max\{d(x, Ty), d(y, Tx)\}.$$

(c) *For each $x, y \in X, x \neq y$,*

$$d(Tx, Ty) < \max\{d(x, Ty), d(y, Tx)\}.$$

(vii) *(Hardy-Rogers [8]) There exist nonnegative constants, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 < 1$ such that, for each $x, y \in X$,*

$$\begin{aligned} d(Tx, Ty) \leqslant &\lambda_1 d(x, y) + \lambda_2 d(x, Tx) + \lambda_3 d(y, Ty) \\ &+ \lambda_4 d(x, Ty) + \lambda_5 d(y, Tx) \end{aligned}$$

(viii) *(Ćirić [9]) There exists a constant $\lambda, 0 \leqslant \lambda < 1$, such that, for each $x, y \in X$*

$$d(Tx, Ty) \leqslant \lambda \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Theorem 1 (Fisher [10] and Khan [11]). *Let (X, d) be a metric space and T be a self map on X satisfying the following:*

$$d(Tx, Ty) \leq \lambda \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}, \lambda \in [0, 1),$$

if

$$d(x, Ty) + d(y, Tx) \neq 0,$$

and

$$d(Tx, Ty) = 0 \quad \text{if} \quad d(x, Ty) + d(y, Tx) = 0.$$

Then T has a unique fixed point $x^* \in X$. Also, for each $x_0^* \in X$, the sequence $\{T^n x_0^*\}$ converges to x^*.

Theorem 2 (Gupta and Saxena, [12]). *Let (X, d) be a complete metric space and let T be a continuous mapping from X into itself satisfying*

$$d(Tx, Ty) \leq \lambda_1 \frac{(1 + d(x, Tx))d(y, Ty)}{1 + d(x, y)} + \lambda_2 \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \lambda_3 d(x, y)$$

for all $x, y \in X, x \neq y$, where $\lambda_1, \lambda_2, \lambda_3$ are constants with $\lambda_1, \lambda_2, \lambda_3 > 0$ and $\lambda_1 + \lambda_2 + \lambda_3 < 1$. Then T has a unique fixed point $x^* \in X$. Also, for all $x \in X$, the sequence $\{T^n x\}$ converges to x^*

Here, we give notations of admissible mappings.

Definition 2. *For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$.*

(i) *(Samet et al. [13]) We say that T is α-admissible if*

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \text{implies that} \quad \alpha(Tx, Ty) \geq 1.$$

(ii) *(Karapınar et al. [14]) A selfmapping T is called triangular α-admissible if*
   - *(1)* *T is α-admissible and*
   - *(2)* *$\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$* implies that *$\alpha(x, z) \geq 1$, for any $x, y, z \in X$*

(iii) *(Aydi et al. [15]) Let $S : X \to X$ be a mapping. We say that (T, S) is a generalized α-admissible pair if for all $x, y \in X$, we have*

$$\alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Sy) \geq 1 \quad \text{and } \alpha(STx, TSy) \geq 1$$

(iv) *(Popescu [16]) We say that T is α-orbital admissible if*

$$\alpha(x, Tx) \geq 1 \Rightarrow \alpha\left(Tx, T^2 x\right) \geq 1.$$

Also, T is called triangular α-orbital admissible if T is α-orbital admissible and

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, Ty) \geq 1 \Rightarrow \alpha(x, Ty) \geq 1.$$

1.2. 1969, Meir and Keeler, a Theorem on Contraction Mappings, [1]

We presume that (X, d) is a complete metric space. For a self mapping T on X, if there is a real number $\lambda \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \lambda \cdot d(x, y),$$

for all $x, y \in X$, then T possesses a unique fixed point (i.e., there exists a point $x^* \in X$ such that $Tx^* = x^*$).

This is the initial metric fixed-point result that was stated and proved by Banach [2]. Meir and Keeler [1] extended Banach's metric fixed point theorem by replacing contraction condition with "weakly uniformly strict contraction" as follows:

Theorem 3. *We presume that (X, d) is a complete metric space. For a self mapping T on X, if for each given $\varepsilon > 0$, there is a $\delta > 0$ so that*

$$\varepsilon \leqslant d(x, y) < \varepsilon + \delta \quad \textit{implies} \quad d(Tx, Ty) < \varepsilon, \tag{1}$$

then T has a unique fix point x^*. Moreover, we have

$$\lim_{n\to\infty} T^n x = x^*. \tag{2}$$

for any initial point $x \in X$.

Proof. We first observe that (1) trivially implies that T is a strict contraction, i.e.,

$$x \neq y \quad \text{implies} \quad d(Tx, Ty) < d(x, y). \tag{3}$$

Thus, T is continuous and it has at most one fixed point. □

In particular, we have the following known result that can be proved, easily, see e.g., Chu and Diaz [17].

Lemma 1 ([17])**.** *If $T : X \to X$ is a strict contraction and if, for every $x \in X$, $\{T^n(x)\}$ is a Cauchy sequence, then T has a unique fixed point and (2) holds.*

Lemma 2. *Condition (1) implies that*

$$\lim d(x_n, x_{n+1}) = 0.$$

The proof is by contradiction. Let $c_n = d(x_n, x_{n+1})$. From (3), c_n is decreasing with n. If $c_n \to \varepsilon > 0$, then (1) fails for c_{m+1} where c_m is chosen less than $\varepsilon + \delta$.

Having proved Lemma 2, we now suppose that some sequence is not a Cauchy sequence. Then there exists $2\varepsilon > 0$ such that $\limsup d(x_m, x_n) > 2\varepsilon$. By hypothesis, there exists a $\delta > 0$, such that

$$\varepsilon \leqslant d(x, y) < \varepsilon + \delta \quad \text{implies } d(Tx, Ty) < \varepsilon. \tag{4}$$

Formula (4) will remain true with δ replaced by $\delta' = \min(\delta, \varepsilon)$. From Lemma 2 we can find M so that $c_M < \delta'/3$. Pick $m, n > M$ so that $d(x_m, x_n) > 2\varepsilon$. For j in $[m, n]$,

$$\left| d(x_m, x_j) - d(x_m, x_{j+1}) \right| \leqslant c_j < \frac{\delta'}{3}.$$

This implies, since $d(x_m, x_{m+1}) < \varepsilon$ and $d(x_m, x_n) > \varepsilon + \delta'$, that there exists j in $[m, n]$ with

$$\varepsilon + \frac{2\delta'}{3} < d(x_m, x_j) < \varepsilon + \delta' \tag{5}$$

However, for all m and j,

$$d(x_m, x_j) \leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{j+1}) + d(x_{j+1}, x_j).$$

Therefore, by (4) and (5),

$$d(x_m, x_j) \leqslant c_m + \varepsilon + c_j < \frac{\delta'}{3} + \varepsilon + \frac{\delta}{3}$$

which contradicts (5). This contradiction proves that x_n must be a Cauchy sequence, and establishes our theorem.

Other authors have extended Banach's theorem in other ways. We will show that our theorem implies some of their results.

In [4], Edelstein considers locally contractive mappings and derives as a corollary that any strict contraction of a compact space has a unique fixed point. This result also follows from our theorem, since in a compact space, any strict contraction $T : X \to X$ is weakly uniformly strict. To prove this, we consider

$$\inf_{\varepsilon \leqslant d(x,y)} [d(x,y) - d(Tx, Ty)] = \delta(\varepsilon).$$

Since X is compact, this infimum is achieved for some pair of points (a, b) with $d(a, b) \geqslant \varepsilon$. Since T is a strict contraction $\delta(\varepsilon) > 0$.

Rakotch [3] and Boyd and Wong [18] work with a function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions $\psi(0) = 0, \psi(t) < t$ for all $t > 0$ and ψ is right upper semicontinuous, assume the inequalities

$$d(Tx, Ty) < \psi(d(x,y)) \text{ and } \psi(d) < d \tag{6}$$

(as well as other conditions).

The following example shows that (6) may be violated while the hypothesis (1) of our theorem is fulfilled.

Example 1. *Let* $X = [0,1] \cup \{3, 4, 6, 7, \ldots, 3n, 3n+1, \ldots\}$ *with the Euclidean distance, and let* Tx *be defined as follows:*

$$\begin{array}{lll} Tx = \frac{x}{2} & & 0 \leqslant x \leqslant 1 \\ Tx = 0 & \text{if} & x = 3n \\ Tx = 1 - \frac{1}{n+2} & \text{if} & x = 3n+1. \end{array}$$

Although T satisfies our condition (1), $\psi(1)$ would have to be 1. On the other hand, Rakotch's [3] in Corollary ($\psi(d)$ monotone), and Boyd and Wong's [18] in Theorem 1 ($\psi(d)$ upper semicontinuous) and Theorem 2 (X metrically convex) follow easily from our theorem.

The following example shows that (6) may be satisfied in a complete metric space, while the mapping T has no fixed point. This resolves the question posed by Boyd and Wong.

Example 2. *Let* $s_n = \sum_{k=1}^{n}(1 + 1/k)$*, and let* $X = \{s_n\}$*. Let* $Ts_n = s_{n+1}$ *for all* n*. Then*

$$d(Tx, Ty) \leqslant \psi(d(x,y)) \text{ with } \psi(1 + 1/n) = 1 + 1/(n+1),$$

but there is no fixed point.

1.3. Partial Metric Spaces

First, we give the partial metric space concept.

The concept of metric space is first formulated axiomatically as "L-space" by Frechét (1906) [19] and later used as "metric space" (or standard metric) by Hausdorff. Metric space is used effectively in solving problems in many fields, both in natural sciences and applied sciences, beyond mathematics.

To overcome the difficulties of the problems, the concept of metric spaces has been expanded and improved in various ways. Some generalized metric spaces are partial metric, b-metric, metric-like, M-metric, modular metric, quasi metric, etc. In this paper, we focus on very interesting and real generalization metric spaces among them, namely partial metric spaces.

The idea of partial metric spaces, a generalization of metric spaces, is introduced by Matthews [20] in the early 1990s to handle computer science problems. The most important difference of the partial metric rather than the metric is that in the partial metric, the self-distance, $p(x,x)$, need not have to be zero. In other words, partial metric spaces have the existing probability of non-zero self-distance. Furthermore, the topologies of these spaces are quite different from each other. The partial metric space limit is not unique. Recently, some interesting research and survey papers for fixed-point theory on partial metric spaces are published in [21–32].

We first recall the definition of partial metric spaces and then some of their critical properties and examples.

Definition 3 ([20])**.** *Let X be a nonempty set and let $p : X \times X \to \mathbb{R}_0^+$ satisfy*

(p1) $p(x,y) = p(y,x)$ *(symmetry);*
(p2) *if* $0 \le p(x,x) = p(y,y) = p(x,y)$ *then* $x = y$ *(equality);*
(p3) $p(x,x) \le p(x,y)$ *(small self-distances);*
(p4) $p(x,z) + p(y,y) \le p(y,z) + p(x,y)$ *(triangularity);*

for all $x,y,z \in X$. Then the pair (X,p) is called a partial metric space and p is called a partial metric on X.

Remark 1 ([20])**.** *Let a partial metric p on X, the function $d_p : X \times X \to \mathbb{R}_0^+$ given by*

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X. A partial metric p on X generates a T_0 topology τ_p on X, having as base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ $\forall x \in X$ and $\varepsilon > 0$.

Example 3 ([20,33])**.** *(i)* *Let $X = \{[a,b] : a,b \in \mathbb{R}, a \le b\}$ and define*

$$p([a,b],[c,d]) + \min\{a,c\} = \max\{b,d\}.$$

Then (X,p) is a partial metric space.

(ii) *Let $X = \mathbb{R}_0^+$ and p on X defined by*

$$p(x,y) = \max\{x,y\}, \forall x,y \in X.$$

Then (X,p) is a partial metric space.

(iii) *Let $X = [0,1] \cup [2,3]$ and given $p : X \times X \to \mathbb{R}_0^+$ by*

$$p(x,y) = \begin{cases} \max\{x,y\}, & \{x,y\} \cap [2,3] \neq \emptyset, \\ |x-y|, & \{x,y\} \subset [0,1] \end{cases}.$$

Then (X,p) is a partial metric space.

Example 4 (see [34])**.** *Let (X,d) and (X,p) be a metric space and partial metric space, respectively. Functions $\rho_i : X \times X \to \mathbb{R}_0^+ (i \in \{1,2,3\})$ given by*

$$\rho_1(x,y) = p(x,y) + d(x,y)$$
$$\rho_2(x,y) = \max\{\omega(x), \omega(y)\} + d(x,y)$$
$$\rho_3(x,y) = a + d(x,y)$$

defined partial metrics on X, where $\omega : X \to \mathbb{R}_0^+$ *is an arbitrary function and* $a \geq 0$.

Definition 4 (see e.g., [20–23,32,35]). *Let* (X,p) *be a partial metric space and* $\{x_n\}$ *be a sequence in* (X,p).

- *(i)* *A sequence* $\{x_n\}$ *converges to* $x \in X$ *if* $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
- *(ii)* *A sequence* $\{x_n\}$ *is called Cauchy if* $\lim_{n,m\to\infty} p(x_n,x_m)$ *exists (and is finite).*
- *(iii)* *A partial metric spaces* (X,p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in X converges, with respect to* τ_p, *to a point* $x \in X$ *such that* $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
- *(iv)* *A mapping* $f : X \to X$ *is said to be continuous at* $x_0 \in X$ *if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $f(B(x_0,\delta)\ ) \subset B(f(x_0),\varepsilon)$
- *(v)* *A sequence* $\{x_n\}$ *in* (X,p) *is called 0-Cauchy sequence if* $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. *The space* (X,p) *said to be 0-complete if every 0-Cauchy sequence in X converges with respect to* τ_p *to a point* $x \in X$ *such that* $p(x,x) = 0$.

Lemma 3 (see e.g., [20,21,23,24,26]). *Let* (X,p) *be a partial metric space and* $\{x_n\}$ *be a sequence in* (X,p).

- *(a)* *If* $x_n \to z$ *as* $n \to \infty$ *in* (X,p) *with* $p(z,z) = 0$, *then we have*

$$\lim_{n\to\infty} p(x_n,y) = p(z,y) \ \ \forall y \in X.$$

- *(b)* *A partial metric spaces* (X,p) *is complete if and only if the metric space* (X,d_p) *is complete. Furthermore,*

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m_\to\to\infty} p(x_n,x_m).$$

- *(c)* *A sequence* $\{x_n\}$ *is Cauchy in* (X,p) *if and only if* $\{x_n\}$ *is Cauchy in* (X,d_p).
- *(d)* *Consider* $X = [0,\infty)$ *endowed with the partial metric* $p : X \times X \to [0,\infty)$ *given by* $p(x,y) = \max\{x,y\}$ *for all* $x,y \geq 0$. *Let* $T : X \to X$ *be a nondecreasing function. If T is continuous with respect to the metric spaces* $d(x,y) = |x-y|$ *for all* $x,y \geq 0$, *then T is continuous with respect to the partial metric p.*

We recall Banach fixed-point theorem in partial metric spaces, as follows;

Theorem 4 ([20]). *Let T be a mapping of a partial metric space* (X,p) *into itself such that:*

$$p(Tx,Ty) \leqslant \lambda p(x,y).$$

for all $x,y \in X$, *where* λ *is real number with* $0 \leqslant \lambda < 1$. *Then T has a unique fixed point.*

1.4. Metric-like Spaces

The idea of dislocated metric space was first defined by Hitzler and Seda [36]. In 2012, Amini-Harandi [37] reintroduced the idea of the dislocated metric space under the name "metric-like", for more details, see e.g., [37–42]. In this study, we prefer the metric-like spaces. The notation of metric-like space is a generalization of that of partial metric space. Now, we recall metric-like (or dislocated metric) notations.

Definition 5 ([36,37]). *Let X be a nonempty set. A function* $\sigma : X \times X \to \mathbb{R}_0^+$ *is called a metric-like on X if* $\forall\ x,y,z \in X$, *the following conditions hold:*

- (σ_0) $\sigma(x,y) \geq 0$ *(nonnegativity),*
- (σ_1) $\sigma(x,y) = \sigma(x,x) = 0 \Longrightarrow x = y$ *(pseudo-indistancy),*

(σ_2) $\sigma(x,y) = \sigma(y,x)$ *(symmetry),*
(σ_3) $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$ *(triangularity).*
The pair (X,σ) *is then called a metric-like space.*

Each partial metric space is a metric-like spaces. But the converse is not true. In addition, we give a metric-like which is not a partial metric and so nor metric. Examples that support these remarks are as follows:

Example 5 (See [37,41,43])**.**
(i) *The pair* $(\mathbb{R}_0^+,\sigma)$, *where* $\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is given by*

$$\sigma(x,y) = \max\{x,y\}.$$

is metric-like space. Also, σ *is a partial metric.*
(ii) *On* $X = \mathbb{R}$ *define the metric-like* σ *by*

$$\sigma(x,y) = \frac{|x-y| + |x| + |y|}{2} \quad \text{for all } x,y \in X.$$

where σ *is not a metric. Restricted to* $\mathbb{R}_0^+$ *becomes* $\sigma(x,y) = \max\{x,y\}$, *so we return to (i).*
(iii) *Let* $X = \{1,2,3\}$ *and defined the metric-like* $\sigma : X^2 \to \mathbb{R}_0^+$ *given by*

$$\sigma(1,1) = 0, \quad \sigma(2,2) = 1, \quad \sigma(3,3) = \frac{2}{3},$$
$$\sigma(1,2) = \sigma(2,1) = \frac{9}{10}, \quad \sigma(2,3) = \sigma(3,2) = \frac{4}{5},$$
$$\sigma(1,3) = \sigma(3,1) = \frac{7}{10}.$$

where $\sigma(2,2) \neq 0, \sigma$ *is not a metric and* $\sigma(2,2) > \sigma(1,2), \sigma$ *is not a partial metric.*

Definition 6 (See [37,42])**.** *Let* (X,σ) *be a metric-like space.*
(a) *A sequence* $\{x_n\}$ *in* X *is a Cauchy sequence if* $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ *exists and is finite.*
(b) (X,σ) *is complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges with respect to* τ_σ *to a point* $x \in X$*; that is,*

$$\lim_{n\to\infty} \sigma(x,x_n) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n,x_m).$$

1.5. M-Metric SPACES

In 2014, Asadi et al. [44] defined the idea of M-metric spaces. M-metric spaces are generalizations of standard metric spaces and partial metric spaces.

Definition 7 ([44])**.** *Let* X *be a non-empty set. A function* $m : X \times X \to \mathbb{R}_0^+$ *is called a M-metric if the following holds are satisfied:*
(m_1) $m(x,x) = m(y,y) = m(x,y) \Longleftrightarrow x = y,$
(m_2) $m_{xy} \leq m(x,y),$
(m_3) $m(x,y) = m(y,x),$
(m_4) $\big(m(x,y) - m_{xy}\big) \leq \big(m(x,z) - m_{xz}\big) + \big(m(z,y) - m_{zy}\big).$
Here we have used the following notations:
(a) $m_{x,y} = \min\{m(x,x), m(y,y)\}$
(b) $M_{x,y} = \max\{m(x,x), m(y,y)\}$
Then the pair (X,m) *is called an M-metric space.*

Each partial metric space is an M-metric space, but the converse is not true. Examples that support the remark are as follows:

Example 6 ([44])**.**
(i) *Let* $X := [0, \infty)$*. Then* $m(x,y) = \frac{x+y}{2}$ *on* X *is an M-metric.*
(ii) *Let* $X = \{1,2,3\}$*. Given* m *on* $X \times X$ *as follows:*

$$m(1,1) = 1, \quad m(2,2) = 9, \quad m(3,3) = 5,$$
$$m(1,2) = m(2,1) = 10, \quad m(1,3) = m(3,1) = 7, \quad m(2,3) = m(3,2) = 7.$$

So, (X,m) *is an M-metric space but it is not a partial metric space.*
(iii) *Let* (X,d) *be a metric space. Let* $\phi : [0,\infty) \to [\phi(0), \infty)$ *be a one to one and nondecreasing or strictly increasing mapping, with* $\phi(0)$ *given such that*

$$\phi(x+y) \leq \phi(x) + \phi(y) - \phi(0), \quad \forall x, y \geq 0$$

Then $m(x,y) = \phi(d(x,y))$ *is an M-metric.*

Example 7 ([45])**.** *Let* $X = \{1,2,3,4\}$*; define the function* m *on* $X \times X$ *as follows* $m(1,1) = 1$*,* $m(2,2) = 3, m(3,3) = 5, m(4,4) = 3, m(1,2) = m(2,1) = 10, m(1,3) = m(3,1) = m(3,2) = m(2,3) = 7, m(1,4) = m(4,1) = 8, m(2,4) = m(4,2) = 6, m(3,4) = m(4,3) = 6$*. Then it is easy to verify that* m *is an M-metric space but it is not a partial metric space because it does not satisfy the triangle inequality* $m(1,2) \not\leq m(1,3) + m(3,2) - m(3,3)$*.*

We said that $m(x,x) \neq 0$*, for all* $x \in X$*. Let* $\phi : X \to \mathbb{R}_0^+$ *defined by* $\phi(x) = 10x$ *and define the self-map* $T : X \to X$ *by* $T(x) = 1$ *for all* $x \neq 4$ *and* $T(4) = 4$*. Take* $\varepsilon = 0.1$ *and define the corresponding open balls* $B(x, 0.1) = \{y \in X : m(x,y) < m_{x,y} + 0.1$*. The open balls are single sets;* $B(x, 0.1) = \{x\} \ \forall\, x \in X$*. So, the M- metric topology on* X *is the discrete topology and, thus, each map defined on* X *is lower semicontinuous. Also,* $\forall\, x \in X$*, we notice that* $m(x, Tx) \leq m_{x,Tx} + \phi(x) - \phi(Tx)$*. So, the function* T *satisfies the hypotheses of Theorem 20 of [45] and so it has a fixed point. Then,* $x = 1, 4$ *are fixed points.*

2. Meir–Keeler Contractions on Partial Metric Spaces

2.1. 2012, Aydi, Karapınar and Rezapour, a Generalized Meir–Keeler-Type Contraction on Partial Metric Spaces, [46]

In [46], the authors proposed a new notion, namely, a generalization of the Meir–Keeler type contractions on partial metric spaces. First, we give main definition.

Definition 8. *Let* (X,p) *be a partial metric space. A selfmapping* T *on* (X,p) *is said to be a generalized Meir–Keeler-type contraction if for any* $\varepsilon > 0$ *there is a* $\delta > 0$ *so that*

$$\varepsilon \leq M(x,y) < \varepsilon + \delta \Longrightarrow p(Tx,Ty) < \varepsilon \tag{7}$$

where $M(x,y) = \max\{p(x,y), p(Ty,y), p(Tx,x), (1/2)[p(x,Ty) + p(Tx,y)]\}$*.*

Remark 2. *It is evident that for the generalized Meir–Keeler-type contraction* T*, we always have*

$$p(Tx,Ty) \leq M(x,y) \quad \forall x, y \in X. \tag{8}$$

In case $M(x,y) = 0$*, we have* $p(Tx,Ty) = 0$ *due to (8). Further, if* $M(x,y) > 0$*, by (7) we derive the strict inequality* $p(Tx,Ty) < M(x,y)$*.*

Proposition 1. *Let* (X,p) *be a partial metric space and* $T : X \to X$ *a generalized Meir–Keeler-type contraction. Then,* $\lim_{n\to\infty} p\left(T^{n+1}x, T^n x\right) = 0$ *for all* $x \in X$*.*

Theorem 5. *Let* (X,p) *be a 0-complete partial metric space, and let* $T : X \to X$ *be an orbitally continuous generalized Meir–Keeler-type contraction. Then,* T *possesses a unique fixed point* $x^* \in X$*. In addition,* $\lim_{n\to\infty} p(T^n x, x^*) = p(x^*, x^*)$ *for each* $x \in X$ *and* $p(x^*, x^*) = 0$*.*

Example 8. *Let (X,p) be the set $[0,\infty)$ equipped with the partial metric $p(x,y) = \max\{x,y\}$. Clearly, (X,p) is a 0 -complete partial metric space. Consider $T : X \to X$ defined by $Tx = x/3(1+x)$. Given $\varepsilon > 0$, we will show that there exists $\delta = \delta(\varepsilon) \geq 0$ such that (7) holds for all $x, y \in X$. Without loss of generality, take $x \leq y$. Then, it is easy to show that*

$$p(Tx,Ty) = \frac{y}{3(1+y)};$$
$$M(x,y) = \max\left\{p(x,y), p(Ty,y), p(Tx,x), \frac{1}{2}[p(x,Ty) + p(Tx,y)]\right\} = y.$$

By letting $\delta(\varepsilon) = 2\varepsilon$, we find that (7) holds. Further, due to Lemma 3, we conclude that T is continuous. Consequently, it is orbitally continuous. As a result, all conditions of Theorem 5 are fulfilled. Thus, $x^* = 0$ is the required unique fixed point of T.

Example 9. *Let (X,p) be the interval $[0,2]$ equipped with the partial metric $p(x,y) = \max\{x,y\}$. Consider $T : X \to X$ defined by*

$$Tx = \begin{cases} \frac{x}{2} & \text{if } 0 \leq x < 1 \\ \frac{1}{2} & \text{if } 1 \leq x \leq 2 \end{cases}$$

Take $x \leq y$. Given $\varepsilon > 0$, we have the two following cases.
Case 1 $(0 \leq x \leq y < 1)$. We have

$$p(Tx,Ty) = \frac{y}{2}, \quad M(x,y) = y.$$

Case 2 $((0 \leq x < 1$ and $1 \leq y < 2)$ or $(1 \leq x \leq y \leq 2))$. We have

$$p(Tx,Ty) = \frac{1}{2}, \quad M(x,y) = y$$

In each case, it suffices to take $\delta = \varepsilon$ in order that (7) holds. Again, by Lemma 3, the mapping T is continuous, and hence it is orbitally continuous. All hypotheses of Theorem 5 are satisfied and $x^* = 0$ is the unique fixed point of T.

2.2. 2012, Aydi and Karapınar, a Meir–Keeler Common Type Fixed-Point Theorem on Partial Metric Spaces, [47]

In this section, we introduce the common fixed-point results of two pairs of weakly compatible self-mappings for the Meir–Keeler type contraction in partial metric space. Now, we give the following results.

Theorem 6. *Let A, B, S, and T be the self maps defined on a complete partial metric space (X,p) satisfying the following conditions:*
(C1) $AX \subseteq TX$ and $BX \subseteq SX$,
(C2) for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$

$$\varepsilon < M(x,y) < \varepsilon + \delta \Rightarrow p(Ax,By) \leq \varepsilon$$

where

$$M(x,y) = \max\left\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}[p(Sx,By) + p(Ax,Ty)]\right\},$$

(C3) for all

$$x, y \in X \text{ with } M(x,y) > 0 \Rightarrow p(Ax,By) < M(x,y),$$

(C4) $p(Ax, By) \leq \max\{a[p(Sx, Ty) + p(Ax, Sx) + p(By, Ty)], b[p(Sx, By) + p(Ax, Ty]\}$ *for all*

$$x, y \in X, 0 \leq a < \frac{1}{2} \text{ and } 0 \leq b < \frac{1}{2}.$$

If one of the ranges AX, BX, TX, and SX is a closed subset of (X, p), then

(I) *A and S have a coincidence point,*
(II) *B and T have a coincidence point.*

Moreover, if A and S, as well as, B and T are weakly compatible, then A, B, S, and T have a unique common fixed point.

Corollary 1. *Let A, B, S, and T be the self maps defined on a partial metric space (X, p) satisfying the following conditions:*

(C1) $AX \subseteq TX$ *and* $BX \subseteq SX$,
(C2) *for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$*

$$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow p(Ax, By) \leq \varepsilon,$$

where

$$M(x, y) = \max\left\{p(Sx, Ty), p(Ax, Sx), p(By, Ty), \frac{1}{2}[p(Sx, By) + p(Ax, Ty)]\right\},$$

(C3) *for all $x, y \in X$ with $M(x, y) > 0 \Rightarrow p(Ax, By) < M(x, y)$,*
(C4)

$$p(Ax, By) < k[p(Sx, Ty) + p(Ax, Sx) + p(By, Ty) + p(Sx, By) + p(Ax, Ty]$$

for all $x, y \in X$ and $0 \leq k < \frac{1}{3}$.

If one of AX, BX, SX, or TX is a complete subspace of X, then

(I) *A and S have a coincidence point,*
(II) *B and T have a coincidence point.*

In addition, if A and S, as well as, B and T are weakly compatible, then A, B, S, and T have a unique common fixed point.

Denote with Φ the family of nondecreasing functions $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for all $t > 0$, where φ^n is the nth iterate of φ. The following lemma is obvious.

Lemma 4. *If $\varphi \in \Phi$, then $\varphi(t) < t$ for all $t > 0$.*

Some Equivalence Statements of Meir–Keeler Contraction

We need the following lemma established by Jachymski [48].

Lemma 5. *Let Q be a subset of $[0, \infty) \times [0, \infty)$. Then the following statements are equivalent:*

(J1) *There exists a function $\delta : (0, \infty) \to (0, \infty)$ such that for any $\varepsilon > 0, \delta(\varepsilon) > \varepsilon$ and*
(J1a) $\sup\{\delta(s) : s \in (0, \varepsilon)\} \geq \delta(\varepsilon)$ *and*
(J1b) $(s, t) \in Q$ *and* $0 \leq s < \delta(\varepsilon)$ *imply* $t < \varepsilon$.
(J2) *There exist functions $\beta, \eta : (0, \infty) \to (0, \infty)$ such that, for any $\varepsilon > 0, \beta(\varepsilon) > \varepsilon, \eta(\varepsilon) < \varepsilon$, and $(s, t) \in Q$ and $0 \leq s < \beta(\varepsilon)$ imply $t < \eta(\varepsilon)$.*
(J3) *There exists an upper semi continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that φ is non-decreasing, $\varphi(s) < s$ for $s > 0$, and $(s, t) \in Q$ implies $t \leq \varphi(s)$.*
(J4) *There exists a lower-semi continuous function $\delta : (0, \infty) \to (0, \infty)$ such that for any δ is non-decreasing for any $\varepsilon > 0, \delta(\varepsilon) > \varepsilon$, and $(s, t) \in Q$ and $0 \leq s < \delta(\varepsilon)$ imply $t < \varepsilon$.*
(J5) *There exists a lower-semi continuous function $w : [0, \infty) \to [0, \infty)$ such that for any w is non-decreasing, $w(s) > s$ for $s > 0$ and $(s, t) \in Q$ implies $w(t) \leq s$.*

Theorem 7. *Let (X, p) be a partial metric space, and $S, T, A_i (i \in \mathbb{N})$ be self-mappings on X. For $x, y \in X$ and for $i, j \in \mathbb{N}$, we define*

$$M_{ij}(x,y) = \left\{ p(Sx,Ty), p(Sx,A_ix), p(Ty,A_jy), \frac{[p(Sx,A_jy) + p(Ty,A_ix)]}{2} \right\}$$

Then the following statements are equivalent.

(JT1) There exists a lower-semi continuous function $\delta : (0, \infty) \to (0, \infty)$ such that, for any $\varepsilon > 0, \delta(\varepsilon) > \varepsilon$ and for any $x, y \in X$ and distinct $i, j \in \mathbb{N}$

$$\varepsilon \leq M_{ij}(x,y) < \delta(\varepsilon) \text{ implies } p(A_ix, A_jy) < \varepsilon.$$

(JT2) There exists an upper-semi continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that, φ is non-decreasing, $\varphi(t) < t$, and

$$p(A_ix, A_jy) \leq \varphi(M_{ij}(x,y)).$$

for any $x, y \in X$ and distinct $i, j \in \mathbb{N}$.

(JT3) There exists a lower-semi continuous function $w : [0, \infty) \to [0, \infty)$ such that, w is non-decreasing, $w(s) > s$ for $s > 0$, and

$$w(p(A_ix, A_jy)) \leq M_{ij}(x,y)$$

for any $x, y \in X$ and distinct $i, j \in \mathbb{N}$.

Remark 3. *Ćirić et al. [49] assumed that the hypothesis $p(Ax, By) \leq \varphi\ (M(x,y))$ is satisfied for all $x, y \in X$ with $\varphi \in \Phi$ and obtained a common fixed-point result.*

In particular from the assumptions on that φ, $(JT2)$ holds for $A_1 = A$ and $A_2 = B$. So, by Theorem 7, (JT1) holds, that is; for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$

$$\varepsilon \leq M(x,y) < \varepsilon + \delta \Rightarrow p(Ax, By) < \varepsilon \tag{9}$$

By Lemma 5 of Jachymski [50], (9) implies (as in metric cases) that the conditions (C2) and (C3) are satisfied, but nothing on the condition (C4). Conversely, in Theorem 6 we have assumed that (C2) and (C3) hold, but we added another condition which is (C4) in order to get a common fixed-point result.

Remark 4. *Theorem 6 is the analogous of Theorem 1 of Rana et al. [51] on partial metrics, except that the conditions (9) and the fact that $a, b \in [0,1]$, are replaced by the weaker conditions (C2), (C3) and $a, b \in \left[0, \frac{1}{2}\right]$. The condition on a and b is modified due to the fact that $p(x,x)$ may not equal to 0 for $x \in X$. Also, Corollary 1 extends Theorem 2.1 of Bouhadjera and Djoudi [52] on partial metric cases. Note that Theorem 2.1 in [52] was improved recently by Akkouchi ([53], Corollary 4.4). Indeed, the Lipschitz constant k is allowed to take values in the interval $\left[0, \frac{1}{2}\right]$ instead of the case studied in [52], where the constant k belongs to the smaller interval $\left[0, \frac{1}{3}\right]$.*

2.3. 2012, Erduran and Imdad, Coupled Fixed-Point Theorems for Generalized Meir–Keeler Contractions in Ordered Partial Metric Spaces [54]

Bhaskar and Lakshmikantham [55] introduced the concept of coupled fixed points and studied some coupled fixed-point theorems.

We achieve this by first considering a function $T : X \times X \to X$ having the mixed monotone property:

Definition 9 ([55])**.** *Let $(X, \leq)$ be a partially ordered set and $T : X \times X \to X$. We say that T has the mixed monotone property if $T(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$,*

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow T(x_1, y) \leq T(x_2, y)$$

and, $\quad y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow T(x, y_1) \geq T(x, y_2).$

This definition coincides with the notion of a mixed monotone function on $\mathbb{R}^2$ and $\leq$ represents the usual total order in $\mathbb{R}$.

Definition 10 ([55])**.** *We call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping T if*

$$T(x, y) = x, \quad T(y, x) = y.$$

We assume that f and T are related by the relation $f(x) = T(x, x)$.

Definition 11 ([55])**.** *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X such that (X, d) is a complete metric space. Further, we endow the product space $X \times X$ with the following partial order:*

$$\text{for } \ (x, y), (u, v) \in X \times X, \ \ (u, v) \leq (x, y) \Leftrightarrow x \geq u, \quad y \leq v.$$

Example 10 ([56])**.** *Let $X = [0, \infty)$ and $T : X \times X \to X$ be defined by*

$$T(x, y) = x + y$$

for all $x, y \in X$. It is easy to see that T has a unique coupled fixed point $(0, 0)$.

Theorem 8 ([55])**.** *Let $T : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(T(x, y), T(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \forall x \geq u, y \leq v.$$

If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq T(x_0, y_0) \quad \text{and} \quad y_0 \geq T(y_0, x_0).$$

Then, there exist $x, y \in X$ such that

$$x = T(x, y) \quad \text{and} \quad y = T(y, x).$$

Denote with Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ. The following lemma is obvious.

Lemma 6. *If $\psi \in \Psi$, then $\psi(t) < t$ for all $t > 0$.*

We recall an important example of coupled fixed point.

Example 11 ([57])**.** *Let $X = \mathbb{R}$ and $d : X \times X \to [0, \infty)$ be the Euclidean metric. Define a mapping $\alpha : X^2 \times X^2 \to [0, \infty)$ defined by*

$$\alpha((x, y), (u, v)) = \begin{cases} 1, & \text{if } x \geq u, y \leq v \\ 0, & \text{otherwise} \end{cases}$$

Given a function $T : X \times X \to X$ as

$$T(x, y) = \frac{3x - y}{5} \quad \text{for all } x, y \in X.$$

Here, T is mixed monotone, but we said that it does not satisfy Theorem 3.4 condition in [58] . We have $\psi \in \Psi$ such that

$$\alpha((x,y),(u,v))d(T(x,y),T(u,v)) \leq \psi\left(\frac{d(x,u)+d(y,v)}{2}\right) \tag{10}$$

equations $\forall\, x \geq u, v \geq y$. Let, we give $x \neq u, y = v$ in the previous inequality. So, $t = |x-u| > 0$ and inequality (10) turns into

$$\frac{3t}{5} = \frac{3|x-u|}{5} = d(T(x,y),T(u,v)) \leq \psi\left(\frac{|x-u|}{2}\right) = \psi\left(\frac{t}{2}\right) \tag{11}$$

Such that $\psi(t) < t$ for any $t > 0$. So, inequality (11) turns into

$$\frac{3t}{5} \leq \psi\left(\frac{t}{2}\right) < \frac{t}{2},$$

which is a contradiction. So, ([58] Theorem 3.4) is not applicable to the operator T in order to prove that $(0,0)$ is the unique coupled fixed point of T.

Now, we present coupled fixed-point results for Meir–Keeler contraction.

Theorem 9 ([59])**.** *Let $(X,\preceq)$ be a partially ordered set and p be a partial metric on X such that (X,p) is a complete partial metric space. Let $T : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that for $x,y,u,v \in X$ with $x \leq u$ and $v \geq y$, we have*

$$p(T(x,y),T(u,v)) \leq \frac{k}{2}[p(x,u)+p(y,v)] \tag{12}$$

where $k \in [0,1)$. If there exists $(x_0,y_0) \in X \times X$ such that $x_0 \leq T(x_0,y_0)$ and $T(x_0,y_0) \leq y_0$, then T has coupled fixed point. Furthermore, $p(x,x) = p(y,y) = 0$.

Definition 12. *Let (X,p) be a partially ordered partial metric space and $T : X \times X \to X$ be a given mapping. We say that T is a generalized Meir–Keeler type function if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that*

$$u \leq x, y \leq v, \varepsilon \leq \frac{1}{2}[p(x,u)+p(y,v)] < \varepsilon + \delta(\varepsilon) \Longrightarrow p(T(x,y),T(u,v)) < \varepsilon$$

Proposition 2. *Let (X,p) be a partially ordered partial metric space and $T : X \times X \to X$ be a mapping. If (12) is satisfied, then T is a generalized Meir–Keeler type function.*

Lemma 7. *Let (X,p) be a partially ordered partial metric space and $T : X \times X \to X$ be a given mapping. If T is a generalized Meir–Keeler type function, then*

$$p(T(x,y),T(u,v)) < \frac{1}{2}[p(x,u)+p(y,v)]$$

for all $x > u, y \leq v$ or for all $x \geq u, y < v$.

Lemma 8. *Let (X,p) be a partially ordered partial metric space and $T : X \times X \to X$ be a given mapping. Assume that the following hypotheses hold:*

(i) T has the mixed strict monotone property,
(ii) T is a generalized Meir–Keeler type function,
(iii) $\exists (x,y) \in X \times X, \exists (u,v) \in X \times X$ such that $x < u$ and $y \geq v$.

Then

$$\eta((T^n(x,y),T^n(y,x)),(T^n(u,v),T^n(v,u))) \to 0, \text{ as } n \to \infty.$$

Remark 5. *Lemma 8 holds if we replace (iii) by:*

$$\exists(x,y) \in X \times X, \exists(u,v) \in X \times X \text{ such that } x \leq u \text{ and } y > v.$$

Theorem 10. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric p on X such that (X, p) is complete partial metric space. Let $T : X \times X \to X$ be mapping satisfying the following hypotheses:*

(i) *T is continuous,*
(ii) *T has the mixed strict monotone property,*
(iii) *T is a generalized Meir–Keeler type function,*
(iv) *$\exists x_0, y_0 \in X$ such that $x_0 < T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$.*

Then, there exists $(x, y) \in X \times X$ such that $x = T(x, y)$ and $y = T(y, x)$.

Theorem 11. *Let $(X, \preceq)$ be a partially ordered set. Suppose that there is a metric p on X such that (X, p) is complete partial metric space. Assume that X has the following properties:*

(a) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*
(b) *if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n.*

Let $T : X \times X \to X$ be mapping satisfying the following hypotheses:

(c) *T is continuous,*
(d) *T has the mixed strict monotone property,*
(e) *T is a generalized Meir–Keeler type function,*
(f) *$\exists x_0, y_0 \in X$ such that $x_0 < T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$.*

Then, there exists $(x, y) \in X \times X$ such that $x = T(x, y)$ and $y = T(y, x)$. Furthermore, $p(x, x) = p(y, y) = 0$.

2.4. 2012, Erhan, Karapınar and Türkoğlu, Different Types Meir–Keeler Contractions on Partial Metric Spaces, [60]

In this section, we prove the existence fixed point for Meir Keeler type contraction of self-mapping T defined in complete partial metric spaces

Firstly, using Hardy-Rogers [8], we give the Definition 13.

Definition 13. *Suppose that $T : X \to X$ is a self-mapping satisfying the following condition: Given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq H(x,y) < \varepsilon + \delta \Rightarrow p(Tx, Ty) < \varepsilon$$

where

$$H(x,y) = \frac{1}{5}[p(x,y) + p(Tx,x) + p(Ty,y) + p(Tx,y) + p(x,Ty)].$$

Then T is called Hardy-Rogers type Meir–Keeler contraction.

Proposition 3. *Let (X, p) be a complete partial metric space. Suppose that $T : X \to X$ is a Hardy-Rogers type Meir–Keeler contraction. Then, for $x \in X$ we have $p\left(T^{n+1}x, T^n x\right) \to 0$, as $n \to \infty$.*

The theorem from below generalizes Meir–Keeler contraction.

Theorem 12. *Let (X, p) be a complete partial metric space. Suppose that $T : X \to X$ is a Hardy-Rogers type Meir–Keeler contraction. Then, T has a unique fixed point, say $x^* \in X$. Moreover, $\lim_{n\to\infty} p(T^n x, x^*) = p(x^*, x^*)$ for all $x \in X$.*

In Definition 14, we define Kannan [5] type Meir Keeler contraction, Reich [6] type Meir Keeler contraction and Chatterjee [7] type Meir Keeler contraction on partial metric space respectively.

Definition 14. *Let (X, p) be a partial metric space. Let $T : X \to X$ be a self-mapping satisfying the following:*

(1) Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq K(x,y) < \varepsilon + \delta \Rightarrow p(Tx,Ty) < \varepsilon$$

where

$$K(x,y) = \frac{1}{2}[p(Tx,x) + p(Ty,y)]$$

Then T is called Kannan type Meir–Keeler contraction.

(2) Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq R(x,y) < \varepsilon + \delta \Rightarrow p(Tx,Ty) < \varepsilon$$

where

$$R(x,y) = \frac{1}{3}[p(x,y) + p(Tx,x) + p(Ty,y)].$$

Then T is called Reich type Meir–Keeler contraction.

(3) Given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq C(x,y) < \varepsilon + \delta \Rightarrow p(Tx,Ty) < \varepsilon,$$

where

$$C(x,y) = \frac{1}{2}[p(x,Ty) + p(y,Tx)]$$

Then T is called Chatterjee type Meir–Keeler contraction.

Theorem 12 is valid also for the notations in Definition 14 .

Corollary 2. *Let (X, p) be a complete partial metric space. Suppose that $T : X \to X$ is Meir–Keeler contraction of Kannan type or Reich type or Chattergee type. Then, T has a unique fixed point, say $x^* \in X$. Moreover, $\lim_{n\to\infty} p(T^n x, x^*) = p(x^*, x^*)$ for all $x \in X$.*

2.5. 2012, Hussain, Kadelburg, Radenović and Al-Solamy, Comparison Functions and Fixed-Point Results in Partial Metric Spaces, [61]

Contractive conditions with comparison function φ of the give form

$$d(Tx,Ty) \leq \varphi(M(x,y))$$

have been used for obtaining (common) fixed-point theorems of mappings in metric spaces until now the celebrated result of Boyd and Wong [18]. Also, interesting and different assumptions for the comparison function φ can be given as follows.

We consider the following properties of functions $\varphi : [0,+\infty) \to [0,+\infty)$. φ^n will denote the nth iteration of φ :

(I) $\varphi(t) < t$ for each $t > 0$ and $\varphi^n(t) \to 0, n \to \infty$ for each $t \geq 0$,
(II) φ is nondecreasing and $\varphi^n(t) \to 0, n \to \infty$ for each $t \geq 0$,
(III) φ is right-continuous, and $\varphi(t) < t$ for each $t > 0$,
(IV) φ is nondecreasing and $\sum_{n\geq 1} \varphi^n(t) < +\infty$ for each $t \geq 0$.

Lemma 9.

(1) (II) $\Rightarrow$ (I).
(2) (III) $+\varphi$ is nondecreasing $\Rightarrow$ (II).
(3) (IV) $\Rightarrow$ (II).
(4) (III) and (IV) are not comparable (even if φ is nondecreasing).

Now, we define Meir–Keeler function via a comparison function in partial metric spaces.

The function $\varphi : [0, +\infty) \to [0, +\infty)$ will be called a Meir–Keeler function if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall\, t \;\; \varepsilon \le t < \varepsilon + \delta \Longrightarrow \varphi(t) < \varepsilon$$

Theorem 13. *Let (X, p) be a complete partial metric space. If $T : X \to X$ satisfies the following condition*

$$\forall x, y \in X \quad p(Tx, Ty) \le \varphi(p(x, y)) \tag{13}$$

for a Meir–Keeler function φ. Then T has a unique fixed point, say x^*, and for each $x \in X$, the Picard sequence $\{T^n x\}$ converges to x^*, satisfying $p(x^*, x^*) = 0$.

Example 12. *Let X, p and d be as in (Example 5.2 in [61]). Consider mapping $T : X \to X$ and function $\varphi : [0, +\infty) \to [0, +\infty)$ given by*

$$T = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi(t) = \frac{3}{4}t$$

Then φ is a Meir–Keeler function. Indeed, for arbitrary $\varepsilon > 0$ choose

$$\delta = (1/3)\varepsilon \text{ and } \varepsilon \le \; t < \varepsilon + \delta = (4/3)\varepsilon \text{ implies that } \varphi(t) < \varepsilon.$$

We will check that T satisfies condition (13) of Theorem 13.

In the cases $x = y = 0; x = y = 1$; and $x = 0, y = 1$, the left-hand side of (13) is equal to zero. In all other cases $(x = y = 2; x = 0, y = 2$; and $x = 1, y = 2)$, it is

$$p(Tx, Ty) = 1 \text{ and } \varphi(p(x, y)) = \varphi(2) = 3/2.$$

Hence, condition (13) always holds true, and mapping T has a unique fixed point $(x^* = 0)$.

Note again that in the case when standard metric d is used instead of partial metric p, this conclusion cannot be obtained. Indeed, for $x = 1, y = 2$ we have that

$$d(T1, T2) = d(0, 1) = 1 > \frac{3}{4} = \varphi(1) = \varphi(d(1, 2)).$$

2.6. 2013, Chen and Chen, Fixed-Point Results for Meir–Keeler-Type $\phi - \alpha$-Contractions on Partial Metric Spaces, [62]

In [62], Chen and Chen introduce generalized Meir–Keeler-type ϕ - α-contractions on partial metric spaces.

In the section, we denote by Φ the class of functions $\phi : \mathbb{R}_0^{+4} \to \mathbb{R}_0^+$ satisfying the below holds:

(ϕ_1) ϕ is an increasing and continuous function in each coordinate;

(ϕ_2) for $t \in \mathbb{R}^+, \phi(t, t, t, t) \le t, \phi(t, 0, 0, t) \le t, \phi(0, 0, t, \frac{t}{2}) \le t$; and $\phi(t_1, t_2, t_3, t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$.

We define generalized Meir–Keeler-type ϕ-contractions and new Meir- Keeler-type $\phi - \alpha$-contractions in partial metric spaces respectively.

Definition 15. *Let (X, p) be a partial metric space, $T : X \to X$ and $\phi \in \Phi$. Then T is called a generalized Meir–Keeler-type ϕ-contraction whenever, for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \le \phi\left(p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2}[p(x, Ty) + p(y, Tx)] \right) < \varepsilon + \delta$$
$$\Longrightarrow p(Tx, Ty) < \varepsilon.$$

Definition 16. *Let (X, p) be a partial metric space, $\alpha : X \times X \to \mathbb{R}_0^+$, $T : X \to X$ and $\phi \in \Phi$. Then T is called a generalized Meir–Keeler-type $\phi - \alpha$-contraction if the following conditions hold:*

(1) *T is α-admissible;*
(2) *for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \le \phi\left(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\right) < \varepsilon + \delta$$
$$\Longrightarrow \alpha(x,x)\alpha(y,y)p(Tx,Ty) < \varepsilon \tag{14}$$

Remark 6. *Note that if T is a generalized Meir–Keeler-type $\phi - \alpha$-contraction, then we have that for all $x, y \in X$,*

$$\alpha(x,x)\alpha(y,y)p(Tx,Ty)$$
$$\le \phi\left(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\right).$$

Also, if $\phi\left(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\right) = 0$, then $p(Tx,Ty) = 0$. So, if
$\phi\left(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\right) > 0$,
then
$\alpha(x,x)\alpha(y,y)p(Tx,Ty) < \phi\left(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\right)$.

Theorem 14. *Let (X, p) be a complete partial metric space, and $\phi \in \Phi$. If $\alpha : X \times X \to \mathbb{R}_0^+$ satisfies the following conditions:*

(α_1) *there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \ge 1$;*
(α_2) *if $\alpha(x_n, x_n) \ge 1$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \alpha(x_n, x_n) \ge 1$;*
(α_3) *$\alpha : X \times X \to \mathbb{R}_0^+$ is a continuous function in each coordinate.*

Suppose that $T : X \to X$ is a generalized Meir–Keeler-type $\phi - \alpha$-contraction. Then T has a fixed point in X.

Theorem 15. *Let (X, p) be a complete partial metric space and $\phi \in \Phi$. If $T : X \to X$ is a generalized Meir–Keeler-type ϕ-contraction, then T has a fixed point in X.*

Here, we present Example 13 to support Theorem 15.

Example 13. *Let $X = [0, 1]$. We define the partial metric p on X by*

$$p(x,y) = \max\{x,y\}.$$

Let $\alpha : [0,1] \times [0,1] \to \mathbb{R}_0^+$ be defined as

$$\alpha(x,y) = 1 + x + y,$$

let $T : X \to X$ be defined as

$$T(x) = \frac{1}{16}x^2,$$

and, let $\phi : \mathbb{R}_0^{+4} \to \mathbb{R}_0^+$ denote

$$\psi(t_1, t_2, t_3, t_4) = \frac{1}{2} \cdot \max\left\{t_1, t_2, t_3, \frac{1}{2}t_4\right\}.$$

Then T is α-admissible.

Without loss of generality, we assume that $x > y$ and verify the inequality (14). For all $x, y \in [0,1]$ with $x > y$, we have

$$\begin{aligned}
&\alpha(x,x)\alpha(y,y)p(Tx,Ty) \geq \frac{1}{16}x^2, \\
&p(x,y) = x, \quad p(x,Tx) = x, \quad p(y,Ty) = y \quad \text{and} \\
&\frac{1}{2}[p(x,Ty) + p(y,Tx)] = \frac{1}{2}\left[\max\left\{x, y^2\right\} + \max\left\{y, x^2\right\}\right] \\
&\qquad \leq \frac{1}{2}[\max\{x,y\} + \max\{y,x\}] \\
&\qquad < x
\end{aligned}$$

and hence

$$\phi\left(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\right) = \frac{1}{2}x.$$

Therefore, all the conditions of Theorem 14 are satisfied, and we obtained that 0 is a fixed point of T. If we let

$$\alpha(x,y) = 1 \quad \text{for } x, y \in X$$

then it is easy to get the following theorem.

2.7. 2014, Imdad, Sharma and Erduran, Generalized Meir–Keeler Type n-Tupled Fixed-Point Theorems in Ordered Partial Metric Spaces, [63]

In this section, we investigate n-tupled (for even n) fixed-point theorems on ordered partial metric spaces. We use the Meir–Keeler type contraction besides the mixed monotone property to obtain these fixed-point results.

We think n to be an even integer.

Let (X, p) be a partial metric. We endow $X \times X \times \cdots \times X, n$ times $(= X^n)$ with the partial metric η defined for $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ by

$\eta((x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n)) = \max\{p(x^1, y^1), p(x^2, y^2), \ldots, p(x^n, y^n)\}$.

Let $T : X^n \to X$ be a given mapping. Then for all $(x^1, x^2, \ldots, x^n) \in X^n$ and for all $m \in \mathbb{N}, m \geq 2$, we denote

$$\begin{aligned}
&T^m\left(x^1, x^2, \ldots, x^n\right) \\
&\quad = T\left(T^{m-1}\left(x^1, x^2, \ldots, x^n\right), T^{m-1}\left(x^2, \ldots, x^n, x^1\right), \ldots, T^{m-1}\left(x^n, x^1, \ldots, x^{n-1}\right)\right).
\end{aligned}$$

We give main results, as follows:

Definition 17. *Let $(X, \preceq)$ be a partially ordered set and $T : X^n \to X$ be a mapping. The mapping T is said to have the mixed strict monotone property if T is nondecreasing in its odd position arguments and nonincreasing in its even position arguments, that is, if,*

(i) $\forall x_1^1, x_2^1 \in X, x_1^1 \prec x_2^1 \Rightarrow T(x_1^1, x^2, x^3, \ldots, x^n) \prec T(x_2^1, x^2, x^3, \ldots, x^n)$,
(ii) $\forall x_1^2, x_2^2 \in X, x_1^2 \prec x_2^2 \Rightarrow T(x^1, x_2^2, x^3, \ldots, x^n) \prec T(x^1, x_1^2, x^3, \ldots, x^n)$,
(iii) $\forall x_1^3, x_2^3 \in X, x_1^3 \prec x_2^3 \Rightarrow T(x^1, x^2, x_1^3, \ldots, x^n) \prec T(x^1, x^2, x_2^3, \ldots, x^n), \vdots$
$\forall x_1^n, x_2^n \in X, x_1^n \prec x_2^n \Rightarrow \quad T(x^1, x^2, x^3, \ldots, x_2^n) \prec T(x^1, x^2, x^3, \ldots, x_1^n)$.

Definition 18. *Let (X, p) be a partially ordered partial metric space and $T : X^n \to X$ be a given mapping. We say that T is a generalized Meir–Keeler type function if for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ with $x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \ldots, y^n \preceq x^n$*

$$\left\{\begin{array}{c}
\varepsilon \leq \max\{p(x^1, y^1), p(x^2, y^2), p(x^3, y^3), \ldots, p(x^n, y^n)\} < \varepsilon + \delta(\varepsilon) \\
\Rightarrow p\left(T(x^1, x^2, x^3, \ldots, x^n), T(y^1, y^2, y^3, \ldots, y^n)\right) < \varepsilon
\end{array}\right.$$

Lemma 10. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $T : X^n \to X$ be a given mapping. If T is a generalized Meir–Keeler type function, then for $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$*

$$p\Big(T\Big(x^1, x^2, \ldots, x^n\Big), T\Big(y^1, y^2, \ldots, y^n\Big)\Big) < \max\Big\{p\Big(x^1, y^1\Big), p\Big(x^2, y^2\Big), \ldots, p(x^n, y^n)\Big\}$$

with $x^1 \prec y^1, y^2 \preceq x^2, x^3 \prec y^3, \ldots, y^n \preceq x^n$ or $x^1 \preceq y^1, y^2 \prec x^2, x^3 \preceq y^3, \ldots, y^n \prec x^n$.

Lemma 11. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $T : X^n \to X$ be a given mapping. Assume that the following hypotheses hold:*

(1) T has the mixed strict monotone property,
(2) T is a generalized Meir–Keeler type function,
(3) there exist $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ with $x^1 \prec y^1, y^2 \preceq x^2, x^3 \prec y^3, \ldots, y^n \preceq x^n$.

Then

$$\begin{aligned}&\eta(\Big(T^m\Big(x^1, x^2, x^3, \ldots, x^n\Big), T^m\Big(x^2, x^3, \ldots, x^n, x^1\Big), \ldots, T^m\Big(x^n, x^1, x^2, \ldots, x^{n-1}\Big)\Big)\\ &\quad\Big(T^m\Big(y^1, y^2, y^3, \ldots, y^n\Big), T^m\Big(y^2, y^3, \ldots, y^n, y^1\Big), \ldots, T^m\Big(y^n, y^1, y^2, \ldots, y^{n-1}\Big)\Big)\Big)\\ &\quad\to 0 \quad \text{as } m \to \infty\end{aligned}$$

Theorem 16. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $T : X^n \to X$ be a given mapping satisfying the following hypotheses:*

(1) T is continuous,
(2) T has the mixed strict monotone property,
(3) T is a generalized Meir–Keeler type function,
(4) there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ such that

$$\begin{cases} x_0^1 \prec T(x_0^1, x_0^2, x_0^3, \ldots, x_0^n) \\ T(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \preceq x_0^2 \\ x_0^3 \prec T(x_0^3, \ldots, x_0^n, x_0^1, x_0^2) \\ \vdots \\ T\Big(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}\Big) \preceq x_0^n \end{cases}$$

Then there exist $(x^1, x^2, x^3, \ldots, x^n) \in X^n$ such that $x^1 = T(x^1, x^2, x^3, \ldots, x^n)$, $x^2 = T(x^2, x^3, \ldots, x^n, x^1), \ldots, x^n = T(x^n, x^1, x^2, \ldots, x^{n-1})$.

Theorem 17. *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $T : X^n \to X$ be a given mapping. Assume that there exists a function θ from $[0, \infty)$ into itself satisfying the following:*

(1) $\theta(0) = 0$ and $\theta(t) > 0$ for every $t > 0$,
(2) θ is nondecreasing and right continuous,
(3) for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned}&\varepsilon < \theta\Big(\max\Big\{p\Big(x^1, y^1\Big), p\Big(x^2, y^2\Big), \ldots, p(x^n, y^n)\Big\}\Big) < \varepsilon + \delta(\varepsilon)\\ &\quad\Rightarrow \quad \theta\Big(p\Big(T\Big(x^1, x^2, \ldots, x^n\Big), T\Big(y^1, y^2, \ldots, y^n\Big)\Big)\Big) < \varepsilon\end{aligned}$$

for all $y^1 \prec x^1, x^2 \preceq y^2, y^3 \prec x^3, \ldots, x^n \preceq y^n$. Then T is a generalized Meir–Keeler type function.

Example 14. *Let $X = [0,1]$. Then $(X, \preceq)$ is a partially ordered set under the natural ordering of real numbers. Define $p : [0,1] \times [0,1] \to \mathbb{R}_0^+$ by $p(x,y) = \max\{x,y\}, x,y \in [0,1]$. Then (X,p) is a complete partial metric space.*

Now for any fixed even integer $n > 1$, consider the product space $X^n = [0,1] \times [0,1] \times \cdots \times [0,1]$, n times (in short we write $X^n = [0,1]^n$). Define $T : X^n \to X$ by

$T(x^1, x^2, x^3, \dots, x^n) = \frac{x^1}{n}$ *for $x^1, x^2, \dots, x^n \in [0,1]$.*

Then T has the mixed strict monotone property. Also T is a generalized Meir–Keeler type function. The proof follows in two parts, that is, we prove the following:

For $(x^1, x^2, \dots, x^n), (y^1, y^2, \dots, y^n) \in X^n$ with $x^1 \prec y^1, y^2 \preceq x^2, x^3 \prec y^3, \dots, y^n \preceq x^n$,

$$\begin{aligned}(1)p\Big(T\Big(x^1, x^2, \dots, x^n\Big), T\Big(y^1, y^2, \dots, y^n\Big)\Big)\\ < \max\Big\{p\Big(x^1, y^1\Big), p\Big(x^2, y^2\Big), \dots, p(x^n, y^n)\Big\},\end{aligned}$$

$$\begin{aligned}(2)\eta((T^m\Big(x^1, x^2, x^3, \dots, x^n\Big),\\ T^m\Big(x^2, x^3, \dots, x^n, x^1\Big), \dots, T^m\Big(x^n, x^1, x^2, \dots, x^{n-1}\Big))),\end{aligned}$$

$$\begin{aligned}&\Big(T^m\Big(y^1, y^2, y^3, \dots, y^n\Big), T^m\Big(y^2, y^3, \dots, y^n, y^1\Big), \dots, T^m\Big(y^n, y^1, y^2, \dots, y^{n-1}\Big)\Big)\Big)\\ &\to 0 \quad \text{as } m \to \infty\end{aligned}$$

The first part is trivial. For second part, we have

$$\begin{aligned}
&\eta\Big(\Big(T^m\Big(x^1, x^2, x^3, \dots, x^n\Big), T^m\Big(x^2, x^3, \dots, x^n, x^1\Big), \dots, T^m\Big(x^n, x^1, x^2, \dots, x^{n-1}\Big)\Big),\\
&\Big(T^m\Big(y^1, y^2, y^3, \dots, y^n\Big), T^m\Big(y^2, y^3, \dots, y^n, y^1\Big), \dots, T^m\Big(y^n, y^1, y^2, \dots, y^{n-1}\Big)\Big)\Big)\\
=&\eta\Big(\Big(T\Big(T^{m-1}\Big(x^1, x^2, \dots, x^n\Big), T^{m-1}\Big(x^2, \dots, x^n, x^1\Big), \dots, T^{m-1}\Big(x^n, x^1, \dots, x^{n-1}\Big)\Big),\\
&T\Big(T^{m-1}\Big(x^2, \dots, x^n, x^1\Big), \dots, T^{m-1}\Big(x^n, x^1, \dots, x^{n-1}\Big), T^{m-1}\Big(x^1, x^2, \dots, x^n\Big)\Big), \dots\\
&T\Big(T^{m-1}\Big(x^n, x^1, \dots, x^{n-1}\Big), T^{m-1}\Big(x^1, x^2, \dots, x^n\Big), \dots, T^{m-1}\Big(x^{n-1}, \dots, x^1, x^n\Big)\Big)\Big)\\
&\Big(T\Big(T^{m-1}\Big(y^1, y^2, \dots, y^n\Big), T^{m-1}\Big(y^2, \dots, y^n, y^1\Big), \dots, T^{m-1}\Big(y^n, y^1, \dots, y^{n-1}\Big)\Big),\\
&T\Big(T^{m-1}\Big(y^2, \dots, y^n, y^1\Big), \dots, T^{m-1}\Big(y^n, y^1, \dots, y^{n-1}\Big), T^{m-1}\Big(y^1, y^2, \dots, y^n\Big)\Big), \dots\\
&T\Big(T^{m-1}\Big(y^n, y^1, \dots, y^{n-1}\Big), T^{m-1}\Big(y^1, y^2, \dots, y^n\Big), \dots, T^{m-1}\Big(y^{n-1}, \dots, y^1, y^n\Big)\Big)\Big)\Big)\\
&\vdots\\
=&\eta\Bigg(\Bigg(T\bigg(\frac{x^1}{n^{m-1}}, \frac{x^2}{n^{m-1}}, \frac{x^3}{n^{m-1}}, \dots, \frac{x^n}{n^{m-1}}\bigg), T\bigg(\frac{x^2}{n^{m-1}}, \frac{x^3}{n^{m-1}}, \dots, \frac{x^n}{n^{m-1}}, \frac{x^1}{n^{m-1}}\bigg), \dots,\\
&T\bigg(\frac{x^n}{n^{m-1}}, \frac{x^1}{n^{m-1}}, \frac{x^2}{n^{m-1}}, \dots, \frac{x^{n-1}}{n^{m-1}}\bigg)\Bigg), \Bigg(T\bigg(\frac{y^1}{n^{m-1}}, \frac{y^2}{n^{m-1}}, \frac{y^3}{n^{m-1}}, \dots, \frac{y^n}{n^{m-1}}\bigg)\\
&T\bigg(\frac{y^2}{n^{m-1}}, \frac{y^3}{n^{m-1}}, \dots, \frac{y^n}{n^{m-1}}, \frac{y^1}{n^{m-1}}\bigg), \dots, T\bigg(\frac{y^n}{n^{m-1}}, \frac{y^1}{n^{m-1}}, \frac{y^2}{n^{m-1}}, \dots, \frac{y^{n-1}}{n^{m-1}}\bigg)\Bigg)\Bigg)\\
=&\eta\bigg(\bigg(\frac{x^1}{n^m}, \frac{x^2}{n^m}, \frac{x^3}{n^m}, \dots, \frac{x^n}{n^m}\bigg), \bigg(\frac{y^1}{n^m}, \frac{y^2}{n^m}, \frac{y^3}{n^m}, \dots, \frac{y^n}{n^m}\bigg)\bigg)\\
=&\max\bigg\{p\bigg(\frac{x^1}{n^m}, \frac{y^1}{n^m}\bigg), p\bigg(\frac{x^2}{n^m}, \frac{y^2}{n^m}\bigg), p\bigg(\frac{x^3}{n^m}, \frac{y^3}{n^m}\bigg), \dots, p\bigg(\frac{x^n}{n^m}, \frac{y^n}{n^m}\bigg)\bigg\}\\
=&\max\bigg\{\frac{y^1}{n^m}, \frac{x^2}{n^m}, \frac{y^3}{n^m}, \dots, \frac{x^n}{n^m}\bigg\} \to 0 \quad \text{as } m \to \infty
\end{aligned}$$

Hence all the hypotheses of Theorem 16 are satisfied. Therefore, T has a unique n-tupled fixed point. Here $(0, 0, \ldots, 0)$ *is an n-tupled fixed point of T.*

2.8. 2014, Nashine and Kadelburg, Fixed-Point Theorems Using Cyclic Weaker Meir–Keeler Functions in Partial Metric Spaces, [64]

In this section, we give fixed-point results via cyclic weaker Meir Keeler functions in 0-complete partial metric spaces.

First, we recall the definition and fixed-point theorem of the cyclic, as follows;

Definition 19 (Kirk et al. [65]). *Let* X *be a nonempty set,* $m \in \mathbb{N}$ *and let* $T : X \to X$ *be a self-mapping. Then* $X = \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* X *with respect to* T *if*

(a) $A_i, i = 1, \ldots,$ *mare non-empty subsets of* X*;*
(b) $T(A_1) \subset A_2, f(A_2) \subset A_3, \ldots, f(A_{m-1}) \subset A_m, f(A_m) \subset A_1$.

Theorem 18 ([65]). *Let* (X, d) *be a complete metric space,* $T : X \to X$ *and let* $X = \bigcup_{i=1}^{m} A_i$ *be a cyclic representation of* X *with respect to* T*. Suppose that* T *satisfies the following condition*

$$d(Tx, Ty) \leq \psi(d(x, y)), \text{ for all } x \in A_i, y \in A_{i+1}, i \in \{1, 2, \ldots, m\},$$

where $A_{m+1} = A_1$ *and* $\psi : [0, 1) \to [0, 1)$ *is a function, upper semi-continuous from the right and* $0 \leq \psi(t) < t$ *for* $t > 0$*. Then,* T *has a fixed point* $x^* \in \bigcap_{i=1}^{m} A_i$*.*

Definition 20 (Păcurar and Rus [66]). *Let* (X, d) *be a metric space,* $m \in \mathbb{N}, A_1, A_2, \ldots, A_m$ *be closed nonempty subsets of* X *and* $X = \bigcup_{i=1}^{m} A_i$*. An operator* $T : X \to X$ *is called a cyclic weaker* φ*-contraction if*

(1) $X = \bigcup_{i=1}^{m} A_i$ *is a cyclic representation of* X *with respect to* T*;*
(2) there exists a continuous, non-decreasing function $\varphi : [0, 1) \to [0, 1)$ *with* $\varphi(t) > 0$ *for* $t \in (0, 1)$ *and* $\varphi(0) = 0$ *such that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m,$ *where* $A_{m+1} = A_1$*.*

Theorem 19 ([66]). *Suppose that* T *is a cyclic weaker* φ*-contraction on a complete metric space* (X, d)*. Then,* T *has a fixed point* $x^* \in \bigcap_{i=1}^{m} A_i$*.*

Here, we introduce fixed-point results via cyclic weaker $(\psi \circ \varphi)$ Meir Keeler conditions in 0-complete partial metric spaces.

Definition 21. *As in [67], we assume in this section the following conditions for a weaker Meir–Keeler function* $\psi : [0, +\infty) \to [0, +\infty)$

(ψ_1) $\psi(t) > 0$ *for* $t > 0$ *and* $\psi(0) = 0$*;*
(ψ_2) *for all* $t \in [0, \infty), \{\psi^n(t)\}_{n \in \mathbb{N}}$ *is decreasing;*
(ψ_3) *for* $t_n \in [0, \infty)$*, we have that*

(a) if $\lim_{n \to \infty} t_n = \gamma > 0$*, then* $\lim_{n \to \infty} \psi(t_n) < \gamma$*, and*
(b) if $\lim_{n \to \infty} t_n = 0$*, then* $\lim_{n \to \infty} \psi(t_n) = 0$*.*

Also suppose that $\varphi : [0, +\infty) \to [0, +\infty)$ *is a non-decreasing and continuous function satisfying:*

(φ_1) $\varphi(t) > 0$ *for* $t > 0$ *and* $\varphi(0) = 0$*;*
(φ_2) φ *is subadditive, that is, for every* $\mu_1, \mu_2 \in [0, +\infty), \varphi(\mu_1 + \mu_2) \leq \varphi(\mu_1) + \varphi(\mu_2)$*;*
(φ_3) *for all* $t \in (0, \infty), \lim_{n \to \infty} t_n = 0$ *if and only if* $\lim_{n \to \infty} \varphi(t_n) = 0$*.*

The version of Chen [67]'s definition (in metric space) of cyclic weaker $(\psi \circ \varphi)$-contraction in partial metric spaces is as follows.

Definition 22. *Let (X, p) be a partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty subsets of X and $X = \bigcup_{i=1}^{m} A_i$. An operator $T : X \to X$ is called a cyclic weaker $(\psi \circ \varphi)$-contraction if*

(1) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T;
(2) for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m$

$$\varphi(p(Tx, Ty)) \le \psi(\varphi(p(x, y))), \tag{15}$$

where $A_{m+1} = A_1$.

Theorem 20. *Let (X, p) be a 0-complete partial metric space, $m \in \mathbb{N}$, $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of (X, p) and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T : Y \to Y$ is a cyclic weaker $(\psi \circ \varphi)$-contraction. Then, T has a unique fixed point $x^* \in Y$. Moreover, $x^* \in \bigcap_{i=1}^{m} A_i$.*

Now, we give an example that supports Theorem 20.

Example 15. *Let $X = [0, 1]$ and a partial metric $p : X \times X \to \mathbb{R}_0^+$ be given by*

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1) \\ 1, & \text{if } x = 1 \text{ or } y = 1. \end{cases}$$

If a mapping $T : X \to X$ is given by

$$Tx = \begin{cases} 1/2, & \text{if } x \in [0, 1) \\ 0, & \text{if } x = 1 \end{cases}$$

and $A_1 = \left[0, \frac{1}{2}\right], A_2 = \left[\frac{1}{2}, 1\right]$, then $A_1 \cup A_2 = X$ is a cyclic representation of X with respect to T. Moreover, mapping T is a cyclic weaker $(\psi \circ \varphi)$-contraction, where $\varphi(t) = t$ and $\psi(t) = \frac{3}{4}t$. Indeed, consider the following cases:

Case 1 $x \in \left[0, \frac{1}{2}\right], y \in \left[\frac{1}{2}, 1\right)$ or $y \in \left[0, \frac{1}{2}\right], x \in \left[\frac{1}{2}, 1\right)$. Then $p(Tx, Ty) = p\left(\frac{1}{2}, \frac{1}{2}\right) = 0$ and relation (15) is trivially satisfied.

Case 2 $x \in \left[0, \frac{1}{2}\right], y = 1$ or $y \in \left[0, \frac{1}{2}\right], x = 1$. Then $p(Tx, Ty) = p\left(\frac{1}{2}, 0\right) = \frac{1}{2}$ and $p(x, y) = 1$. Relation (15) holds as it reduces to $\frac{1}{2} < \frac{3}{4}$.

We conclude that T has a unique fixed point (which is $x^* = \frac{1}{2}$).

Note that, if instead of the given partial metric p its associated metric

$$p^s(x, y) = \begin{cases} 2|x - y|, & \text{if } x, y \in [0, 1), \\ 1, & \text{if } x \in [0, 1), y = 1 \text{ or } x = 1, y \in [0, 1), \\ 0, & \text{if } x = y = 1, \end{cases}$$

is used, then for $x = \frac{1}{2}$ and $y = 1$ the respective condition (i.e., condition (ii) of Definition 4 from [67]) is not satisfied since it reduces to

$$\varphi\left(p^s\left(\frac{1}{2}, 0\right)\right) = 1 < \frac{3}{4} = \psi\left(\varphi\left(p^s\left(\frac{1}{2}, 1\right)\right)\right).$$

Similar conclusion is obtained if the standard Euclidean metric is used.

Hence, this example shows that Theorem 20 is a proper extension of ([67], Theorem 3).

Later, we introduce fixed-point results via cyclic weaker (ψ, φ) Meir Keeler conditions. The results in this part generalized from [66–68].

We assume $\psi : [0, +\infty) \to [0, +\infty)$ to be a weaker Meir–Keeler function satisfying conditions $(\psi_1), (\psi_2)$ and (ψ_3). Also consider $\varphi : [0, +\infty) \to [0, +\infty)$ to be a non-decreasing and continuous function satisfying $(\varphi_1)((\varphi_2)$ and (φ_3) are not needed). To complete the results, we need the following notion of a cyclic weaker (ψ, φ)-contraction, which is the counterpart of the respective notion from [67]:

Definition 23. *Let (X, p) be a partial metric space, $m \in \mathbb{N}$, and let $A_1, A_2, \ldots, A_m$ be nonempty subsets of X such that $X = \bigcup_{i=1}^{m} A_i$. An operator $T : X \to X$ is called a cyclic weaker (ψ, φ)-contraction if*

(1) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T;
(2) for any $x \in A_i, y \in A_{i+1}, i = 1, 2, \ldots, m$

$$p(Tx, Ty) \leq \psi(p(x, y)) - \varphi(p(x, y)), \tag{16}$$

where $A_{m+1} = A_1$.

Theorem 21. *Let (X, p) be a 0-complete partial metric space, $m \in \mathbb{N}$, let $A_1, A_2, \ldots, A_m$ be nonempty closed subsets of (X, p) and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that $T : Y \to Y$ is a cyclic weaker (ψ, φ)-contraction. Then, T has a unique fixed point $x^* \in Y$. Moreover, $x^* \in \bigcap_{i=1}^{m} A_i$.*

We give an example that supports Theorem 21.

Example 16. *Let $X = [0, 2] = [0, 1] \cup [1, 2] = A_1 \cup A_2$ be equipped by the usual partial metric $p(x, y) = \max\{x, y\}$. Let $T : X \to X$ be given by $Tx = 2 - x$. Then $A_1 \cup A_2 = X$ is a cyclic representation of X with respect to T. If $\psi(t) = 4t$ and $\varphi(t) = 2t$, we will prove that T is a cyclic weaker (ψ, φ)-contraction. Indeed let, e.g., $x \in [0, 1]$ and $y \in [1, 2]$, Then*

$$p(Tx, Ty) = \max\{2 - x, 2 - y\} = 2 - x \leq 2 \leq 2y = 4y - 2y = \psi(p(x, y)) - \varphi(p(x, y)),$$

and condition (16) is fulfilled. All other conditions of Theorem 21 are also satisfied and T has a unique fixed point $(x^* = 1)$.

We also provide another example that supports the main results. The results reveal that it is stronger than from [67].

Example 17. *Let $X = [0, 1]$ and a partial metric $p : X \times X \to \mathbb{R}_0^+$ be given by*

$$p(x, y) = \begin{cases} |x - y|, & \text{if } x, y \in [0, 1) \\ 1, & \text{if } x = 1 \text{ or } y = 1 \end{cases}$$

If a mapping $T : X \to X$ is given by

$$Tx = \begin{cases} 1/8, & \text{if } x \in [0, 1) \\ 0, & \text{if } x = 1 \end{cases}$$

and $A_1 = \left[0, \frac{1}{8}\right], A_2 = \left[\frac{1}{8}, 1\right]$, then $A_1 \cup A_2 = X$ is a cyclic representation of X with respect to T. Moreover, mapping T is a cyclic weaker (ψ, φ)-contraction, where

$$\psi(t) = \frac{t^2}{1 + t} \quad \text{and} \quad \varphi(t) = \frac{t^2}{2 + t}.$$

Indeed, consider the following cases:

Case 1 $x \in \left[0, \frac{1}{8}\right], y \in \left[\frac{1}{8}, 1\right)$ or $y \in \left[0, \frac{1}{8}\right], x \in \left[\frac{1}{8}, 1\right)$. Then $p(Tx, Ty) = p\left(\frac{1}{8}, \frac{1}{8}\right) = 0$ and relation (16) is trivially satisfied.

Case 2 $x \in \left[0, \frac{1}{8}\right], y = 1$ or $y \in \left[0, \frac{1}{8}\right], x = 1$. Then $p(Tx, Ty) = p\left(\frac{1}{8}, 0\right) = \frac{1}{8}$ and $p(x, y) = 1$. Relation (16) holds as it reduces to $\frac{1}{8} < \frac{1}{6} = \frac{1}{2} - \frac{1}{3}$.

We conclude that T has a unique fixed point (which is $x^* = \frac{1}{8}$).

Note again that, if instead of the given partial metric p its associated metric p^s is used, then for $x = \frac{1}{2}$ and $y = 1$ the respective condition (i.e., condition (ii) of Definition 5 from [67]) is not satisfied since it reduces to

$$\varphi\left(p^s\left(\frac{1}{8},0\right)\right) = \frac{1}{4} < \frac{1}{6} = \frac{1}{2} - \frac{1}{3} = \psi\left(p^s\left(\frac{1}{2},1\right)\right) - \varphi\left(p^s\left(\frac{1}{2},1\right)\right).$$

Similar conclusion is obtained if the standard Euclidean metric is used.

So, this example shows that Theorem 21 is a proper extension of ([67], Theorem 4).

2.9. 2015, Choudhury and Bandyopadhyay, Suzuki Type Common Fixed-Point Theorem in Complete Metric Space and Partial Metric Space, [69]

In this section, we show that a pair of compatible mappings have unique common fixed point in partial metric spaces respectively. First we need the following Suzuki theorem.

Theorem 22 ([70])**.** *Define a function θ from $[0,1)$ onto $\left(\frac{1}{2},1\right]$ by*

$$\theta(r) = \begin{cases} 1, & \text{if} \quad 0 \le r \le \frac{(\sqrt{5}-1)}{2}; \\ \frac{1-r}{r^2}, & \text{if} \quad \frac{(\sqrt{5}-1)}{2} \le r \le 2^{-\frac{1}{2}}; \\ \frac{1}{1+r}, & \text{if} \quad \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Let (X,d) be a complete metric space. T is a mapping on X. If T satisfy the following

$$\theta(r)d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le rd(x,y) \text{ for all } x,y \in X,$$

then T has a fixed point.

Definition 24 ([71])**.** *Let S and T be mappings from a metric space (X,d) into itself. Then S and T are said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some z in X. Thus, if $d(STx_n, TSx_n) \to 0$ as $d(Sx_n, Tx_n) \to 0$, then S and T are compatible.*

Definition 25 ([24])**.** *Let (X,p) be a partial metric space and $T,S : X \to X$ are mappings of X into itself. We say that the pair $\{T,S\}$ is partial compatible if the following conditions hold:*

(b_1) $p(x,x) = 0 \Rightarrow p(Sx,Sx) = 0;$
(b_2) $p\lim_{n\to\infty} p(TSx_n, STx_n) = 0$, *whenever $\{x_n\}$ is a sequence in X such that $Tx_n \to t$ and $Sx_n \to t$ for some $t \in X$.*

Theorem 23. *Let (X,d) be a complete metric space. Let S be a continuous mappings on X and T be another mapping on X such that $\{T,S\}$ is compatible and $T(X) \subset S(X)$. Also let for all $x,y \in X$ and for any $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ such that*

$$\text{(i) } \frac{1}{2}d(Sx,Tx) < d(Sx,Sy) \Rightarrow d(Tx,Ty) < \max\left\{d(Sx,Sy), \frac{1}{2}(d(Sx,Tx) + d(Sy,Ty))\right\};$$

$$\text{(ii) } \frac{1}{2}d(Sx,Tx) < d(Sx,Sy)$$

and

$$\max\left\{d(Sx,Sy), \frac{1}{2}(d(Sx,Tx) + d(Sy,Ty))\right\} < \varepsilon + \delta(\varepsilon) \Rightarrow d(Tx,Ty) \le \varepsilon.$$

Then there exists a unique common fixed point of S and T.

Lemma 12. *Let (X, p) be a partial metric space, T a self map on X, d the constructed metric in [10] and $x, y \in X$. Then*

$$\max\left\{d(Sx, Sy), \frac{1}{2}(d(Sx, Tx) + d(Sy, Ty))\right\}$$
$$= \max\left\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\right\}$$

for all $x, y \in X$ with $x \neq y$.

Theorem 24. *Let (X, p) be a complete partial metric space. Let S be a continuous mappings on X, and T be another mapping on X such that $\{T, S\}$ is partial compatible and $T(X) \subset S(X)$. Also let for all $x, y \in X$ and for any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that*

$$\text{(i) } \frac{1}{2}p(Sx, Tx) < p(Sx, Sy) \Rightarrow p(Tx, Ty)$$
$$< \max\left\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\right\};$$
$$\text{(ii) } \frac{1}{2}p(Sx, Tx) < p(Sx, Sy)$$

and

$$\max\left\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\right\} < \varepsilon + \delta(\varepsilon) \Rightarrow p(Tx, Ty) \leq \varepsilon.$$

Then there exist a unique common fixed point of S and T.

Also, we present an example.

Example 18. *Let*

$$X = [0, 2] \quad p(x, y) = \max\{x, y\} \text{ for all } x \in X.$$

Therefore (X, p) be a complete partial metric space. Define two functions S, T as follows:

$$Sx = \begin{cases} 2x, & \text{if} \quad x \in [0, 1] \\ 3 - x, & \text{if} \quad x \in [1, 2] \end{cases}$$

$$Tx = \begin{cases} 0, & \text{if} \quad 0 \leq x < 1 \\ \frac{1}{x}, & \text{if} \quad 1 \leq x \leq 2 \end{cases}$$

It is clear that $T0 = 0 = S0$, otherwise $Sx \neq Tx$ for all $x \in X$.
Case I: Let $0 \leq x < 1$. Then

$$\frac{1}{2}p(Sx, Tx) = \frac{1}{2}p(2x, 0) = x < \max\{2x, 2y\} = p(Sx, Sy).$$

Therefore,

$$\max\left\{p(Sx, Sy), \frac{1}{2}(p(Sx, Tx) + p(Sy, Ty))\right\}$$
$$= \max\left\{p(2x, 2y), \frac{1}{2}(p(2x, 0) + p(2y, 0))\right\}$$
$$= \max\left\{\max\{2x, 2y\}, \frac{1}{2}(2x + 2y)\right\}$$
$$= \max\{2x, 2y\}.$$

For $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ such that

$$\max\left\{p(Sx,Sy),\frac{1}{2}(p(Sx,Tx)+p(Sy,Ty))\right\}$$
$$< \varepsilon+\delta(\varepsilon) \text{ implies } p(Tx,Ty)=0<\varepsilon.$$

Hence the result is true for $0 \le x < 1$.
Case II: Let $1 \le x \le 2$. $\frac{1}{2}p(Sx,Tx)=\frac{1}{2}(3-x)<\max\{3-x,3-y\}=p(Sx,Sy)$. *Now,*

$$\max\left\{p(Sx,Sy),\frac{1}{2}(p(Sx,Tx)+p(Sy,Ty))\right\} \tag{17}$$
$$=\max\left\{\max\{3-x,3-y\},\frac{1}{2}(3-x+3-y)\right\}$$
$$=\max\{3-x,3-y\} \tag{18}$$

We also have

$$p(Tx,Ty)=\max\left\{\frac{1}{x},\frac{1}{y}\right\}. \tag{19}$$

Now for the given $\varepsilon>0$, *there exist* $\delta(\varepsilon)>0$ *such that*

$$\max\left\{p(Sx,Sy),\frac{1}{2}(p(Sx,Tx)+p(Sy,Ty))\right\}<\varepsilon+\delta(\varepsilon).$$

Using (18), (19) and the above inequality implies $p(Tx,Ty)<\varepsilon$.
Hence the result satisfied for $1 \le x \le 2$. *0 is the unique common fixed point of S and T.*

2.10. 2016, Choudhury and Bandyopadhyay, Coupled Meir–Keeler Type Contraction in Metric Spaces with an Application to Partial Metric Spaces, [72]

In this section, we establish a Meir–Keeler type coupled fixed-point results in partial metric spaces.

Definition 26 ([73])**.** *Let* (X,d) *be a metric space and* $T: X\times X\to X$ *be a given mapping. T is a generalized Meir–Keeler type function if for all* $\varepsilon>0$, *there exists* $\delta(\varepsilon)>0$ *such that*

$$\varepsilon \le \frac{1}{2}(d(x,u)+d(y,v))<\varepsilon+\delta(\varepsilon)\Rightarrow d(T(x,y),T(u,v))<\varepsilon.$$

First, we give coupled Meir–Keeler contraction in metric spaces.

Definition 27. *Let* (X,d) *be a metric space. Let* $T: X\times X\to X$ *be a given mapping. We say that T is a coupled Meir–Keeler type contraction if it satisfies the following:* $(x,y),(u,v)\in X\times X$ *and for all* $\varepsilon>0$, *there exists* $\delta(\varepsilon)>0$ *such that*

$$\varepsilon \le \frac{1}{4}[d(x,u)+d(y,v)+\max\{d(x,T(x,y))+d(y,T(y,x)),(d(u,T(u,v))$$
$$+d(v,T(v,u)))\}]<\varepsilon+\delta(\varepsilon)\Rightarrow d(T(x,y),T(u,v))<\varepsilon \tag{20}$$

Definition 28 ([55])**.** *Let X be a non-empty set and* $T: X\times X\to X$ *be a given mapping. An element* $(x,y)\in X\times X$ *is said to be a coupled fixed point of the mapping T if* $T(x,y)=x$ *and* $T(y,x)=y$

Now, we give coupled Meir–Keeler type contraction in partial metric spaces.

Definition 29. *Let* (X,p) *be any partial metric space and* $T: X\times X\to X$ *be a continuous mapping. T is a coupled Meir–Keeler type contraction in partial metric space if T satisfies the following: for all* $\varepsilon>0$ *there exists* $\delta(\varepsilon)>0$ *such that*

$$\varepsilon \leq \frac{1}{4}[p(x,u) + p(y,v) + \max\{p(x,T(x,y)) + p(y,T(y,x)), (p(u,T(u,v)) + p(v,T(v,u)))\}] < \varepsilon + \delta(\varepsilon) \Rightarrow p(T(x,y),T(u,v)) < \varepsilon. \tag{21}$$

Lemma 13. *Let (X,d) be a metric space and $T : X \times X \to X$ satisfy (20), then*

$$d(T(x,y),T(u,v)) \leq \frac{1}{4}[d(x,u) + d(y,v) + \max\{(d(x,T(x,y)) + d(y,T(y,x))), (d(u,T(u,v)) + d(v,T(v,u)))\}]. \tag{22}$$

Lemma 14. *Let (X,p) be a partial metric space, T a self map on X, d the constructed metric in [25,74] and $x,y,u,v \in X$. Then*

$$\frac{1}{4}[d(x,u) + d(y,v) + \max\{d(x,T(x,y)) + d(y,T(y,x)), (d(u,T(u,v)) + d(v,T(v,u)))\} = \frac{1}{4}[p(x,u) + p(y,v) + \max\{p(x,T(x,y)) + p(y,T(y,x)), (p(u,T(u,v)) + p(v,T(v,u)))\}$$

for all $x,y,u,v \in X$ with $(x,y) \neq (u,v)$.

Theorem 25. *Let X be any non-empty set and p be a partial metric on X such that (X,p) is a complete partial metric space. Let $T : X \times X \to X$ be a continuous mapping and T be a coupled Meir–Keeler type contraction in partial metric space, that is, T satisfies (21). Then T has a unique coupled fixed point.*

Example 19. *Let $X = [0,2], p(x,y) = \max\{x,y\}$. $T : [0,2] \times [0,2] \to [0,2]$ is defined by $\begin{cases} T(x,y) = 0 & \text{for all } (x,y) \in [0,1) \times [0,1); \\ \max\left\{1 - \frac{1}{x}, 1 - \frac{1}{y}\right\} & \text{for all } (x,y) \in [1,2] \times [1,2]. \end{cases}$ It is clear that T is a continuous function.*

Case 1. For $(x,y),(u,v) \in [0,1) \times [0,1)$ then (22) is trivially satisfied.

Case 2. Let $(x,y),(u,v) \in [1,2] \times [1,2]$ for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq p(x,u) + p(y,v) + \max\{p(x,T(x,y)), p(y,T(y,x)), p(u,T(u,v)), p(v,T(v,u))\} < \varepsilon + \delta. \tag{23}$$

Now,

$$\begin{aligned} & p(x,u) + p(y,v) + \max\{p(x,T(x,y)) + p(y,T(y,x)), p(u,T(u,v)) + p(v,T(v,u))\} \\ & = \max\{x,u\} + \max\{y,v\} + \max\left\{\max\left\{x, \max\left\{1 - \frac{1}{x}, 1 - \frac{1}{y}\right\}\right.\right. \\ & + \max\left\{y, \max\left\{1 - \frac{1}{x}, 1 - \frac{1}{y}\right\}, \max\left\{u, \max\left\{1 - \frac{1}{u}, 1 - \frac{1}{v}\right\}\right.\right. \\ & \left.\left.\left. + \max\left\{v, \max\left\{1 - \frac{1}{v}, 1 - \frac{1}{u}\right\}\right\}\right\}\right\} \\ & = \max\{x+y, x+v, u+y, u+v\} + \max\{x+y, u+v\} \\ & = \max\{2(x+y), x+y+u+v, 2x+y+v, x+u+2v, u+x+2y, \\ & 2u+y+v, 2(u+v)\}, \end{aligned} \tag{24}$$

$$\begin{aligned} p(T(x,y),T(u,v)) & = \max\left\{\max\left\{1 - \frac{1}{x}, 1 - \frac{1}{y}\right\}, \max\left\{1 - \frac{1}{u}, 1 - \frac{1}{v}\right\}\right\} \\ & = \max\left\{1 - \frac{1}{x}, 1 - \frac{1}{y}, 1 - \frac{1}{u}, 1 - \frac{1}{v}\right\}. \end{aligned} \tag{25}$$

Hence, from (23)–(25), we get $p(T(x,y),T(u,v)) < \varepsilon$. So, T is a coupled Meir–Keeler type contraction in partial metric space. T has a unique coupled fixed point $(0,0)$.

2.11. 2017, Popa and Patriciu, a General Fixed-Point Theorem of Meir–Keeler Type for Mappings Satisfying an Implicit Relation in Partial Metric Spaces, [75]

In this section, Meir–Keeler type for mappings satisfying an implicit relation in partial metric spaces, which generalize (Theorem 2.3 and Corollary 2.4 [47]) is proved.

Let T, S be self mappings that are defined on a nonempty set X. We say that $x \in X$ is a coincidence point of T and S if $Tx = Sx$. We denote by $\mathcal{C}(T,S)$ the set of all coincidence points of T and S.

Let (X,d) be a metric space. Jungck [71] defined T and S to be compatible if

$$\lim_{n\to\infty} d(TSx_n, STx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

The idea of pointwise R - weakly commuting mappings is introduced by Pant [76]. Moreover, Pant [76] showed that the pointwise R-weakly commuting is equivalent to commutativity in the coincidence points.

Definition 30 ([77])**.** *Two self mappings T and S that are defined on a nonempty set X are said to be weakly compatible if $TSx^* = STx^*$ for each $x^* \in \mathcal{C}(T,S)$.*

Definition 31. *Let $\mathcal{F}_{MK}$ be the set of all real continuous mappings $F(t_1,\ldots,t_6) : \mathbb{R}_0^{+6} \to \mathbb{R}$ satisfying the following conditions:*

(F_1) : F is nonincreasing in variables $t_2, t_3, \ldots, t_6$,
(F_2) : $F(t,t,0,t,t,t) \le 0$ implies $t = 0$,
(F_3) : $F(t,t,t,0,t,t) \le 0$ implies $t = 0$.

Theorem 26. *Let A, B, S and T be self mappings on a complete partial metric space (X,p) satisfying the following conditions:*

$AX \subset TX$ and $BX \subset SX$,

for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } p(Ax,By) \le \varepsilon,$$

where $M(x,y) = \max\Big\{p(Sx,Ty), p(Sx,Ax), p(Ty,By), \frac{1}{2}[p(Sx,By) + p(Ty,Ax)]\Big\}$,

for all $x, y \in X$ with $M(x,y) > 0$ implies $p(Ax,By) < M(x,y)$,

$$F(p(Ax,By), p(Sx,Ty), p(Sx,Ax), p(Ty,By), p(Sx,By), p(Ty,Ax)) \le 0,$$

for all $x, y \in X$ and $F \in \mathcal{F}_{MK}$.

If one of AX, BX, SX, TX is a closed subset of (X,p), then

(a) $\mathcal{C}(A,S) \ne \varnothing$,
(b) $\mathcal{C}(B,T) \ne \varnothing$.

Moreover, if A and S, as well B and T are weakly compatible, then A, B, S and T have a unique common fixed point.

Theorem 27. *Let A, B, S and T be self mappings on a partial metric space (X,p) satisfying the following conditions:*

$AX \subset TX$ and $BX \subset SX$

for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$,

$$\varepsilon < M(x,y) < \varepsilon + \delta \text{ implies } p(Ax,By) \le \varepsilon,$$

where $M(x,y) = \max\{p(Sx,Ty), p(Ax,Sx), p(By,Ty), \frac{1}{2}[p(Sx,By) + p(Ty,Ax)]\Big\}$, *for all*

$$x,y \in X \text{ with } M(x,y) > 0 \text{ implies } p(Ax,By) < M(x,y),$$

$F(p(Ax,By), p(Sx,Ty), p(Sx,Ax), p(Ty,By), p(Sx,By), p(Ty,Ax)) \leq 0$, *for all* $x,y \in X$ *and* $F \in \mathcal{F}_{MK}$.

If one of AX, BX, SX, TX *is a complete subspace of* (X,p)*, then*

(a) $\mathcal{C}(A,S) \neq \varnothing$,
(b) $\mathcal{C}(B,T) \neq \varnothing$.

Moreover, if A *and* S*, as well* B *and* T *are weakly compatible, then* A, B, S *and* T *have a unique common fixed point.*

2.12. 2018, Gunasekar, Karpagam and Zlatanov, on ρ-Cyclic Orbital MK Contractions in a Partial Metric Space, [78]

A cyclic map with a contractive type of hold named ρ-cyclic orbital Meir–Keeler contraction is established on partial metric space. We provide fixed-point results and best proximity point results in complete partial metric spaces for these maps.

Lim [79] introduced the concept of L-functions. We use this function to obtain the main results.

Definition 32 ([79])**.** *A function* $\phi : [0,\infty) \to [0,\infty)$ *is called an* L*-function if* $\phi(0) = 0$; $\phi(s) > 0$ *for every* $s > 0$ *and for every* $s > 0$*, there exists* $u > s$ *such that* $\phi(t) \leq s$ *for* $t \in [s,u]$.

Note that every L-function satisfies the condition $\phi(s) \leq s$ for every $s \geq 0$. Suzuki [80] generalized the L-function as follows.

Lemma 15 ([80])**.** *Let* X *be a nonempty set, and let* $T, S : X \to [0,\infty)$*. Then, the following are equivalent:*

(1) *For each* $\varepsilon > 0$*, there exists a* $\delta > 0$*, such that* $T(x) < \varepsilon + \delta \Rightarrow S(x) < \varepsilon$.
(2) *There exists an* L*-function* ϕ *(which may be chosen to be a non-decreasing and continuous) such that* $T(x) > 0 \Rightarrow S(x) < \phi(T(x))$, $x \in X$ *and* $T(x) = 0 \Rightarrow S(x) = 0, \quad x \in X$.

Suzuki et al. [81] established the uniform convexity (UC) property. For the investigation of the best proximity points, the notion of the UC of a Banach space plays a crucial role. We introduce UC notion in partial metric space as follows.

Definition 33. *Let* (X,p) *be a partial metric space and* A *and* B *be subsets. The pair* (A,B) *is said to satisfy the property UC if the following holds: If* $\{x_n\}, \{z_n\}$ *are sequences in* A *and* $\{y_n\}$ *is a sequence in* B*, such that* $\lim_{n\to\infty} p(x_n,y_n) = \operatorname{dist}(A,B)$ *and for any* $\varepsilon > 0$*, there is* $n \in \mathbb{N}$*, so that* $p(z_m,y_n) < \operatorname{dist}(A,B) + \varepsilon$ *for all* $n \geq N$*, then for any* $\varepsilon > 0$*, there is* $N_1 \in \mathbb{N}$*, so that* $p(x_n,z_m) < \varepsilon$ *for* $m,n \geq N_1$.

Now, let us recall the notion of cyclic maps.

Definition 34. *Let* X *be a nonempty set and* $A_1, A_2, \ldots, A_\rho$ *be nonempty subsets of* X*. A map* $T : \cup_{i=1}^{\rho} A_i \to \cup_{i=1}^{\rho} A_i$ *is called a* ρ*-cyclic map if* $T(A_i) \subseteq A_{i+1}$*, for all* $i = 1,2,\ldots,\rho$*, where we use the convention* $A_{\rho+1} = A_1$.

Definition 35. *Let* (X,p) *be a partial metric space and* $A_1, A_2, \ldots, A_\rho$ *be nonempty subsets of* X*. A point* $x \in A_i$ *is said to be a best proximity point of* T *in* A_i*, if* $p(x,Tx) = \operatorname{dist}(A_i, A_{i+1})$, $1 \leq i \leq \rho$.

We also give the definition of ρ-cyclic orbital Meir–Keeler contraction.

Definition 36. *Let (X, p) be a partial metric space, $A_1, A_2, \ldots, A_\rho$ be nonempty subsets of X and $T : \cup_{i=1}^{\rho} A_i \to \cup_{i=1}^{\rho} A_i$ be a ρ-cyclic map. The map T is called a ρ-cyclic orbital Meir–Keeler contraction if for some $x \in A_i$, for each $\varepsilon > 0$, there exists $\delta > 0$, such that the following condition:*

$$p\left(T^{\rho n+k-1}x, T^k y\right) < D_k + \varepsilon + \delta \Rightarrow p\left(T^{\rho n+k}x, T^{k+1}y\right) < D_{k+1} + \varepsilon, \tag{26}$$

holds for all $n \in \mathbb{N}_0$ and for all $y \in A_i$, where $D_k \geq 0$, for $k = 1, 2, \ldots, \rho$.

Theorem 28. *Let (X, p) be a complete partial metric space. Let $A_1, A_2, \ldots, A_\rho$ be nonempty and closed subsets of X. Let $T : \cup_{i=1}^{\rho} A_i \to \cup_{i=1}^{\rho} A_i$ be a ρ-cyclic orbital Meir–Keeler contraction map with constants D_k equal to zero or $D_k = \operatorname{dist}(A_{i+k-1}, A_{i+k}), k = 1, 2, \ldots, \rho$.*

(1) If $D_k = 0$ for all $k = 1, 2, \ldots \rho$, then $\cap_{i=1}^{\rho} A_i$ is nonempty, and T has a unique fixed point $x^* \in \cap_{i=1}^{\rho} A_i$. For any $x \in \cup_{i=1}^{\rho} A_i$, satisfying (26) with $D_k = 0, \lim_{n\to\infty} T^n x = x^*$ holds.

(2) If $D_k = \operatorname{dist}(A_{i+k-1}, A_{i+k})$ for all $k = 1, 2, \ldots, \rho$ and (X, p) is a partial metric space with property UC, then for every $x \in A_i$ satisfying (26), the sequence $\{T^{\rho n} x\}$ converges to a unique point $z \in A_i$, which is the best proximity point, as well as the unique periodic point of T in A_i. Furthermore, $T^k z$ is a best proximity point of T in A_{i+k}, which is also a unique periodic point of T in A_{i+k}, for each $k = 1, 2, \ldots, (\rho - 1)$.

Without loss of generality, let us assume that $x \in A_1$.

Lemma 16. *Let (X, p) be a partial metric space. Let $A_1, A_2, \ldots, A_\rho$ be nonempty subsets of X. Let $T : \cup_{i=1}^{\rho} A_i \to \cup_{i=1}^{\rho} A_i$ be a ρ-cyclic orbital Meir–Keeler contraction map with constants D_k equal to zero or $D_k = \operatorname{dist}(A_k, A_{1+k})$. Then, there exists an L-function ϕ such that for an $x \in A_1$ satisfying (26), the following holds: if $p\left(T^{\rho n+k-1}x, T^k y\right) > D_k$, then:*

$$p\left(T^{\rho n+k}x, T^{k+1}y\right) - D_{k+1} < \phi\left(p\left(T^{\rho n+k-1}x, T^k y\right) - D_k\right)$$

and if $p\left(T^{\rho n+k-1}x, T^k y\right) = D_k$, then $p\left(T^{\rho n+k}x, T^{k+1}y\right) = D_{k+1}$, for each $k = 1, 2, \ldots, \rho$, for all $n \in \mathbb{N}$ and for all $y \in A_1$.

Remark 7. *From Lemma 16, it follows that for a ρ-cyclic orbital Meir–Keeler contraction map T, the sequence $\left\{p\left(T^{\rho n+k-1}x, T^k y\right) - D_k\right\}_{n=1}^{\infty}, k = 1, 2, \ldots, \rho$ is non-increasing.*

Lemma 17. *Let (X, p) be a partial metric space. Let $A_1, A_2, \ldots, A_\rho$ be nonempty subsets of X. Let $T : \cup_{i=1}^{\rho} A_i \to \cup_{i=1}^{\rho} A_i$ be a ρ-cyclic orbital Meir–Keeler contraction map with constants D_k equal to zero or $D_k = \operatorname{dist}(A_k, A_{1+k})$. Then, for any $x \in A_1$ satisfying (26), for all $y \in A_1$ and for each $k \in \{0, 1, 2, \ldots, \rho - 1\}$, the sequence $\left\{p\left(T^{\rho n+k}x, T^{\rho n+k+1}y\right)\right\}_{n=1}^{\infty}$ converges to D_{k+1}.*

Lemma 18. *Let (X, p) be a complete partial metric space with property UC and $A_1, A_2, \ldots, A_\rho$ be non-empty and closed subsets of X. Let $T : \cup_{i=1}^{\rho} A_i \to \cup_{i=1}^{\rho} A_i$ be a ρ-cyclic orbital Meir–Keeler contraction map with constants $D_k = \operatorname{dist}(A_k, A_{k+1})$. Let $x \in A_1$ satisfy (26). Suppose that for each $k = 0, 1, 2, \ldots, \rho - 1$, the sequence $\left\{T^{\rho n+k}x\right\}$ converges to $z_k \in A_{1+k}$. Then:*

(a) $\operatorname{dist}(A_1, A_2) = \operatorname{dist}(A_2, A_3) = \cdots = \operatorname{dist}(A_{\rho-1}, A_\rho) = \operatorname{dist}(A_\rho, A_1)$;

(b) z_k is a best proximity point of T in A_{1+k} and $z_k = T^k z_0$, for $i = 1, 2, \ldots, \rho$;

(c) z_k is a periodic point of T with period ρ in A_{1+k}.

We use Lemma 19 to check whether a partial metric space is complete.

Lemma 19. *Let (X,d) be a complete metric space and (X,p) be a partial metric space. Let $\omega : X \to [0,+\infty)$ and $p(x,y) = d(x,y) + \max\{\omega(x), \omega(y)\}$. The partial metric space (X,p) is complete if and only if ω satisfies the condition: if $\limsup_{d(x_n,x)\to 0} \omega(x_n) < \infty$, then $\lim_{n\to\infty} \omega(x_n) = \omega(x)$.*

Corollary 3. *Let (X,d) be a complete metric space and $\omega : X \to [0,+\infty)$ be a continuous function with respect to metric d, and let us consider the partial metric space (X,p), where $p(x,y) = d(x,y) + \max\{\omega(x), \omega(y)\}$. Then, if $A \subset X$ is closed in (X,d), then it is closed in (X,p).*

Example 20. *Let us consider the metric space $([0,+\infty), d)$, endowed with the metric $d(x,y) = |x-y|$. Let us consider the function*

$$\omega(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 1, & x = 0 \end{cases}.$$

Then, $([0,+\infty), p)$ is a complete partial metric space, where $p(x,y) = |x-y| + \max\{\omega(x), \omega(y)\}$.

Let $\lim_{n\to\infty}|x_n - x| = 0$. Then, $\limsup_{n\to\infty} \omega(x_n) < \infty$ if and only if $x \neq 0$. By the continuity of ω at any different point from zero, it follows that $\lim_{n\to\infty} \omega(x_n) = \omega(x)$, provided that $\lim_{n\to\infty} x_n = x \neq 0$. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $([0,+\infty), p)$.

Thus, the limit $\lim_{n,m\to\infty}(|x_n - x_m| + \max\{\omega(x_n), \omega(x_m)\})$ exists and is finite. This limit is finite if and only if $\lim_{n\to\infty} x_n = x \neq 0$. Consequently, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $([0,+\infty), p)$ if and only if $\lim_{n\to\infty}|x_n - x| = 0$ for some $x \neq 0$ and $\lim_{n\to\infty} p(x_n, x) = \omega(x) = p(x,x)$. Consequently, $([0,+\infty), p)$ is a complete partial metric space.

Example 21. *Let us consider the metric space $([0,+\infty), d)$, endowed with the metric $d(x,y) = |x-y|$. Let us consider the function*

$$\omega(x) = \begin{cases} 1, & x \neq 1 \\ a, & x = 1 \end{cases}.$$

Then, $([0,+\infty), p)$ is a complete partial metric space with $p(x,y) = |x-y| + \max\{\omega(x), \omega(y)\}$ if and only if $a = 1$. Let $\lim_{n\to\infty}|x_n - x| = 0$. Then, $\limsup_{n\to\infty} \omega(x_n) < \infty$ and $\lim_{n\to\infty} \omega(x_n) = \omega(x)$, provided that $x \neq 1$ or $a = 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence, such that $x_n \neq 1$, which is convergent to one with respect to the metric d. Then, it is a Cauchy sequence in $([0,+\infty), p)$, because the limit

$$\lim_{n,m\to\infty}(|x_n - x_m| + \max\{\omega(x_n), \omega(x_m)\}) = a$$

exists and is finite. From

$$a = p(1,1) \text{ and } p(x_n, 1) = (|x_n - 1| + \max\{\omega(x_n), \omega(1)\} = \max\{1, a\}$$

it follows that $\{x_n\}_{n=1}^{\infty}$ is convergent in $([0,+\infty), p)$ if and only if $a = 1$.

Example 22. *Let us consider the Banach space $\left(\mathbb{R}^2, \|\cdot\|_2\right)$, where $\mathbb{R}^2 = \{x = (u,v) : u, v \in \mathbb{R}\}$ and $\|x\|_2 = \|(u,v)\|_2 = \sqrt{u^2 + v^2}$. Let us endow $\mathbb{R}^2$ with the partial metric $p(x,y) = \|x - y\|_2 + \max\left\{\|x\|_2^2, \|y\|_2^2\right\}$. From Example 4, $p(x,y)$ is a partial metric. From Lemma 19 , it follows that $\left(\mathbb{R}^2, p\right)$ is a complete partial metric space. We consider the sets A_1, A_2, A_3, A_4 defined by $A_1 = \{(u,v) \in \mathbb{R} : u \geq 0, v \geq 0\}$, $A_2 = \{(u,v) \in \mathbb{R} : u \leq 0, v \geq 0\}$, $A_3 = \{(u,v) \in \mathbb{R} : u \leq 0, v \leq 0\}$, $A_4 = \{(u,v) \in \mathbb{R} : u \geq 0, v \leq 0\}$. From Corollary 3 , it follows that A_1, A_2, A_3, A_4 are closed sets in $\left(\mathbb{R}^2, p\right)$. Let us define a cyclic map $T : \cup_{i=1}^{\rho} A_i \to \cup_{i=1}^{\rho} A_{i+1}$ by:*

$$T(u,v) = \left(\frac{-|v|}{1+2\|(u,v)\|_2}, \frac{|u|}{1+2\|(u,v)\|_2}\right), \textit{ for } (u,v) \in A_1;$$
$$T(u,v) = \left(\frac{-|v|}{1+2\|(u,v)\|_2}, \frac{-|u|}{1+2\|(u,v)\|_2}\right), \textit{ for } (u,v) \in A_2;$$
$$T(u,v) = \left(\frac{|v|}{1+2\|(u,v)\|_2}, \frac{-|u|}{1+2\|(u,v)\|_2}\right), \textit{ for } (u,v) \in A_3; \textit{ and}$$
$$T(u,v) = \left(\frac{|v|}{1+2\|(u,v)\|_2}, \frac{|u|}{1+2\|(u,v)\|_2}\right), \textit{ for } (u,v) \in A_4.$$

Let $x = (0,0)$. *Let us choose an arbitrary* $y_0 = (u_0, v_0) \in A_i$. *Let us denote* $T^k y_0 = y_k = (u_k, v_k)$. *Then,* $p\left(T^k(u_0,v_0), T^{4n+k-1}x\right) = p((u_k,v_k),(0,0)) = \|(u_k,v_k)\|_2 + \|(u_k,v_k)\|_2^2 = \sqrt{u_k^2+v_k^2} + u_k^2 + v_k^2$. *Now:*

$$\begin{aligned} R_4 &= p\left(T^{k+1}y_0, T^{4n+k}x\right) = p\left(T^{k+1}(u_0,v_0), T^{4n+k}x\right) \\ &= p(T(u_k,v_k),(0,0)) \\ &= \frac{\sqrt{u_k^2+v_k^2}}{(1+2\|(u_k,v_k)\|_2)} + \frac{u_k^2+v_k^2}{(1+2\|(u_k,v_k)\|_2)^2} \\ &\le \frac{\sqrt{u_k^2+v_k^2} + u_k^2+v_k^2}{(1+2\|(u_k,v_k)\|_2)} = \frac{\|(u_k,v_k)\|_2 + \|(u_k,v_k)\|_2^2}{(1+2\|(u_k,v_k)\|_2)} = \frac{\|y_k\|_2 + \|y_k\|_2^2}{(1+2\|y_k\|\|_2)}. \end{aligned}$$

Now, $p(x,y) = \|y_k\|_2 + \|y_k\|_2^2$.
By solving the equation $\|y_k\|_2^2 + \|y_k\|_2 + p(x,y) = 0$, *we get*

$$\|y_k\|_2 = \frac{\sqrt{1+4p(x,y_k)}-1}{2}.$$

Hence:

$$p\left(T^{k+1}y_0, T^{4n+k}x\right) \le \frac{p(x,y_k)}{\sqrt{1+4p(x,y_k)}}.$$

The function $\frac{t}{\sqrt{1+4t}}$ *is a continuous function in the interval* $[0,+\infty)$. *From* $\frac{\varepsilon}{\sqrt{1+4\varepsilon}} < \varepsilon$, *we get the condition that there exists* $\delta(\varepsilon) > 0$ *such that the inequality* $p\left(T^{k+1}y_0, T^{4n+k}x\right) \le \frac{p(x,y_k)}{\sqrt{1+4p(x,y_k)}} < \varepsilon$ *holds whenever the inequality holds* $p\left(T^k y_0, T^{4n+k-1}x\right) = p(y_k,x) < \varepsilon + \delta(\varepsilon)$. *Consequently, T is a 4-cyclic orbital Meir–Keeler contraction, and x is the unique fixed point.*

2.13. 2013, Chen and Karapınar, Fixed-Point Results for the α-Meir–Keeler Contraction on Partial Hausdorff Metric Spaces, [82]

We introduce fixed-point results for a multi-valued mapping satisfying the α-Meir–Keeler contraction with respect to the partial Hausdorff metric H in complete partial metric spaces.

Let (X,d) be a metric space, and let $CB(X)$ denote the collection of all nonempty, closed and bounded subsets of X. For $A, B \in CB(X)$, we define

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(x,B) := \inf\{d(x,b) : b \in B\}$, and it is well known that H is called the Hausdorff metric induced by the metric d. A multi-valued mapping $T : X \to CB(X)$ is called a contraction if

$$H(Tx,Ty) \le \lambda d(x,y)$$

for all $x, y \in X$ and $\lambda \in [0, 1)$. The study of fixed points for multi-valued contractions using the Hausdorff metric was introduced in Nadler [83].

Theorem 29 ([83])**.** *Let (X, d) be a complete metric space, and let $T : X \to CB(X)$ be a multivalued contraction. Then there exists $x \in X$ such that $x \in Tx$.*

In [84], authors investigated the partial Hausdorff metric H_p induced by the partial metric p. Let (X, p) be a partial metric space, and let $CB^p(X)$ be the collection of all nonempty, closed and bounded subset of the partial metric space (X, p). Note that closedness is taken from (X, τ_p), and boundedness is given as follows: A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \in \mathbb{R}$ such that for all $a \in A$, we have

$$a \in B_p(x_0, M), \text{ that is, } p(x_0, a) < p(a, a) + M.$$

For $A, B \in CB^p(X)$ and $x \in X$, they define

$$\begin{aligned} p(x, A) &:= \inf\{p(x, a) : a \in A\} \\ \delta_p(A, B) &:= \sup\{p(a, B) : a \in A\} \\ \delta_p(B, A) &:= \sup\{p(b, A) : b \in B\} \\ H_p(A, B) &= \max\{\delta_p(A, B), \delta_p(B, A)\} \end{aligned}$$

It is immediate to get that if $p(x, A) = 0$, then $d_p(x, A) = 0$, where $d_p(x, A) = \inf\{d_p(x, a) : a \in \text{A}\}$.

Aydi et al. [84] also proved the following properties;

$$\delta_p : CB^p(X) \times CB^p(X) \to \mathbb{R} \text{ and } H_p : CB^p(X) \times CB^p(X) \to \mathbb{R}.$$

Proposition 4 ([84])**.** *Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$, the following properties hold:*

(1) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$;
(2) $\delta_p(A, A) \leq \delta_p(A, B)$;
(3) $\delta_p(A, B) = 0$ *implies that* $A \subset B$;
(4) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Proposition 5 ([84])**.** *Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$, the following properties hold:*

(1) $H_p(A, A) \leq H_p(A, B)$;
(2) $H_p(A, B) = H_p(B, A)$;
(3) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$;
(4) $H_p(A, B) = 0$ *implies that* $A = B$.

Lemma 20 ([84])**.** *Let (X, p) be a partial metric space, $A, B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that*

$$p(a, b) \leq h H_p(A, B).$$

Now, we give the notions of a strictly α-admissible and an α-Meir Keeler contraction with respect to the partial Hausdorff metric H_p.

Definition 37. *Let (X, p) be a partial metric space, $T : X \to CB^p(X)$ and $\alpha : X \times X \to \mathbb{R}^+$. We say that T is strictly α-admissible if*

$$\alpha(x, y) > 1 \text{ implies that } \alpha(y, z) > 1, \quad x \in X, y \in Tx, z \in Ty.$$

Definition 38. *Let (X, p) be a partial metric space and $\alpha : X \times X \to \mathbb{R}^+$. We call $T : X \to CB^p(X)$ an α-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p if the following conditions hold:*

$(c_1)T$ is strictly α-admissible;

$(c_2\ )$ for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq p(x,y) < \varepsilon + \delta \text{ implies that } \alpha(x,y)H_p(Tx,Ty) < \varepsilon.$$

Remark 8. *Note that if $T : X \to CB^p(X)$ is a α-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p, then we have that for all $x, y \in X$*

$$\alpha(x,y)H_p(Tx,Ty) \leq p(x,y).$$

Further, if $p(x,y) = 0$, then $H_p(Tx,Ty) = 0$.
So, if $p(x,y) = 0$, then $\alpha(x,y)H_p(Tx,Ty) < p(x,y)$.

The main theorem is as follows;

Theorem 30. *Let (X, p) be a complete partial metric space. Suppose that $T : X \to CB^p(X)$ is an α-Meir–Keeler contraction with respect to the partial Hausdorff metric H and that there exists $x_0 \in X$ such that $\alpha(x_0, y) > 1$ for all $y \in Tx_0$. Then T has a fixed point in X (that is, there exists $x^* \in X$ such that $x^* \in Tx^*$).*

2.14. 2015, Jen, Chen and Peng, Some New Fixed-Point Theorems for the Meir–Keeler Contractions on Partial Hausdorff Metric Spaces, [85]

In this section, we introduce fixed-point results for a multi-valued mapping concerning with three classes of Meir–Keeler contractions with respect to the partial Hausdorff metric H in partial metric spaces.

Remark 9. *It is clear that, if the function $\psi : \mathbb{R}_0^+ \to [0,1)$ is a Reich function ($\mathcal{R}$-function), then ψ is also a stronger Meir–Keeler-type function.*

Now, we denote by Φ the class of functions $\phi : \mathbb{R}_0^{+4} \to \mathbb{R}_0^+$ satisfying the following conditions:

(1) ϕ is an increasing and continuous function in each coordinate;
(2) for $t \in \mathbb{R}^+, \phi(t,t,t,t) \leq t$ and $\phi(t_1,t_2,t_3,t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$.

We now introduce the notion of (ψ, ϕ)-Meir–Keeler contraction on partial Hausdorff metric spaces.

Definition 39. *Let (X, p) be a partial metric space, $\psi : \mathbb{R}_0^+ \to [0,1)$ and $\phi \in \Phi$. We call $T : X \to CB^p(X)$ a (ψ, ϕ)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p, if the following conditions hold:*

(c_1) *ψ is a stronger Meir–Keeler-type function;*
(c_2) *for all $x, y \in X$, we have*

$$H_p(Tx,Ty) \leq \psi(p(x,y))\phi\left(p(x,y), p(x,Tx), p(y,Ty), \frac{p(x,Ty) + p(y,Tx)}{2}\right)$$

We state and prove the main fixed-point result for the (ψ, ϕ)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p

Theorem 31. *Let (X, p) be a complete partial metric space. Suppose $T : X \to CB^p(X)$ is a (ψ, ϕ)-Meir–Keeler contraction with respect to the partial Hausdorf metric H_p. Then T has a fixed point in X, that is, there exists $x^* \in X$ such that $x^* \in Tx^*$.*

Inspired by the Reich function and stronger Meir–Keeler function, we establish the following notion of (ψ, ϕ)-Reich's contraction with respect to the partial Hausdorff metric H_p.

Definition 40. *Let (X, p) be a partial metric space, $\psi : \mathbb{R}_0^+ \to [0, 1)$, and $\phi \in \Phi$. We call $T : X \to CB^p(X)$ a (ψ, ϕ)-Reich's contraction with respect to the partial Hausdorff metric H_p if the following conditions hold:*

(1) ψ is a Reich function ($\mathcal{R}$-function);
(2) for all $x, y \in X$, we have

$$H_p(Tx, Ty) \le \psi(p(x, y))\phi\left(p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right)$$

Using Remark 9, Definition 2.14, and Theorem 31, we are easy to get the following theorem.

Theorem 32. *Let (X, p) be a complete partial metric space. Suppose $T : X \to CB^p(X)$ is a (ψ, ϕ)-Reich's contraction with respect to the partial Hausdorff metric H_p. Then T has a fixed point in X, that is, there exists $x^* \in X$ such that $x^* \in Tx^*$.*

We let Ξ be the class of all non-decreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions:

(1) φ is a weaker Meir–Keeler-type function;
(2) for all $t \in (0, \infty)$, $\{\varphi^n(t)\}_{n \in \mathbb{N}}$ is decreasing;
(3) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$,
(4) for $t > 0$, if $\lim_{n\to\infty} \varphi^n(t) = 0$, then $\lim_{n\to\infty} \sum_{i=n}^{m} \varphi^i(t) = 0$, where $m > n$;
(5) for $t_n \in \mathbb{R}_0^+$, if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \varphi(t_n) = 0$.

In [82] , the authors introduce notion of strictly α-admissible.

We now introduce the notion of (α, φ)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p, as follows:

Definition 41. *Let (X, p) be a partial metric space, $\varphi \in \Xi$, and $\alpha : X \times X \to \mathbb{R}^+$. We call $T : X \to CB^p(X)$ a (α, φ)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p if the following conditions hold:*

(c_1) *T is strictly α-admissible;*
(c_2) *for each $x, y \in X$,*

$$\alpha(x, y)H_p(Tx, Ty) \le \varphi(p(x, y)).$$

We now state main result for the (α, φ)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p

Theorem 33. *Let (X, p) be a complete partial metric space. Suppose $T : X \to CB^p(X)$ is an (α, φ)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p. Suppose also that*

(i) there exists $x_0 \in X$ such that $\alpha(x_0, y) > 1$ for all $y \in Tx_0$;
(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \ge 1$ for all n.

Then T has a fixed point in X (that is, there exists $x^* \in X$ such that $x^* \in Tx^*$).

Note: It is clear that if Meir–Keeler type function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ then, we have for all $t \in (0, \infty)$, $\gamma(t) < t$.

We consider the family

$$\Omega = \left\{(\gamma_1, \gamma_2, \gamma_3, \gamma_4) \mid \gamma_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+, i = 1, 2, 3, 4\right\}$$

such that:

(1) $\gamma_1(t), \gamma_2(t), \gamma_3(t) \leq \gamma_4(t)$ for all $t > 0$;
(2) $\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t)$ are continuous;
(3) $\gamma_1(t_1) = \gamma_2(t_2) = \gamma_3(t_3) = \gamma_4(t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$;
(4) γ_4 is a Meir–Keeler-type function;
(5) $\gamma_4(t_1 + t_2) \leq \gamma_4(t_1) + \gamma_4(t_2)$ for all $t_1, t_2 > 0$.

We now introduce the notion of $(\alpha, \phi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$-Meir–Keeler contraction on partial Hausdorff metric spaces.

Definition 42. *Let (X, p) be a partial metric space, $\phi \in \Phi, (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \Omega$, and $\alpha : X \times X \to \mathbb{R}^+$. We call $T : X \to CB^p(X)$ a $(\alpha, \phi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p if the following conditions hold:*
(1) T is strictly α-admissible;
(2) for all $x, y \in X$, we have

$$\alpha(x,y)H_p(Tx,Ty) \leq \phi\left(\gamma_1(p(x,y)), \gamma_2(p(x,Tx)), \gamma_3(p(y,Ty)), \frac{\gamma_4(p(x,Ty) + p(y,Tx))}{2}\right).$$

We now state main result for the $(\alpha, \phi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p.

Theorem 34. *Let (X, p) be a complete partial metric space. Suppose $T : X \to CB^p(X)$ is a $(\alpha, \phi, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p. Suppose also that*
(i) there exists $x_0 \in X$ such that $\alpha(x_0, y) > 1$ for all $y \in Tx_0$;
(ii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.
Then T has a fixed point in X (that is, there exists $x^* \in X$ such that $x^* \in Tx^*$).

2.15. 2018, Chen, Karapınar and O'Regan, on $(\alpha - \phi)$-Meir–Keeler Contractions on Partial Hausdorff Metric Spaces, [86]

In this section, we present a new $(\alpha - \phi)$-Meir–Keeler contraction for multi-valued mappings and we investigate the existence of fixed-point theorems in partial metric space.

Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(i) ψ is nondecreasing;
(ii) there exist $k_0 \in \mathbb{N}$ and $a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that
$$\psi^{k+1}(t) \leq a\psi^k(t) + v_k,$$
for $k \geq k_0$ and any $t \in \mathbb{R}_0^+$.

In the literature such functions are called (c)-comparison functions (see [87]).

Lemma 21 (See e.g., [87]). *If $\psi \in \Psi$, then the following hold:*
(i) $(\psi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}^+$;
(ii) $\psi(t) < t$, for any $t \in \mathbb{R}^+$;
(iii) ψ is continuous at 0 ;
(iv) the series $\sum_{k=1}^{\infty} \psi^k(t)$ converges for any $t \in \mathbb{R}^+$.

We denote by Φ the class of functions $\phi : \left(\mathbb{R}_0^+\right)^4 \to \mathbb{R}_0^+$ satisfying the following conditions:
$(\phi_1)\phi$ is an increasing and continuous function in each coordinate;
(ϕ_2) for each ϕ there exists $\psi \in \Psi$ such that $\phi(t, t, t, t) = \psi(t)$ for all $t \in \mathbb{R}_0^+$,

We establish the concept of a $(\alpha - \phi)$-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p induced by the partial metric.

Definition 43. *Let (X, p) be a partial metric space, $\alpha : X \times X \to \mathbb{R}_0^+$ be a mapping and $\phi \in \Phi$. A multi-valued mapping $T : X \to CB^p(X)$ is called a $(\alpha - \phi)$-Meir–Keeler contraction with respect to the associated partial Hausdorff metric H_p if the following conditions hold:*

(C) For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \varepsilon \leq &\phi\Big(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\Big) < \varepsilon + \delta \\ &\Longrightarrow \alpha(x,y)H_p(Tx,Ty) < \varepsilon \end{aligned} \tag{27}$$

for all $x, y \in X$.

A multi-valued mapping T is called a ϕ-Meir–Keeler-type contraction if $\alpha(x, y) = 1$ for all $x, y \in X$ in (27), that is,

$$\begin{aligned} \varepsilon \leq \phi\Big(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\Big) < \varepsilon + \delta \\ \Longrightarrow H_p(Tx,Ty)) < \varepsilon, \end{aligned}$$

for all $x, y \in X$.

Remark 10. *Note that if $T : X \to CB^p(X)$ is a $(\alpha - \phi)$-Meir–Keeler contraction with respect to the associated partial Hausdorff metric H_p, then we have*

$$\alpha(x,y)H_p(Tx,Ty) \leq \phi\Big(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\Big),$$

for all $x, y \in X$.

Notice that if $\phi\Big(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\Big) > 0$, then we have

$$\alpha(x,y)H_p(Tx,Ty) < \phi\Big(p(x,y), p(x,Tx), p(y,Ty), \frac{1}{2}[p(x,Ty) + p(y,Tx)]\Big).$$

We present main result.

Theorem 35. *Let (X, p) be a complete partial metric space and $\alpha : X \times X \to \mathbb{R}_0^+$ be a mapping. Suppose that a multi-valued mapping $T : X \to CB^p(X)$ is a $(\alpha - \phi)$-Meir–Keeler contraction with respect to the associated partial Hausdorff metric H_p. Also assume that*

(i) T is strictly α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, y) > 1$ for all $y \in Tx_0$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) > 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) > 1$ for all n.

Then T has a fixed point in X, that is, there exists $x^* \in X$ such that $x^* \in Tx^*$.

Example 23. *Let $X = \{0, 1, 2\}$ be endowed with the partial metric $p : X \times X \to \mathbb{R}^+$ defined by*

$$p(x,y) = \frac{1}{2}|x - y| + \max\{x,y\} \text{ for all } x,y \in X.$$

Then (X, p) is a complete partial metric space, and we have

$$\begin{gathered} p(0,0) = 0; \quad p(1,1) = 1; \quad p(2,2) = 2 \\ p(0,1) = p(1,0) = \frac{3}{2}; \quad p(0,2) = p(2,0) = 3; \quad p(1,2) = \frac{5}{2}. \end{gathered}$$

We next define $T : X \to CB(X)$ by

$$T(0) = T(1) = \{0\} \text{ and } T(2) = \{0,1\}$$

Then we have

(1) *if* $x, y \in \{0,1\}$, *then* $H_p(T(x), T(y)) = H_p(\{0\}, \{0\}) = 0$;
(2) *if* $x \in \{0,1\}$ *and* $y = 2$; *then* $H_p(T(0), T(2)) = H_p(T(1), T(2)) = H_p(\{0\}, \{0,1\}) = \frac{3}{2}$;
(3) *if* $x = y = 2$; *then* $H_p(T(2), T(2)) = H_p(\{0,1\}, \{0,1\}) = \sup\{p(x,x) : x \in \{0,1\}\} = 1$;
(4) $p(0, T(1)) = 0, p(1, T(0)) = \frac{3}{2}, p(0, T(2)) = 0, p(2, T(0)) = 3, p(1, T(2)) = 1,$ $p(2, T(1)) = 3$.

Now, we put $\phi(t_1, t_2, t_3, t_4) = \frac{2}{3}\max\{t_1, t_2, t_3, t_4\}$. *Then all of the hypotheses of Theorem 35 are satisfied. Note* $x = 0$ *is the unique fixed point of* T.

2.16. 2020, Afassinou and Narain, Fixed-Point and Endpoint Theorems for (α, β)*-Meir–Keeler Contraction on the Partial Hausdorff Metric, [88]*

The purpose of this section is to prove the notion of a multivalued strictly (α, β)-admissible mappings and a multivalued (α, β)-Meir–Keeler contractions with respect to the partial Hausdorff metric H_p in partial metric spaces.

Suzuki [89] established the idea of mappings satisfying condition (C) which is also known as Suzuki-type generalized nonexpansive mapping.

Definition 44. *Let* (X, d) *be a metric space. A mapping* $T : X \to X$ *is said to satisfy condition* (C) *if for all* $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \le d(x, y) \Rightarrow d(Tx, Ty) \le d(x, y).$$

Theorem 36. *Let* (X, d) *be a compact metric space and* $T : X \to X$ *be a mapping satisfying condition* (C) *for all* $x, y \in X$. *Then* T *has a unique fixed point.*

Definition 45 ([90])**.** *Let* $T : X \to X$ *be a mapping and let* $\alpha, \beta : X \to \mathbb{R}_0^+$ *be two functions. Then* T *is called a cyclic* (α, β)*-admissible mapping, if*

(1) $\alpha(x) \ge 1$ *for some* $x \in X$ *implies that* $\beta(Tx) \ge 1$,
(2) $\beta(x) \ge 1$ *for some* $x \in X$ *implies that* $\alpha(Tx) \ge 1$.

In 2019, Mebawondu et al. [91] generalized the concept of an α-admissible mapping by introducing the notion of an (α, β)-cyclic admissible mapping.

Definition 46 ([91])**.** *Let* X *be a nonempty set,* $T : X \to X$ *be a mapping and* $\alpha, \beta : X \times X \to \mathbb{R}_0^+$ *be two functions. We say that* T *is an* (α, β)*-cyclic admissible mapping, if for all* $x, y \in X$

(1) $\alpha(x, y) \ge 1 \Rightarrow \beta(Tx, Ty) \ge 1$,
(2) $\beta(x, y) \ge 1 \Rightarrow \alpha(Tx, Ty) \ge 1$.

Remark 11. *It is easy to see that if* $\alpha(x, y) = \beta(x, y)$, *we obtain the result from [13].*

Inspired by above results, we proved the notion of a multivalued strictly (α, β)-admissible mappings and a multivalued (α, β)-Meir–Keeler contraction with respect to partial Hausdorff metric H_p in the partial metric spaces.

Now, we prove the existence theorems for fixed points of these class of mappings.

Definition 47. *Let* X *be a nonempty set,* $T : X \to CB_p(X)$ *and* $\alpha, \beta : X \times X \to (0, \infty)$ *be three functions. We say that* T *is strictly* (α, β)*-cyclic admissible mapping, if for all* $x, y \in X$ *and* $\hat{x} \in Tx, \hat{y} \in Ty$ *with*

(1) $\alpha(x, y) > 1 \Rightarrow \beta(\hat{x}, \hat{y}) > 1$,
(2) $\beta(x, y) > 1 \Rightarrow \alpha(\hat{x}, \hat{y}) > 1$.

Remark 12. *Clearly if $\beta(x,y) = \alpha(x,y)$, we have $\alpha(x,y) > 1 \Rightarrow \beta(\hat{x},\hat{y}) > 1$, which is the multivalued version of α-admissible.*

Lemma 22. *Let X be a nonempty set and $T : X \to CB_p(X)$ be a strictly (α,β) cyclic admissible mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0,x_1) > 1$ and $\beta(x_0,x_1) > 1$, where $x_1 \in Tx_0$. Define the sequence $\{x_n\}$ by $x_{n+1} \in Tx_n$, then $\alpha(x_m,x_{m+1}) > 1$ implies that $\beta(x_n,x_{n+1}) > 1$ and $\beta(x_m,x_{m+1}) > 1$ implies that $\alpha(x_n,x_{n+1}) > 1$, for all $n,m \in \mathbb{N}_0$ with $m < n$.*

Definition 48. *Let (X,p) be a partial metric space and $\alpha,\beta : X \times X \to (0,\infty)$ be two functions. We say that $T : X \to CB_p(X)$ is an (α,β)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \le p(x,y) < \varepsilon + \delta \Rightarrow \alpha(x,y)\beta(x,y)H_p(Tx,Ty) < \varepsilon,$$

for all $x,y \in X$.

Remark 13. *If $T : X \to CB_p(X)$ is an (α,β)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p, then we have*

$$\alpha(x,y)\beta(x,y)H_p(Tx,Ty) < p(x,y),$$

for all $x,y \in X$ when $p(x,y) > 0$. On the other hand, observe that if $p(x,y) = 0$, we clearly have that $H_p(Tx,Ty) = 0$ and using [84] we obtain that $Tx = Ty$. Thus, for all $x,y \in X$, we get that

$$\alpha(x,y)\beta(x,y)H_p(Tx,Ty) \le p(x,y).$$

Remark 14. *It is also easy to see that if $\alpha(x,y)\beta(x,y) = 1$. Consequently we have obtained the multivalued version of Meir–Keeler type contractions in the framework of partial metric spaces.*

Theorem 37. *Let (X,p) be a complete partial metric space and $T : X \to CB_p(X)$ be an (α,β)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p. Suppose the following conditions hold:*

(1) T is strictly (α,β)-cyclic admissible mapping,
(2) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) > 1$ and $\beta(x_0,x_1) > 1$,
(3) If $\{x_n\}$ is a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) > 1, \beta(x_n,x_{n+1}) > 1$, then $\alpha(x_n,x) > 1, \beta(x_n,x) > 1$ for all $n \in \mathbb{N}$.

Then T has a fixed point.

Definition 49. *Let $T : X \to CB_p(X)$ be a multivalued mapping on a partial metric space (X,p).*

(1) An element $x \in X$ is called an endpoint of T if $Tx = \{x\}$. It is clear that an endpoint of T is also a fixed point of T.
(2) T has the approximate endpoint property if there exists a sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} H_p(\{x_n\},Tx_n) = 0$ or equivalently if $\inf_{x\in X}\sup_{y\in Tx} p(x,y) = 0$.

Theorem 38. *Let (X,p) be a complete partial metric space and $T : X \to CB_p(X)$ be an (α,β)-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p. Suppose the following conditions hold:*

(1) T is strictly (α,β)-cyclic admissible mapping,
(2) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) > 1$ and $\beta(x_0,x_1) > 1$,
(3) If $\{x_n\}$ is a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) > 1, \beta(x_n,x_{n+1}) > 1$, then $\alpha(x_n,x) > 1, \beta(x_n,x) > 1$ for all $n \in \mathbb{N}$.

Then T has an endpoint x if and only if T has the approximate endpoint property.

Definition 50. *Let (X, p) be a partial metric space, $\alpha : X \times X \to (0, \infty)$ be a function. We say that $T : X \to CB_p(X)$ is an α-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p, if for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq p(x,y) < \varepsilon + \delta \Rightarrow \alpha(x,y)H_p(Tx,Ty) < \varepsilon,$$

for all $x, y \in X$.

Remark 15. *If $T : X \to CB_p(X)$ is an α-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p. Then we have*

$$\alpha(x,y)H_p(Tx,Ty) < p(x,y),$$

for all $x, y \in X$ when $p(x,y) > 0$. On the other hand, observe that if $p(x,y) = 0$, we clearly have that $H_p(Tx,Ty) = 0$ and using [84], we obtain that $Tx = Ty$. Thus, for all $x, y \in X$, we get

$$\alpha(x,y)H_p(Tx,Ty) \leq p(x,y).$$

Corollary 4. *Let (X, p) be a complete partial metric space and $T : X \to CB_p(X)$ be an α-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p. Suppose the following conditions hold:*

(1) T is strictly α-admissible mapping,
(2) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) > 1$,
(3) If $\{x_n\}$ is a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$, then $\alpha(x_n, x) > 1$, for all $n \in \mathbb{N}$.

Then T has a fixed point.

Corollary 5. *Let (X, p) be a complete partial metric space and $T : X \to CB_p(X)$ be an α-Meir–Keeler contraction with respect to the partial Hausdorff metric H_p. Suppose the following conditions hold:*

(1) T is strictly α-admissible mapping,
(2) there exists $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) > 1$,
(3) If $\{x_n\}$ is a sequence such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$, then $\alpha(x_n, x) > 1$, for all $n \in \mathbb{N}$.

Then T has an endpoint x if and only if T has the approximate endpoint property.

2.17. Meir–Keeler Contractions on Various Generalized Partial Metric

2.17.1. 2018, Zhou, Zheng and Zhang, Fixed-Point Theorems in Partial b-Metric Spaces, [92]

We investigate some fixed-point theorem for Meir–Keeler mappings in partial p_b-metric [93] space which generalizes and motivate the result of Chatterjea [7] and Shukla [93].

Definition 51 ([94,95])**.** *Let X be a (nonempty) set and $s \geq 1$ a real number. A function $b : X \times X \to [0, \infty)$ is a b-metric space on X if following conditions are satisfied:*

(i) $b(x,y) = 0$ if and only if $x = y$
(ii) $b(x,y) = b(y,x)$;
(iii) $b(x,z) \leq s[b(x,y) + b(y,z)]$, for every $x, y, z \in X$.

Definition 52. *Let X be a nonempty set , $s \geq 1$ be a given real number and let $p_b : X \times X \longrightarrow [0, \infty)$ be a mapping such that for all $x, y, z \in X$, the following conditions hold:*

(Pb1)$x = y$ if and only if $p_b(x,x) = p_b(x,y) = p_b(y,y)$
(Pb2)$p_b(x,x) \leq p_b(x,y)$
(Pb3)$p_b(x,y) = p_b(y,x)$
(Pb4)$p_b(x,y) \leq s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$

Then the pair (X, p_b) is called a partial p_b-metric space. The number s is call the coefficient of (X, p_b, s).

Theorem 39. *Let (X, p_b, s) be a complete partial p_b-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a mapping satisfying the following condition:*

$$p_b(Tx, Ty) \leq \lambda[p_b(x, Ty) + p_b(y, Tx)] \quad x, y \in X,$$

where $\lambda \in \left[0, \frac{1}{2s}\right)$. Then T has unique fixed point $x^* \in X$ and $p_b(x^*, x^*) = 0$.

Remark 16 ([7]). *Let (X, b) be a metric space, a mapping $T : X \to X$ is said to be a C-contraction if there exists $\beta \in \left(0, \frac{1}{2}\right)$ such that*

$$b(Tx, Ty) \leq \beta(b(x, Ty) + p_b(y, Tx))$$

holds for all $x, y \in X$.

Taking $s = 1$ in Theorem 39, we can get the C-contraction fixed-point theorem in partial metric spaces, taking $p_b(x^*, x^*) = 0$ in Theorem 39. we can get the C-contraction fixed-point theorem in p_b- metric spaces.

Theorem 40. *Let (X, p_b, s) be a complete partial p_b-metric space with coefficient $s > 1$ and $T : X \to X$ be a mapping satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq p_b(x, x^*) < \varepsilon + \delta \Rightarrow sp_b(Tx, Tx^*) < \varepsilon$$

Then T has a unique fixed point $x^* \in X$ and $p_b(x^*, x^*) = 0$.

Remark 17. *Let (X, b, s) be a complete b-metric space with coefficient $s \geq 1, T : X \to X$ be a mapping satisfying the following condition: for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq b(x, x^*) < \varepsilon + \delta \Rightarrow sb(Tx, Tx^*) < \varepsilon$$

Then T has a unique fixed point $x^* \in X$.

2.17.2. 2019, Vujaković, Aydi, Radenović and Mukheimer, Some Remarks and New Results in Ordered Partial b-Metric Spaces, [96]

We present a new Meir–Keeler type result in partial b-metric spaces.

Theorem 41. *Let (X, p_b, s) be a p_b-complete partial b-metric space and let T be a self-mapping on X verifying:*

For $\varepsilon > 0$ there is $\delta > 0$ so that

$$\varepsilon \leq p_b(x, y) < \varepsilon + \delta \text{ implies } s \cdot p_b(Tx, Ty) < \varepsilon$$

Then T has a unique fixed point $x^* \in X$, and for each $x \in X, \lim_{n\to\infty} p_b(T^n x, x^*) = p_b(x^*, x^*) = 0$.

In [96], authors announce an open question:

Prove or disprove the following:

Let (X, p_b, s) be a p_b-complete partial b-metric space and let T be a self-mapping on X satisfying:

Given $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in X$

$$\varepsilon \leq p_b(x, y) < \varepsilon + \delta \text{ implies } p_b(Tx, Ty) < \varepsilon$$

Then T has a unique fixed point $x^* \in X$, and for each $x \in X$, $\lim_{n\to\infty} p_b(T^n x, x^*) = p_b(x^*, x^*) = 0$.

2.17.3. 2020, Hosseinzadeh and Parvaneh, Meir–Keeler Type Contractive Mappings in Modular and Partial Modular Metric Spaces, [97]

In this section, we investigate the new $\alpha\varphi - \omega$-Meir–Keeler contraction and establish some fixed-point results. Also, we introduce some fixed-point results in modular metric and partial modular metric space.

Definition 53 ([98–100])**.** *A function $\omega : (0, +\infty) \times X \times X \to [0, +\infty]$ is called a modular metric on X if the following axioms hold:*

(i) $x = y$ *if and only if* $\omega_\lambda(x, y) = 0$ *for all* $\lambda > 0$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ *for all* $\lambda > 0$ *and for all* $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x, y) \le \omega_\lambda(x, z) + \omega_\mu(z, y)$ *for all* $\lambda, \mu > 0$ *and for all* $x, y, z \in X$.
If in the Definition 53, we use the condition
(i') $\omega_\lambda(x, x) = 0$ *for all* $\lambda > 0$ *and for all* $x \in X$;

instead of (i) then ω is said to be a pseudomodular metric on X. A modular metric ω on X is called regular if the following weaker version of (i) is satisfied

$$x = y \quad \text{if and only if} \quad \omega_\lambda(x, y) = 0 \text{ for some } \lambda > 0.$$

ω is called convex if for all $\lambda, \mu > 0$ and for all $x, y, z \in X$ the inequality holds:

$$\omega_{\lambda+\mu}(x, y) \le \frac{\lambda}{\lambda+\mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda+\mu}\omega_\mu(z, y)$$

Remark 18. *If ω is a pseudomodular metric on a set X, then the function $\lambda \to \omega_\lambda(x, y)$ is nonincreasing on $(0, +\infty)$ for all $x, y \in X$. Indeed, if $0 < \mu < \lambda$, then*

$$\omega_\lambda(x, y) \le \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y).$$

Definition 54 ([98–100])**.** *Let ω be a pseudomodular on X and $x_0 \in X$ fixed. Consider the following two sets:*

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \quad \text{as} \quad \lambda \to +\infty\}$$

and

$$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \quad \text{such that } \omega_\lambda(x, x_0) < +\infty\}$$

X_ω and X^*_ω are called modular spaces (around x_0).

Now, we define the notion of $\alpha\varphi - \omega$-Meir–Keeler contractive mapping as follow.

Definition 55. *Let X_ω be a modular metric space and let T be a self-mapping on X_ω. Also, suppose that $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ and $\varphi_\lambda : X_\omega \times (0, \infty) \to [0, \infty)$ are two functions where φ_λ is lower ω-semicontinuous in X_ω. We say that T is an $\alpha\varphi - \omega$-Meir–Keeler contraction if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \le \omega_\lambda(x, y) + \varphi_\lambda(x) + \varphi_\lambda(y) < \varepsilon + \delta$$

$$\Rightarrow \alpha(x, y)[\omega_\lambda(Tx, Ty) + \varphi_\lambda(Tx) + \varphi_\lambda(Ty)] < \varepsilon,$$

for all $x, y \in X_\omega$ and for all $\lambda > 0$.

Throughout the section, $\mathrm{Fix}(T)$ will denotes the set of fixed points of T.

Theorem 42. *Let X_ω be an ω-complete modular metric space which is ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there is a function $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ such that the following assertions hold:*

(i) T is a triangular α-admissible mapping,
(ii) T is an $\alpha\varphi - \omega-$ Meir–Keeler contraction,
(iii) there exists $x_0 \in X_\omega$ such that

$$\alpha(x_0, Tx_0) \geq 1$$

(iv) T is an ω-continuous mapping.

Then, T has a fixed point $x^* \in X$ such that $\varphi_\lambda(x^*) = 0$. Further, if $\alpha(x, y) \geq 1$ for all $x, y \in$ $\mathrm{Fix}(T)$, then T has a unique fixed point.

For a self-mapping that is not ω-continuous we have the following result.

Theorem 43. *Let X_ω be a ω-complete modular metric space which is ω regular and let $T : X_\omega \to X_\omega$ be a self-mapping. Assume that there is a function $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ such that the following assertions hold:*

(i) T is a triangular α-admissible mapping,
(ii) T is $\alpha\varphi - \omega$-Meir–Keeler contraction,
(iii) there exists $x_0 \in X_\omega$ such that

$$\alpha(x_0, Tx_0) \geq 1$$

(iv) if $\{x_n\}$ is a sequence in X_ω such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ with $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$.

Then T has a fixed point $x^* \in X$ such that $\varphi_\lambda(x^*) = 0$. Further, if $\alpha(x, y) \geq 1$ for all $x, y \in \mathrm{Fix}(T)$, then T has a unique fixed point.

Fixed-point results in partial modular metric spaces

Definition 56. *A function $p : (0, +\infty) \times X \times X \to [0, +\infty)$ is called a partial modular metric on X if the following conditions hold:*

(p1) $x = y$ if and only if $p_\lambda(x, y) = p_\lambda(x, x) = p_\lambda(y, y)$ for all $\lambda > 0$;
(p2) $p_\lambda(x, y) = p_\lambda(y, x)$ for all $\lambda > 0$ and for all $x, y \in X$;
(p3) $p_\lambda(x, x) \leq p_\lambda(x, y)$ for all $\lambda > 0$ and for all $x, y \in X$;
(p4) $p_{\lambda+\mu}(x, y) \leq p_\lambda(x, z) + p_\mu(z, y) - \left[\frac{p_\lambda(x,x)+p_\lambda(z,z)+p_\mu(z,z)+p_\mu(y,y)}{2}\right]$ for all $\lambda, \mu > 0$ and for all $x, y, z \in X$.

Definition 57. *Let X_p be a partial modular metric space and let T be a self-mapping on X_p. Also, suppose that $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ is a function. We say that T is an $\alpha - p$-Meir–Keeler contractive if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq p_\lambda(x, y) < \varepsilon + \delta \Rightarrow \alpha(x, y)p_\lambda(Tx, Ty) < \varepsilon,$$

for any $x, y \in X_w$ and for all $\lambda > 0$.

Theorem 44. *Let X_p be a p-regular p-complete partial modular metric space and let $T : X_p \to X_p$ be a self-mapping. Assume that there is a function $\alpha : X_\omega \times X_\omega \to [0, +\infty)$ such that the following assertions hold:*

(i) T is a triangular α-admissible mapping,
(ii) T is $\alpha - p$-Meir–Keeler contraction,
(iii) there exists $x_0 \in X_\omega$ such that $\alpha(x_0, Tx_0) \geq 1$.
(iv) if $\{x_n\}$ is a sequence in X_p such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}_0$ with $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x) \geq 1$.

Then T has a unique fixed point $x^* \in X$ such that $p_\lambda(x^*, x^*) = 0$ for all $\lambda > 0$. Further, if $\alpha(x, y) \geq 1$ for all $x, y \in Fix\ (T)$, then T has a unique fixed point.

3. Meir–Keeler Contractions on Metric-like Spaces

3.1. 2014, Al-Mezel, Chen, Karapınar and Rakočević, Fixed-Point Results for Various-Admissible Contractive Mappings on Metric-like Spaces, [101]

In this section, we present fixed-point results for α-admissible mappings on metric-like space via various auxiliary functions. In particular, we prove the existence of a fixed point of the generalized Meir–Keeler type $\alpha - \phi$-contractive self-mapping T defined on a metric-like space X.

Firstly, we present the generalized Meir–Keeler type $\alpha - \phi$-contractive mappings. Later, we proved the existence and uniqueness of such mappings in the context of metric-like spaces.

Let Φ be the class of all function $\phi : \mathbb{R}_0^{+5} \to \mathbb{R}_0^+$ satisfying the following conditions:

> $(\phi_1)\phi$ is an increasing and continuous function in each coordinate;
> (ϕ_2) for $t > 0, \phi(t, t, t, 2t, 2t) < t, \phi(t, 0, 0, t, t) < t$, and $\phi(0, 0, t, t, 0) < t$;
> $(\phi_3)\phi(t_1, t_2, t_3, t_4, t_5) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

Here, we give main result in metric-like spaces as follows.

Definition 58. *Let (X, σ) be a metric-like space and let $\alpha : X \times X \to [0, \infty)$. One says that $T : X \to X$ is called a generalized Meir–Keeler type $\alpha - \phi$-contractive mapping if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\begin{aligned} \varepsilon &\leq \phi(\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \sigma(x, Ty), \sigma(y, Tx)) \\ &< \varepsilon + \delta \longrightarrow \alpha(x, y)\sigma(Tx, Ty) < \varepsilon, \end{aligned}$$

for all $x, y \in X$ and $\phi \in \Phi$.

Remark 19. *Note that if T is a generalized Meir–Keeler type $\alpha - \phi$-contractive mapping, then we have, for all $x, y \in X$ and $\phi \in \Phi$,*

$$\alpha(x, y)\sigma(Tx, Ty) \leq \phi(\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \sigma(x, Ty), \sigma(y, Tx)).$$

Theorem 45. *Let (X, σ) be a complete metric-like space and let $T : X \to X$ be a generalized Meir–Keeler type $\alpha - \phi$-contractive mapping where α is transitive. Suppose that*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.

Then there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 46. *Let (X, σ) be a complete metric-like space and let $T : X \to X$ be a generalized Meir–Keeler type $\alpha - \phi$-contractive mapping, where α is transitive. Suppose that*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then there exists $x^* \in X$ such that $Tx^* = x^*$.

(U) For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x, y) \geq 1$, where Fix (T) denotes the set of fixed points of T.

Next, we will show that x^* is a unique fixed point of T.

Theorem 47. *Adding condition (U) to the hypotheses of Theorem 45 (resp., Theorem 46), one obtains that x^* is the unique fixed point of T.*

Fixed-Point Theorems via the Weaker Meir–Keeler Function μ

Here, we investigate the existence and uniqueness of a fixed point of certain mappings by using the Meir–Keeler function. Now, we recall the notion of the weaker Meir–Keeler function $\mu : [0, \infty) \to [0, \infty)$.

Definition 59 (see [67]). *One calls $\mu : [0, \infty) \to [0, \infty)$ a weaker Meir–Keeler function if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that, for $t \in [0, \infty)$ with $\varepsilon \leq t < \varepsilon + \delta$, there exists $n_0 \in \mathbb{N}$ such that $\mu^{n_0}(t) < \varepsilon$.*

One denotes by $\mathbb{M}$ the class of nondecreasing functions $\mu : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(μ_1) *$\mu : [0, \infty) \to [0, \infty)$ is a weaker Meir–Keeler function;*
(μ_2) *$\mu(t) > 0$ for $t > 0$ and $\mu(0) = 0$;*
(μ_3) *for all $t > 0$, $\{\mu^n(t)\}_{n\in\mathbb{N}}$ is decreasing;*
(μ_4) *if $\lim_{n\to\infty} t_n = \gamma$, then $\lim_{n\to\infty} \mu(t_n) \leq \gamma$.*

And one denotes by Θ the class of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(φ_1) *φ is continuous;*
(φ_2) *$\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$.*

We introduce the generalized weaker Meir–Keeler type (μ, φ) - α-contractive mappings in metric-like spaces.

Definition 60. *Let (X, σ) be a metric-like space, and let $\alpha : X \times X \to \mathbb{R}_0^+$. One says that $T : X \to X$ is called a generalized weaker Meir–Keeler type $\alpha - (\mu, \varphi)$-contractive mapping if T is α-admissible and satisfies*

$$\alpha(x, y)\sigma(Tx, Ty) \leq \mu(M(x, y)) - \varphi(M(x, y))$$

for all $x, y \in X$, where $\mu \in \mathbb{M}$, $\varphi \in \Theta$, and

$$M(x, y) = \max\left\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty) \frac{\sigma(x, Ty) + \sigma(y, Tx)}{4}\right\}$$

The main result of this section is the following.

Theorem 48. *Let (X, σ) be a complete metric-like space and let $T : X \to X$ be a generalized weaker Meir–Keeler type α (μ, φ)-contractive mapping where α is transitive. Suppose that*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) T is continuous.

Then there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 49. *Let (X, σ) be a complete metric-like space and let $T : X \to X$ be a generalized weaker Meir–Keeler type α (μ, φ)-contractive mapping where α is transitive. Suppose that*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then there exists $x^* \in X$ such that $Tx^* = x^*$.

Theorem 50. *Adding condition (U) to the hypotheses of Theorem 48 (resp., Theorem 49), one obtains that x^* is the unique fixed point of T.*

3.2. 2017, Aydi, Felhi,Karapınar and Alshaikh, an Implicit Relation for Meir–Keeler Type Mappings on Metric-like Spaces, [102]

We present an implicit relation for Meir–Keeler type mappings using an auxiliary pair of notations (α, ψ) on the metric-like spaces.

Definition 61. *Let Γ be the set of all continuous functions $F(t_1, \ldots, t_6) : \mathbb{R}_{+^6} \to \mathbb{R}$ such that*
(F1) : $F(t, 0, t, 0, 0, t) \leq 0$ implies that $t = 0$;
(F2) : $F(t, 0, 0, t, t, 0) \leq 0$ implies that $t = 0$.

In [15] , Aydi et al. generalized [13] by introducing a pair of mappings defined generalized α-admissible.

Let Φ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
(Φ_1) ψ is nondecreasing;
(Φ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

These functions are called (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for any $t > 0$.

Now, we establish new generalized α-implicit Meir Keeler contractive pair of mappings in the metric-like spaces.

Definition 62. *Let (X, σ) be a metric-like space and $T, S : X \to X$ be given mappings. We say that (T, S) is a generalized α-implicit Meir–Keeler contractive pair of mappings if there exist $\psi \in \Phi, \alpha : X \times X \to [0, \infty)$ and $F \in \Gamma$ such that*

(d1) For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \psi(M(x,y)) < \varepsilon + \delta \text{ implies } \alpha(x,y)\sigma(Tx, Sy) < \varepsilon \tag{28}$$

where

$$M(x,y) = \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Sy), \frac{\sigma(x,Sy) + \sigma(y,Tx))}{4}\right\}$$

(d2)

$$F(\sigma(Tx,Sy), \sigma(x,y), \sigma(x,Tx), \sigma(y,Sy), \sigma(x,Sy), \sigma(y,Tx)) \leq 0 \tag{29}$$

for all $x, y \in X$.

Remark 20. *If we take $T = S$ in (28) and (29),*

$$\varepsilon \leq \psi(N(x,y)) < \varepsilon + \delta \text{ implies } \alpha(x,y)\sigma(Tx, Ty) < \varepsilon,$$

and

$$F(\sigma(Tx,Ty), \sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \sigma(x,Ty), \sigma(y,Tx)) \leq 0$$

where $F \in \Gamma$ and

$$N(x,y) = \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty) + \sigma(y,Tx))}{4}\right\},$$

then we say that T is a generalized α-implicit Meir–Keeler contractive mapping.

Remark 21. *It is obvious that the condition (28) yields*

$$\alpha(x,y)\sigma(Tx,Sy) \leq \psi(M(x,y)), \text{ for all } x, y \in X.$$

Theorem 51. *Let T and S be a self-mappings defined on a complete metric-like space (X,σ) and (T,S) be a generalized α-implicit Meir–Keeler contractive pair of mappings. Suppose that*

(i) (T,S) is a generalized α-admissible pair;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Then there exists a common fixed point $x^* \in X$ of T and S, that is, $x^* = Tx^* = Sx^*$ with $\sigma(x^*, Tx^*) = \sigma(x^*, Sx^*) = \sigma(x^*, x^*) = 0$.

For the uniqueness of the common fixed point of a generalized $\alpha - \psi$ contractive pair of mappings, we consider the following hypotheses.

(H0) $\alpha(STx, Tx) \geq 1$ for all $x \in X$.
(H1) For all $x, y \in \text{Fix}(T)$, there exists $z \in X$ such that $\min\{\alpha(x,z), \alpha(z,x)\} \geq 1$ and $\min\{\alpha(y,z), \alpha(z,y)\} \geq 1$.

Here, $\text{Fix}(T)$ denotes the set of fixed points of T.

Theorem 52. *Adding conditions (HO) and (H1) to the hypotheses of Theorem 51 we obtain that x^* is the unique common fixed point of T and S.*

Corollary 6. *Let T be a self-mapping defined on a complete metric-like space (X,σ). Suppose that T is an α-implicit Meir–Keeler contractive mapping. Assume that*

(i) T is an α-admissible mapping;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

Then there exists a fixed point $x^* \in X$ of T, that is, $x^* = Tx^*$ with $\sigma(x^*, Tx^*) = \sigma(Tx^*, Tx^*) = \sigma(x^*, x^*) = 0$.

We omit the proof. It is sufficient to take $S = T$ in Theorem 51.

Corollary 7. *Adding conditions (HO) and (H1) to the hypotheses of Corollary 6. we obtain that x^* is the unique fixed point of T.*

As it seen above, the implicit relation can be replaced by the continuity of both S and T. Clearly, the continuity hypothesis is heavier than the implicit relation. For the sake of completeness, we put the corresponding results with the proofs.

Definition 63. *Let (X,σ) be a metric-like space and $T : (X,\sigma) \to (X,\sigma)$ be a given mapping. T is said sequentially continuous at $x^* \in X$ if for each sequence $\{x_n\}$ in X converging to x^*, we have $Tx_n \to Tx^*$, that is, $\lim_{n\to\infty} \sigma(Tx_n, Tx^*) = \sigma(Tx^*, Tx^*)$.*

T is said sequentially continuous on X if T is sequentially continuous at each $x^* \in X$.

Remark 22. *Let (X,σ) be a metric-like space and $T : (X,\sigma) \to (X,\sigma)$ be a given mapping. If T is continuous on X, then it is sequentially continuous on X.*

We introduce new generalized sequentially continuous α contractive Meir–Keeler pair of mappings in the metric-like spaces.

Definition 64. *Let (X,σ) be a metric-like space and $T, S : X \to X$ be given mappings. We say that (T,S) is a generalized sequentially continuous α-contractive Meir–Keeler pair of mappings if there exist $\psi \in \Phi, \alpha : X \times X \to [0,\infty)$ and $F \in \Gamma$ such that*

(d1) For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \psi(M(x,y)) < \varepsilon + \delta \text{ implies } \alpha(x,y)\sigma(Tx,Sy) < \varepsilon,$$

where

$$M(x,y) = \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Sy), \frac{\sigma(x,Sy) + \sigma(y,Tx))}{4}\right\}$$

(d2) $T, S : X \to X$ are sequentially continuous.

Theorem 53. *Let T and S be self-mappings defined on a complete metric-like space (X, σ) and (T, S) be a generalized sequentially continuous α-contractive Meir Keeler pair of mappings. Suppose that*

(i) (T, S) is a generalized $\alpha-$ admissible pair;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) $\alpha(x, x) \geq 1$ for all $x \in X$ satisfying $\sigma(x, x) = 0$.

Then there exists a common fixed point $x^* \in X$ of T and S, that is, $x^* = Tx^* = Sx^*$ with $\sigma(x^*, Tx^*) = \sigma(x^*, Sx^*) = \sigma(x^*, x^*) = 0$.

Theorem 54. *Adding conditions (HO) and (H1) to the hypotheses of Theorem 53. we obtain that x^* is the unique common fixed point of T and S.*

Corollary 8. *Let T be a self-mapping defined on a complete metric-like space (X, p). Assume that T is a sequentially continuous α-contractive Meir–Keeler mapping. Suppose that*

(i) T is α-admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) $\alpha(x, x) \geq 1$ for all $x \in X$ satisfying $\sigma(x, x) = 0$.

Then there exists a fixed point $x^* \in X$ of T, that is, $x^* = Tx^*$ with $\sigma(x^*, Tx^*) = \sigma(Tx^*, Tx^*) = \sigma(x^*, x^*) = 0$.

We omit the proof since it is sufficient to take $S = T$ in Theorem 53.

Theorem 55. *Adding conditions (HO) and (H1) to the hypotheses of Corollary 8, we obtain that x^* is the unique fixed point of T.*

3.3. 2019, Karapınar, Chen and Lee, Best Proximity Point Theorems for Two Weak Cyclic Contractions on Metric-like Spaces, [103]

In this section, we the study two best proximity point theorems in the setting of metric-like spaces that are based on cyclic contraction: Meir–Keeler–Kannan type cyclic contractions.

Here, we present the best proximity point results of Meir–Keeler-Kannan type cyclic contractions, as follows;

A mapping $T : A \cup B \to A \cup B$ is called Kannan type cyclic contraction, if there exists $k \in \left(0, \frac{1}{2}\right)$ such that

$$\sigma(Tx, Ty) \leq k[\sigma(x, Tx) + \sigma(y, Ty)] + (1 - 2k)\sigma(A, B)$$

for all $x \in A$, and $y \in B$ where A, B are nonempty subsets of a metric-like space (X, σ).

In [104], authors introduce the new best proximity point result for a Kannan type cyclic contraction, as follows.

Theorem 56 ([104])**.** *Suppose that a mapping $T : A \cup B \to A \cup B$ is a Kannan type cyclic contraction, where A, B are nonempty, closed subsets of a metric-like space (X, σ) and $k \in \left[0, \frac{1}{2}\right)$. If we set $x_{n+1} = Tx_n$ for each $n \in \mathbb{N}_0$, for an arbitrary $x_0 \in A$, then*

$$\sigma(x_n, x_{n+1}) \to \sigma(A, B), \text{ as } n \to \infty$$

We have the following:

(1) If $x_0 \in A$ and $\{x_{2n}\}$ has a subsequence $\{x_{2n_k}\}$ which converges to $x^* \in A$ with $\sigma(x^*, x^*) = 0$, then, $\sigma(x^*, Tx^*) = \sigma(A, B)$.

(2) *If* $x_0 \in B$ *and* $\{x_{2n-1}\}$ *has a subsequence* $\{x_{2n_k-1}\}$ *which converges to* $x_* \in B$ *with* $\sigma(x_*, x_*) = 0$, *then* $\sigma(x_*, Tx_*) = \sigma(A, B)$.

We shall $\mathcal{M}$ denote set of all Meir–Keeler type function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. Note that if $\gamma \in \mathcal{M}$, for all $t \in (0, \infty)$ we have $\gamma(t) < t$.

Inspired by Meir–Keeler function and Kannan type cyclic contraction, we present the idea of Meir–Keeler-Kannan-type cyclic contraction.

Definition 65. *Let* $\phi \in \mathcal{M}$ *and* $T : A \cup B \to A \cup B$ *be a cyclic mapping, where A and B be nonempty subsets of a metric-like space* (X, σ). *Then, the mapping T is said to be a Meir–Keeler-Kannan type cyclic contraction, if*

$$\sigma(Tx, Ty) - \sigma(A, B) \leq \phi\left(\frac{\sigma(x, Tx) + \sigma(y, Ty)}{2} - \sigma(A, B)\right)$$

for all $x \in A$ *and all* $y \in B$.

Now, we study the best proximity point results of Meir–Keeler-Kannan type cyclic contraction. The following theorem is generalized Theorem 56.

Lemma 23. *Let* $T : A \cup B \to A \cup B$ *be a cyclic Meir–Keeler-Kannan type contraction, where A and B be nonempty closed subsets of a metric-like space* (X, σ), *and* $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \in \mathcal{M}$ *and it is increasing. For* $x_0 \in A \cup B$, *define* $x_{n+1} = Tx_n$ *for each* $n \in \mathbb{N}_0$. *Then*

$$\sigma(x_n, x_{n+1}) \to \sigma(A, B), \text{ as } n \to \infty$$

Using Lemma 23 we prove the following best proximity point theorem.

Theorem 57. *Let* $T : A \cup B \to A \cup B$ *be a cyclic Meir–Keeler-Kannan type contraction, where A and B are nonempty closed subsets of a complete metric-like space* (X, σ). *If we set* $x_{n+1} = Tx_n$ *for each* $n \in \mathbb{N}_0$, *for an arbitrary* $x_0 \in A \cup B$, *then we have the following:*

(1) *If* $x_0 \in A$ *and* $\{x_{2n}\}$ *has a subsequence* $\{x_{2n_k}\}$ *which converges to* $x^* \in A$ *with* $\sigma(x^*, x^*) = 0$, *then* $\sigma(x^*, Tx^*) = \sigma(A, B)$.

(2) *If* $x_0 \in B$ *and* $\{x_{2n-1}\}$ *has a subsequence* $\{x_{2n_k-1}\}$ *which converges to* $x_* \in B$ *with* $\sigma(x_*, x_*) = 0$, *then* $\sigma(x_*, Tx_*) = \sigma(A, B)$.

We provide an example that satisfies Theorem 57.

Example 24. *Let* $X = \mathbb{R}_0^+ \times \mathbb{R}_0^+$ *be endowed with the metric-like* $\sigma : X \times X \to \mathbb{R}_0^+$ *defined by:*

$$\sigma((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| + |y_1 - y_2|, & \text{if } (x_1, y_1), (x_2, y_2) \in [0,1] \times [0,1] \\ x_1 + x_2 + y_1 + y_2, & \text{if not} \end{cases}$$

Let $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *be defined by:*

$$\phi(t) = \frac{3t}{4}.$$

Clearly, (X, σ) *is a complete metric-like space, and* ϕ *is an increasing Meir–Keeler function. Take* $A = [0,1] \times \{0\}$ *and* $B = [0,1] \times \{1\}$, *and let* $T : A \cup B \to A \cup B$ *be defined by*

$$T((x, 0)) = \left(\frac{1}{4}x, 1\right), \text{ for all } x \in [0,1],$$

and

$$T((x, 1)) = \left(\frac{1}{4}x, 0\right), \text{ for all } x \in [0,1].$$

Then we have $\sigma(A, B) = 1$ *and* T *is a cyclic mapping.*
For $(x_1, 0) \in A$ *and* $(x_2, 1) \in B$, *we have that* $x_1, x_2 \in [0, 1]$ *and*

$$\begin{aligned}
&\sigma(T((x_1,0)), T((x_2,1))) - \sigma(A,B) \\
=&\sigma\left(\left(\frac{1}{4}x_1, 1\right), \left(\frac{1}{4}x_2, 0\right)\right) - 1 \\
=&\frac{1}{4}|x_1 - x_2| \\
\leq&\frac{1}{4}(x_1 + x_2),
\end{aligned}$$

and

$$\begin{aligned}
&\phi\left(\frac{\sigma((x_1,0), T((x_1,0))) + \sigma((x_2,1), T((x_2,1))}{2} - \sigma(A,B)\right) \\
=&\phi\left(\frac{\sigma((x_1,0),(x_1,1)) + \sigma((x_2,1),(x_2,0))}{2} - 1\right) \\
=&\phi\left(\frac{\sigma\left(\left(\frac{1}{4}x_1, 0\right), (x_1,1)\right) + \sigma\left(\left(\frac{1}{4}x_2, 1\right), (x_2,0)\right)}{2} - 1\right) \\
=&\phi\left(\frac{3}{8}(x_1 + x_2)\right) = \frac{9}{32}(x_1 + x_2).
\end{aligned}$$

Then T *is a Meir–Keeler-Kannan type cyclic contraction.*
Let $\eta_0 = (a, 0) \in A$. *Then for all* $n \in \mathbb{N}_0$, *we have*

$$\eta 2n + 1 = T^{2n+1}((a,0)) = \left(\frac{1}{4^{2n+1}}a, 1\right) \in B$$

and

$$\eta_{2n+2} = T^{2n+2}((a,0)) = \left(\frac{1}{4^{2n+2}}a, 0\right) \in A$$

Thus, we get that as $n \to \infty$

$$\sigma(\eta_{2n+1}, \eta_{2n+2}) = \left|\frac{1}{4^{2n+2}} - \frac{1}{4^{2n+1}}\right| a + 1 \to 1 = \sigma(A,B)$$

So, Lemma 23 holds and we also get that $(0,0) \in A$ *and* $(0,1) \in B$ *are the two best proximity points of* T.

3.4. 2016, Pasicki, Some Extensions of the Meir–Keeler Theorem, [105]

In this section, we present a new method of proving such Meir–Keeler theorem and give new results.

Lemma 24. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a nonnegative sequence such that*

$$a_{n+1} > 0 \text{ yields } a_{n+1} < a_n, \quad n \in \mathbb{N}. \tag{30}$$

Then $\lim_{n\to\infty} a_n = 0$ *if and only if the following condition is satisfied:*
for each $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\varepsilon < a_n < \varepsilon + \delta \text{ implies } a_{n+1} \leq \varepsilon, \quad n \in \mathbb{N}. \tag{31}$$

We use the term of dislocated metric following Hitzler and Seda [36]; dislocated metric differs from metric since $\sigma(x, y) = 0$ yields $x = y$ (no equivalence). The topology of a dislocated metric space is generated by balls. If σ is a dislocated metric, then the pair (X, σ) was first defined by Matthews as a metric domain (see [20]).

In the present section, we put $x_n = T^n x_0, n \in \mathbb{N}$.

Lemma 25. *Let (X, σ) be a dislocated metric space, and let T be a self mapping satisfying*

$$\sigma(x_{n+2}, x_{n+1}) > 0 \text{ implies } \sigma(x_{n+2}, x_{n+1}) < \sigma(x_{n+1}, x_n), \quad n \in \mathbb{N}. \tag{32}$$

Then $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$ if and only if the following condition holds:
for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < \sigma(x_{n+1}, x_n) < \varepsilon + \delta \text{ implies } \sigma(x_{n+2}, x_{n+1}) \leq \varepsilon, \quad n \in \mathbb{N}. \tag{33}$$

Definition 66. *Let (X, σ) be a dislocated metric space. Then a self mapping T on X is contractive if the following condition is satisfied:*

$$\sigma(Ty, Tx) > 0 \text{ implies } \sigma(Ty, Tx) < \sigma(y, x), \quad x, y \in X. \tag{34}$$

If $T : X \to X$ is a contractive mapping, then (32) holds for each $x_0 \in X$. Now, from Lemma 25 we obtain the following:

Corollary 9. *Let (X, σ) be a dislocated metric space, and let T be a contractive self mapping on X. Then $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$ if and only if (33) holds.*

Let us consider

$$c_T(y, x) = \max\{\sigma(y, x), \sigma(Ty, y), \sigma(Tx, x)\}$$

and

$$\sigma(Ty, Tx) > 0 \text{ implies } \sigma(Ty, Tx) < c_T(y, x), \quad x, y \in X. \tag{35}$$

Then we obtain $\sigma(x_{n+2}, x_{n+1}) > 0 \quad$ implies

$$\begin{aligned}\sigma(x_{n+2}, x_{n+1}) < c_T(x_{n+1}, x_n) &= \max\{\sigma(x_{n+1}, x_n), \sigma(x_{n+2}, x_{n+1})\} \\ &= \sigma(x_{n+1}, x_n), \quad n \in \mathbb{N}.\end{aligned}$$

Corollary 10. *Let (X, σ) be a dislocated metric space, and let T be a self mapping on X satisfying (35) (or (34) or (32)). Then $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$ if and only if (33) holds ($\varepsilon < p(\cdots) < \varepsilon + \delta$ can be also replaced by $\varepsilon < c_T(\cdots) < \varepsilon + \delta$ in (33).*

Also, each partial metric is a dislocated metric.

For a mapping $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ continuous at $(0, 0)$ and such that $\beta(0, 0) = 0$, let us consider

$$D_T(y, x) = \sigma(y, x) + \beta(\sigma(Ty, y), \sigma(Tx, x)) \tag{36}$$

$$\sigma(Ty, Tx) > 0 \quad \text{implies } \sigma(Ty, Tx) < D_T(y, x), \quad x, y \in X$$

Let us consider

$$m_T(y, x) = \max\{\sigma(y, x), \sigma(Ty, y), \sigma(Tx, x), [\sigma(Ty, x) + \sigma(Tx, y)]/2\}$$

and

$$\sigma(Ty, Tx) > 0 \text{ implies } \sigma(Ty, Tx) < m_T(y, x), \quad x, y \in X \tag{37}$$

We have

$$\begin{aligned}&[\sigma(x_{n+2}, x_n) + \sigma(x_{n+1}, x_{n+1})]/2\\ &\quad\le [\sigma(x_{n+2}, x_{n+1}) + \sigma(x_{n+1}, x_n) - \sigma(x_{n+1}, x_{n+1}) + \sigma(x_{n+1}, x_{n+1})]/2\\ &\quad= [\sigma(x_{n+2}, x_{n+1}) + \sigma(x_{n+1}, x_n)]/2 \le \max\{\sigma(x_{n+2}, x_{n+1}), \sigma(x_{n+1}, x_n)\}\\ &\quad= c_T(x_{n+1}, x_n).\end{aligned}$$

Corollary 11. *Let (X, σ) be a partial metric space, and let T be a self mapping on X satisfying (37) (or (35) or (34) or (32)). Then $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$ if and only if (33) holds ($\varepsilon < \sigma(\cdots) < \varepsilon + \delta$ can be also replaced by $\varepsilon < m_T(\cdots) < \varepsilon + \delta$ or by $\alpha < c_T(\cdots) < \varepsilon + \delta$ in (33)).*

Lemma 26. *Let (X, σ) be a dislocated metric space, and let T be a self mapping on X satisfying the following conditions:*

$$\sigma(x_{n+k+1}, x_{k+1}) > 0 \quad \textit{implies } \sigma(x_{n+k+1}, x_{k+1}) < c_T(x_{n+k}, x_k), \quad k, n \in \mathbb{N}, \tag{38}$$

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < c_T(x_{n+k}, x_k) < \varepsilon + \delta \textit{ implies } \sigma(x_{n+k+1}, x_{k+1}) \le \varepsilon, \quad k, n \in \mathbb{N}. \tag{39}$$

Then $\lim_{m,n\to\infty} \sigma(x_n, x_m) = 0$. In addition, c_T can be replaced by σ in (38) or (39) (so also in both of them). Similarly, if σ is a partial metric, then c_T can be replaced by m_T in (38) or (39).

Definition 67. *A self mapping T on a dislocated metric space (X, σ) is 0 -continuous at x if $\lim_{n\to\infty} \sigma(x, x_n) = 0$ implies $\lim_{n\to\infty} \sigma(Tx, Tx_n) = 0$ for each sequence $(x_n)_{n\in\mathbb{N}}$ in X; T is 0 -continuous if it is 0 -continuous at each point $x \in X$.*

Lemma 27. *Let (X, σ) be a dislocated metric space, and let T be a self mapping on X. If is contractive, then T has at most one fixed point; the same holds if and only if satisfies (35) or (37) and σ satisfies if and only if is 0-continuous at x (e.g., if and only if is contractive) and $\lim_{n\to\infty} \sigma(x, T^n x_0) = 0$, then $x = Tx$ and $\sigma(x, x) = 0$.*

Theorem 58. *Let T be a 0-continuous self mapping on a 0 -complete dislocated metric space (X, σ). Assume that (35) or (34) holds and the following condition is satisfied:*

for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon < c_T(y, x) < \varepsilon + \delta$ implies $\sigma(Ty, Tx) \le \varepsilon, \quad x, y \in X$.

Then T has a unique fixed point, say x, and $\lim_{n\to\infty} \sigma(x, T^n x_0) = \sigma(x, x) = 0, x_0 \in X$.

Theorem 59. *Let h be a self mapping on a 0-complete dislocated metric space (X, σ) such that $T = h^s$ (for some $s \in \mathbb{N}$) satisfies the assumptions of Theorem 58. Then h has a unique fixed point, say x, and $\lim_{n\to\infty} \sigma(x, h^n x_0) = \sigma(x, x) = 0, x_0 \in X$.*

Theorem 60. *Let T be a 0-continuous cyclic self mapping on a 0-complete dislocated metric space (X, σ), and let the following conditions be satisfied:*

$$\sigma(Ty, Tx) > 0 \textit{ implies } \sigma(Ty, Tx) < c_T(y, x), \quad x \in X_j, y \in X_{j++}, j = 1, \ldots, t,$$

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < c_T(y, x) < \varepsilon + \delta \textit{ implies } \sigma(Ty, Tx) \le \varepsilon, \quad x \in X_j, y \in X_{j++}, j = 1, \ldots, t.$$

Then T has a unique fixed point, say x, and $\lim_{n\to\infty} \sigma(x, T^n x_0) = \sigma(x, x) = 0, x_0 \in X$.

Theorem 61. *Let h be a self mapping on a 0-complete dislocated metric space (X, σ) such that $T = h^s$ (for some $s \in \mathbb{N}$) satisfies the assumptions of Theorem 60. Then has a unique fixed point, say x, and $\lim_{n\to\infty} \sigma(x, h^n x_0) = \sigma(x, x) = 0, x_0 \in X$.*

Theorem 62. *Let (X, σ) be a 0-complete dislocated metric space, and let T be a 0-continuous cyclic self mapping on X such that $\lim_{n\to\infty} \sigma(T^{n+1}x_0, T^n x_0) = 0, x_0 \in X$. Assume that the following conditions hold:*

$$\sigma(Ty, Tx) > 0 \textit{ implies } \sigma(Ty, Tx) < D_T(y, x), \quad x \in X_j, y \in X_{j++}, j = 1, \ldots, t, \tag{40}$$

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < D_T(y, x) < \varepsilon + \delta \textit{ implies } \sigma(Ty, Tx) \leq \varepsilon, \quad x \in X_j, y \in X_{j++}, j = 1, \ldots, t. \tag{41}$$

Then T has a fixed point, say x, such that $\lim_{n\to\infty} \sigma(x, T^n x_0) = 0, x_0 \in X$, and x is unique if σ is a metric. In addition, $\sigma(y, x)$ can be replaced by $c_T(y, x)$ (or by $m_T(y, x)$ if σ is a partial metric) in (36) for (40) or (41) (so also for both of them).

Theorem 63. *Let (X, σ) be a 0-complete dislocated metric space, and let T be a 0-continuous self mapping on X such that $\lim_{n\to\infty} \sigma(T^{n+1}x_0, T^n x_0) = 0, x_0 \in X$. Assume that the following conditions hold:*

$$\sigma(Ty, Tx) > 0 \textit{ implies } \sigma(Ty, Tx) < D_T(y, x), \quad x, y \in X, \tag{42}$$

for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon < D_T(y, x) < \varepsilon + \delta \textit{ implies } \sigma(Ty, Tx) \leq \varepsilon, \quad x, y \in X. \tag{43}$$

Then T has a fixed point, say x, such that $\lim_{n\to\infty} \sigma(x, T^n x_0) = 0, x_0 \in X$, and x is unique if σ is a metric. In addition, $\sigma(y, x)$ can be replaced by $c_T(y, x)$ (or by $m_T(y, x)$ if σ is a partial metric) in (36) for (42) or (43) (so also for both of them).

3.5. 2017, Pasicki, Meir and Keeler Were Right, [106]

It Is Shown Here that the Celebrated Fixed-Point Theorem of Meir–Keeler Is Equivalent to the Formally More General Result of Matkowski and Ćirić.

Meir–Keeler condition [1] was later extended by Matkowski in [107,108] and by Ćirić [9]. They used two conditions. One of them has the following form:

$$\sigma(Ty, Tx) < \sigma(y, x), \quad x \neq y$$

If σ is a metric, then the above condition is equivalent to the following one:

$$\sigma(Ty, Tx) > 0 \text{ yields } \sigma(Ty, Tx) < \sigma(y, x) \tag{44}$$

and clearly, Meir–Keeler condition implies (44) (for $\varepsilon = \sigma(y, x) > 0$).

The second condition of Matkowski and Ćirić is

for each $\varepsilon > 0$ there exists an $\delta > 0$ for which

$$\varepsilon < \sigma(y, x) < \varepsilon + \delta \text{ yields } \sigma(Ty, Tx) \leq \varepsilon.$$

If we assume that (44) holds, then the above condition is equivalent to the following one:

for each $\varepsilon > 0$ there exists an $\delta > 0$ for which

$$\sigma(y, x) < \varepsilon + \delta \text{ yields } \sigma(Ty, Tx) \leq \varepsilon, \tag{45}$$

as for $\sigma(y, x) \leq \varepsilon$ we have $\sigma(Ty, Tx) < \sigma(y, x) \leq \varepsilon$. Obviously, Meir–Keeler condition implies (44) and (45).

If $T : X \to X$ is a mapping, then for $x_n = T^n x_0, n \in \mathbb{N}$, the set $Z[x_0] = \{x_0, x_1, \ldots, x_n, \ldots\}$ is an orbit of T.

Lemma 28. *Let T be a self mapping on a dislocated metric space (X, σ), and let $Z = Z[x_0]$ be an orbit of T. If the following conditions are satisfied:*

$$\sigma\left(T^2x, Tx\right) > 0 \text{ yields } \sigma\left(T^2x, Tx\right) < \sigma(Tx, x), \quad x \in Z, \tag{46}$$

$$\text{for each } \varepsilon > 0 \text{ there exists an } \delta > 0 \text{ for which}$$

$$\sigma(Tx, x) < \varepsilon + \delta \text{ yields } \sigma\left(T^2x, Tx\right) \le \varepsilon, \quad x \in Z, \tag{47}$$

then $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$. If $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$ and (45) holds for $y = x_n, x = x_m$ and large $m, n \in \mathbb{N}$, then $\lim_{m,n\to\infty} \sigma(x_n, x_m) = 0$.

Corollary 12. *Let T be a self mapping on a dislocated metric space (X, σ), and let $Z = Z[x_0]$ be an orbit of T. Then (46), (47) yield $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$. If $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$, (45) holds for $y = x_n, x = x_m$ and large $m, n \in \mathbb{N}$, and (X, σ) is 0 -complete, then there exists an $x \in X$ such that $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = 0$.*

Lemma 29. *Let T be a self mapping on a dislocated metric space (X, σ), and let $Z = Z[x_0]$ be an orbit of T. If (46), (47) are satisfied, then $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$. If $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$, (45) holds for $y = x_n, x = x_m$ and large $m, n \in \mathbb{N}$, and (X, σ) is 0-complete, then there exists an $x \in X$ such that $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = 0$; if in addition, T is 0 -continuous at x, then $Tx = x$.*

Theorem 64. *Let T be a self mapping on a 0-complete dislocated metric space (X, σ). If (44), (45) are satisfied for all $x, y \in X$, then T has a unique fixed point x, and in addition, $\lim_{n\to\infty} \sigma(T^n x_0, x) = \sigma(Tx, x) = 0, x_0 \in X$.*

Theorem 65. *Let T be a self mapping on a 0 -complete dislocated metric space (X, σ). If Meir–Keeler condition is satisfied for all $x, y \in X$, then T has a unique fixed point x, and $\lim_{n\to\infty} \sigma(T^n x_0, x) = \sigma(Tx, x) = 0, x_0 \in X$.*

Proposition 6. *If a mapping $T : X \to X$ satisfies (44) and (45) for all $x, y \in X$, then Meir–Keeler condition for T replaced byT^2, and all $x, y \in X$*

Corollary 13. *Theorems 64 and 65 are equivalent and the same concerns the classical results of Meir Keeler, and Matkowski, Ćirić.*

The next theorem is a tool in proving fixed-point theorems, and it is a consequence of Lemma 29.

Theorem 66. *Assume that T is a self mapping on a dislocated metric space (X, σ). If (46), (47) are satisfied for an orbit Z of T, then*

$\lim_{n\to\infty} \sigma\left(T^{n+1}z, T^n z\right) = 0, z \in Z$. *If (X, σ) is 0 -complete,*

$\lim_{n\to\infty} \sigma\left(T^{n+1}x_0, T^n x_0\right) = 0$, *and (45) holds for $y = T^n x_0, x = T^m x_0$ and large $m, n \in \mathbb{N}$, then there exists an x such that $\lim_{n\to\infty} \sigma(T^n x_0, x) = \sigma(x, x) = 0$; if in addition, T is 0-continuous at x, then $Tx = x$. If (X, σ) is 0 -complete, T and each $x_0 \in X$ fulfil the above requirements and x is the unique fixed point of T, then $\lim_{n\to\infty} \sigma(T^n x_0, x) = \sigma(Tx, x) = 0$, $x_0 \in X$.*

Theorem 67. *Let T be a self mapping on a 0-complete dislocated metric space (X, σ), and let (44) hold for all $x, y \in X$. If (45) is satisfied for each orbit Z of T, then T has a unique fixed point x, and $\lim_{n\to\infty} \sigma(T^n x_0, x) = \sigma(Tx, x) = 0, x_0 \in X$*

Remark 23. *Let us note that $\sigma(y, x)$ in (44) or (45) for Theorem 67 can be replaced by*

$$c_T(y, x) = \max\{\sigma(y, x), \sigma(Ty, y), \sigma(Tx, x)\}$$

if $\sigma(y,y), \sigma(x,x) \leq \sigma(y,x)$ for c_T used in (44), and if T is 0 -continuous (see the reasoning below).

Let us present a more advanced application of Theorem 66.
Also, each partial metric is a dislocated metric.
Let us consider the following conditions

$$\sigma(Ty,Tx) > 0 \text{ yields } \sigma(Ty,Tx) < m_T(y,x), \quad x,y \in X,$$

$$\text{for each } \varepsilon > 0 \text{ there exists an } \delta > 0 \text{ for which} \tag{48}$$

$$m_T(y,x) < \varepsilon + \delta \text{ yields } \sigma(Ty,Tx) \leq \varepsilon, \quad x,y \in Z \tag{49}$$

where

$$m_T(y,x) = \max\{\sigma(y,x), \sigma(Ty,y), \sigma(Tx,x), [\sigma(Ty,x) + \sigma(Tx,y)]/2\} =$$
$$\max\{c_T(y,x), [\sigma(Ty,x) + \sigma(Tx,y)]/2\}$$

Theorem 68. *Let T be a self mapping on a 0-complete partial metric space (X,σ), and let (48) hold. If (49) is satisfied for each orbit Z of T, and T is 0 -continuous, then T has a unique fixed point x, and $\lim_{n\to\infty} \sigma(T^n x_0, x) = \sigma(Tx,x) = 0, x_0 \in X$.*

Lemma 30. *Let g,h be commuting self mappings on a dislocated metric space (X,σ), and let the following conditions hold:*

$$\sigma(hy,gx) > 0 \text{ yields } \sigma(hy,gx) < \sigma(y,x), \quad x,y \in X \tag{50}$$

for each $\varepsilon > 0$ there exists an $\delta > 0$ for which

$$\sigma(y,x) < \varepsilon + \delta \text{ yields } \sigma(hy,gx) \leq \varepsilon, \quad x,y \in X. \tag{51}$$

Then $h \circ g$ satisfies Meir–Keeler conditions for all $x,y \in X$.

Let us note, that Proposition 65 is a consequence of Lemma 30 for $g = h = T$.

Theorem 69. *Let g,h be commuting self mappings on a 0-complete dislocated metric space (X,σ). If (50), (51) hold, then g,h have a unique and common fixed point x, and $\lim_{n\to\infty} \sigma((h\circ g)^n x_0, x) = \sigma(gx,x) = \sigma(hx,x) = 0$, $x_0 \in X$*

Lemma 31. *Let g,h be commuting cyclic self mappings on a dislocated metric space (X,σ), and let the following conditions hold:*

$$\sigma(hy,gx) > 0 \text{ yields } \sigma(hy,gx) < \sigma(y,x)$$
$$x \in X_j, y \in X_{j++}, j = 1,\ldots,t$$

for each $\varepsilon > 0$ there exists an $\delta > 0$ for which

$$\sigma(y,x) < \varepsilon + \delta \text{ yields } \sigma(hy,gx) \leq \varepsilon$$
$$x \in X_j, y \in X_{j++}, j = 1,\ldots,t$$

Then for $T = h \circ g$ the following condition is satisfied:
for each $\varepsilon > 0$, there exists an $\delta > 0$ such that

$$\varepsilon \leq \sigma(y,x) < \varepsilon + \delta \text{ implies } \sigma(Ty,Tx) < \varepsilon$$
$$x \in X_j, y \in X_{j++}, j = 1,\ldots,t$$

3.6. 2017, Karapınar, a Note on Meir–Keeler Contractions on Dislocated Quasi-b-Metric, [109]

In this section, we show that Meir–Keeler type contractions possess a fixed point in the setting of dislocated quasi-b-metric.

Definition 68 ([110]). *For a nonempty set X, a dislocated quasi-b-metric is a function $\sigma_{qb} : X \times X \to \mathbb{R}_0^+$ such that for all $x, y, w \in X$ and a fixed constant $s \geq 1$:*

(σ_1) *if $\sigma_{qb}(x, y) = 0$ then $x = y$.*
(σ_2) $\sigma_{qb}(x, y) \leq s[\sigma_{qb}(x, w) + \sigma_{qb}(w, y)]$.

Moreover, the pair (X, σ_{qb}, s) is called dislocated quasi-b-metric space (DqbMS).

We recall the notion of (b)-comparison function.

For a fixed real number $s \geq 1$, let Ψ_b be all functions $\varphi_b : [0, +\infty) \to [0, +\infty)$ satisfying the conditions

(b_1) φ_b is increasing,
(b_2) there exist $k_0 \in \mathbb{N}, a \in (0, 1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that $s^{k+1}\varphi_b^{k+1}(t) \leq a s^k \varphi_b^k(t) + v_k$, for $k \geq k_0$ and any $t \in [0, \infty)$.

Any $\varphi_b \in \Psi_b$ is called (b)-comparison function [111]. For $s = 1$, in the definition above, φ_b is known as (c)-comparison functions.

We will need the following results.

Lemma 32 ([87,111,112]). *For a comparison function $\varphi_b : [0, +\infty) \to [0, +\infty)$ the following hold:*

(1) the series $\sum_{k=0}^{\infty} s^k \varphi^k(t)$ converges for any $t \in [0, +\infty)$;
(2) the function $b_s : [0, +\infty) \to [0, +\infty)$ defined by

$$b_s(t) = \sum_{k=0}^{\infty} s^k \varphi^k(t), t \in [0, \infty)$$

is increasing and continuous at 0.

(3) each iterate φ_b^k of $\varphi k \geq 1$ is also a comparison function;
(4) φ_b is continuous at 0;
(5) $\varphi_b(t) < t$ for any $t > 0$.

3.6.1. (α, ψ)-Meir–Keeler Type Contraction

We introduce the following notion which is an improved version of Meir–Keeler contraction.

Definition 69. *Let (X, σ_{qb}, s) be a DqbMS. We say that $T : X \to X$ is an (α, ψ)-Meir–Keeler type contraction if there exist two functions $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{R}_0^+$ satisfying the following condition:*

For each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \psi(\sigma_{qb}(x, y)) < \varepsilon + \delta \text{ implies } \alpha(x, y)\sigma_{qb}(Tx, Ty) < \varepsilon.$$

Notice that for an (α, ψ)-Meir–Keeler type contraction $T : X \to X$, we have $\alpha(x, y)\sigma_{qb}(Tx, Ty) \leq \psi(\sigma_{qb}(x, y))$, for any $x, y \in X$.

In what follows we shall state and prove the first main result of this section.

Theorem 70. *Suppose that (X, σ_{qb}, s) is a complete DqbMS and a self-mapping $T : X \to X$ is a (α, ψ)-Meir–Keeler type contraction. Assume also that*

(i) T is α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; and $\alpha(Tx_0, x_0) \geq 1$;
(iii) T is continuous.

Then, there exists $x \in X$ such that $Tx = x$.

Example 25. *Let $X = [0, \infty)$ endowed with $\sigma_{qb}(x, y) = |x - y| + |x|$ for all $x, y \in [0, \infty)$. It is clear that (X, σ_{qb}, s) is a complete DqbMS. Define $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$ by: $Tx = \frac{x^2}{3}$, and*

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x = y = 0; \\ 0 & \text{otherwise} . \end{cases}$$

We can prove easily T is an (α, ψ)-Meir–Keeler type contraction and it is an α-orbital admissible. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. In fact, for $x_0 = 0$, we have

$$\alpha(0, T0) = 1$$

Now, we show that T is a continuous. Let $\lim_{n\to\infty} x_n = x$ in the context of DqbMS (X, σ_{qb}, s), that is,

$$\lim_{n\to\infty} \sigma_{qb}(x_n, x) = \sigma_{qb}(x, x) = \lim_{n,m\to\infty} \sigma_{qb}(x_n, x_m).$$

We shall show that T is continuous. Indeed,

$$\begin{aligned} \lim_{n\to\infty} \sigma_{qb}(Tx_n, Tx) &= \lim_{n\to\infty} |Tx_n - Tx| + |Tx_n| \\ &= \sigma_{qb}(Tx, Tx) = |Tx - Tx| + |Tx| = Tx = \frac{x^2}{3} \\ &= \lim_{n,m\to\infty} \sigma_{qb}(Tx_n, Tx_m) = \lim_{n,m\to\infty} |Tx_n - Tx_m| + |Tx_n| \\ &= \lim_{n,m\to\infty} \left| \frac{x_n^2}{3} - \frac{x_m^2}{3} \right| + \left| \frac{x_n^2}{3} \right|. \end{aligned}$$

So all hypotheses of Theorem 70 are satisfied. Consequently, T has a fixed point. Notice that $x = 0$ is a fixed point of T.

Theorem 71. *Suppose that (X, σ_{qb}, s) is a complete DqbMS and a self-mapping $T : X \to X$ is a (α, ψ) - Meir–Keeler type contraction. Assume also that*

(i) T is α-orbital admissible;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then, there exists $x \in X$ such that $Tx = x$.

In the following example, a self-mapping T is not continuous.

Example 26. *Let $X = [0, \infty)$ endowed with the dislocated metric $\sigma(x, y) = \max\{x, y\} + |x|$ for all $x, y \in [0, \infty)$. Define $T : X \to X$ and $\alpha : X \times X \to \mathbb{R}_0^+$ by:*

$$Tx = \begin{cases} \frac{1}{2}x^3 - 1 & x > 1, \\ 0 & 0 \leq x \leq 1, \end{cases} \qquad \text{and } \alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

Clearly T is not continuous at 1 which shows that Theorem 70 is not applicable in this case.

We shall prove that a self-mapping T is an $(\alpha - \psi)$-Meir–Keeler type contraction. Let $\varepsilon > 0$ be given. Take $\delta > 0$ and suppose that $\varepsilon \leq \psi(\sigma(x, y)) < \varepsilon + \delta$ we want to show that

$$\alpha(x, y)\sigma(Tx, Ty) < \varepsilon.$$

Suppose that $\alpha(x, y) = 1$, then $x, y \in [0, 1]$ and so $Tx = 0, Ty = 0$. Hence

$$\begin{aligned}\sigma(Tx,Ty) &= \sigma(0,0)\\ &= \max\{0,0\} + |0| = 0\\ &< \varepsilon.\end{aligned}$$

Also, T is an α-orbital-admissible. To see this, let $\alpha(x,y) \geq 1$, then both $x,y \in [0,1]$. Due to definition of T, we have $Tx = 0 \in [0,1]$ and $Ty = 0 \in [0,1]$. Thus, we get $\alpha(Tx,Ty) \geq 1$.

Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Indeed, for $x_0 = 0$ we have

$$\alpha(0,T0) = \alpha(0,0) = 1 = \alpha(0,0) = \alpha(T0,0).$$

Finally, let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$. Since $\alpha(x_n, x_{n+1}) \geq 1$ for all n, by the definition of α, we have $x_n \in [0,1]$ for all n and $x \in [0,1]$, then $\alpha(x_n, x) = 1$.

So, we conclude that all hypotheses of Theorem 71 are fulfilled. So, we proved that T has a fixed point.

3.6.2. Generalized (α,ψ)-Meir–Keeler Type Contraction

Definition 70. *Suppose that (X, σ_{qb}, s) is a DqbMS. A self-mapping $T : X \to X$ is said to be a generalized (α,ψ) Meir–Keeler type contraction if there exist $\psi \in \Psi$ and $\alpha : X \times X \to \mathbb{R}_0^+$ such that for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leq \psi(P(x,y)) < \varepsilon + \delta \quad \textit{implies } \alpha(x,y)\sigma_{qb}(Tx,Ty) < \varepsilon$$

where

$$P(x,y) = \max\{\sigma_{qb}(x,y), \sigma_{qb}(x,Tx), \sigma_{qb}(y,Ty)\}.$$

If a self-mapping $T : X \to X$ is a generalized- (α,ψ)-Meir–Keeler type contraction, then we have $\alpha(x,y)\sigma_{qb}(Tx,Ty) \leq \psi(P(x,y))$, for any $x,y \in X$.

The following is the first main result of this section.

Theorem 72. *Suppose that (X, σ_{qb}, s) is a complete DqbMS, a self-mapping $T : X \to X$ is a generalized-(α,ψ) - Meir Keeler type contraction and the following conditions are fulfilled:*

(i) T is triangular α-orbital admissible mapping;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$;
(iii) T is continuous.

Then, T has a fixed point, that is, there exists $x \in X$ such that $Tx = x$.

Theorem 73. *Suppose that (X, σ_{qb}, s) is a complete DqbMS, a self-mapping $T : X \to X$ is a generalized-(α,ψ) - Meir Keeler type contraction, where $\alpha \in \psi \in$ with $\psi(t) < \frac{t}{s}$ for a constant $s \geq 1$. and the following conditions are fulfilled:*

(i) T is triangular α-orbital admissible mapping;
(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.

Then, T has a fixed point, that is, there exists $x \in X$ such that $Tx = x$.

3.6.3. The Uniqueness of the Fixed Point

Let Fix(T) denotes the set of fixed points of the mapping T.

We, first, recollect the following interesting condition for uniqueness of a fixed point of an $(\alpha - \psi)$-Meir Keeler type contraction.

(H) For all $x, y \in \mathrm{Fix}(T)$, then there exists $w \in X$ such that $\alpha(x,w) \geq 1$ and $\alpha(w,y) \geq 1$, where

Theorem 74. *Putting condition (H) to the statements of Theorem 70 (respectively, Theorem 71), we obtain that u is the unique fixed point of T.*

The following is an alternative uniqueness condition:
(U) For all $x, y \in$ Fix (T), then $\alpha(x, y) \geq 1$.

Theorem 75. *Putting condition (U) to the statements of Theorem 70(resp. Theorem 71), we find that x is the unique fixed point of T.*

In what follows, we propose the conditions for the uniqueness of a fixed point of a generalized $(\alpha - \psi)$ Meir–Keeler type contraction:

(H1) For all $x, y \in$ Fix (T), then there exists $w \in X$ such that $\alpha(x, w) \geq 1, \alpha(y, w) \geq 1$ and $\alpha(w, Tw) \geq 1$.

(H2) Let $x, y \in \text{Fix}(T)$. If there exists a sequence $\{w_n\}$ in X such that $\alpha(x, w_n) \geq 1, \alpha(y, w_n) \geq 1$ and $\alpha(w_n, w_{n+1}) \geq 1$, then

$$\sigma_{qb}(w_n, w_{n+1}) \leq \inf\Big\{\sigma_{qb}(x, w_n), \sigma_{qb}(v, w_n)\Big\}.$$

(H3) For any $x \in \text{Fix}(T)$, then $\alpha(x, x) \geq 1$.

Theorem 76. *Putting conditions (H1), (H2) and (H3) to the statements of Theorem 72 (respectively, Theorem 73), we have that x is the unique fixed point of T.*

If set $\alpha(x, y) = 1$ for all x, y in Theorem 70, we get the following result:

Theorem 77. *Suppose that (X, σ_{qb}, s) is a complete DqbMS and a self-mapping $T : X \to X$ is a (α, ψ) - Meir–Keeler type contraction. Then, there exists $x \in X$ such that $Tx = x$.*

Notice that (α, ψ)-Meir–Keeler type contraction $T : X \to X$ is non-expansive, $\sigma_{qb}(Tx, Ty) \leq \psi(\sigma_{qb}(x, y)) \leq \sigma_{qb}(x, y)$ and hence, it is continuous.
If set $\alpha(x, y) = 1$ for all x, y in Theorem 72 we find the following consequence:

Theorem 78. *Suppose that (X, σ_{qb}, s) is a complete DqbMS, a self-mapping $T : X \to X$ is a generalized-(α, ψ) - Meir Keeler type contraction. If T is continuous T has a fixed point, that is, there exists $x \in X$ such that $Tx = x$.*

3.7. 2016, Gholamian and Khanehgir, Fixed Points of Generalized Meir–Keeler Contraction Mappings in b-Metric-like Spaces, [113]

We recall the following definition.

Definition 71 ([114])**.** *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $\sigma_b : X \times X \to \mathbb{R}_0^+$ is a b-metric-like if, for all $x, y, z \in X$, the following conditions are satisfied:*

$(\sigma_b 1)\sigma_b(x, y) = 0$ *implies* $x = y$,
$(\sigma_b 2)\sigma_b(x, y) = \sigma_b(y, x)$,
$(\sigma_b 3)\sigma_b(x, y) \leq s[\sigma_b(x, z) + \sigma_b(z, y)]$.

A b-metric-like space is a pair (X, σ_b, s) such that X is a nonempty set and σ_b is a b-metric like on X. The number s is called the coefficient of (X, σ_b, s).

We establish fixed-point results for the generalized Meir–Keeler type contraction in b metric-like space.

Definition 72. *Suppose that (X, σ_b, s) is a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be generalized Meir–Keeler contraction if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \beta(\sigma_b(x,y))\sigma_b(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon \tag{52}$$

for all $x, y \in X$ where $\beta : [0,\infty) \to \left(0, \frac{1}{s}\right)$ is a given function.

Remark 24. *Let T be a generalized Meir–Keeler contractive mapping. Then it is intuitively clear that*

$$\alpha(x,y)\sigma_b(Tx,Ty) < \beta(\sigma_b(x,y))\sigma_b(x,y)$$

for all $x, y \in X$ when $x \neq y$.

The definitions of two types (type I and II) of generalized Meir Keeler contractions in b-metric-like spaces are as follows.

Definition 73. *Let (X, σ_b, s) be a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be generalized Meir–Keeler contraction of type (I) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \beta(\sigma_b(x,y))M(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon, \tag{53}$$

where

$$M(x,y) = \max\{\sigma_b(x,y), \sigma_b(Tx,x), \sigma_b(Ty,y)\} \tag{54}$$

for all $x, y \in X$.

Definition 74. *Let (X, σ_b, s) be a b-metric-like space with coefficient s. A triangular α-admissible mapping $T : X \to X$ is said to be generalized Meir–Keeler contraction of type (II) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq \beta(\sigma_b(x,y))N(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon,$$

where

$$N(x,y) = \max\left\{\sigma_b(x,y), \frac{1}{2}[\sigma_b(Tx,x) + \sigma_b(Ty,y)]\right\} \tag{55}$$

for all $x, y \in X$.

We present two important remark for our new generalized contraction.

Remark 25. *Suppose that $T : X \to X$ is a generalized Meir–Keeler contraction of type (I) (respectively, type (II)). Then*

$$\alpha(x,y)\sigma_b(Tx,Ty) < \beta(\sigma_b(x,y))M(x,y) \quad (\text{ respectively, } \beta(\sigma_b(x,y))N(x,y)),$$

for all $x, y \in X$ when $M(x,y) > 0$ (respectively, $N(x,y) > 0$).

Remark 26. *It is readily verified that $N(x,y) \leq M(x,y)$ for all $x, y \in X$, where $M(x,y)$ and $N(x,y)$ are defined in (54) and (55), respectively.*

We also present a new theorem for Meir–Keeler type contractions with a rational expression by generalization of idea of Samet et al. [115].

Theorem 79. *Let (X, σ_b, s) be a complete b-metric-like space and $T : X \to X$ be a triangular α-admissible mapping. Suppose that the following conditions hold:*

(a) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0) \geq 1$,

(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}_0$, then $\alpha(x_n, z) \geq 1$ for all $n \in \mathbb{N}_0$,*
(c) *for each $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following condition:*

$$2s\varepsilon \leq \sigma_b(y, Ty)\frac{1 + \sigma_b(x, Tx)}{1 + M(x, y)} + N(x, y) < s(2\varepsilon + \delta) \quad \text{implies}$$
$$\alpha(x, y)\sigma_b(Tx, Ty) < \varepsilon. \tag{56}$$

Then T has a fixed point in X.

Example 27 shows the validity of Theorem 79.

Example 27. *Let $X = \{0, 1, 2, 3\}$. Define $\sigma_b : X \times X \to \mathbb{R}_0^+$ as follows:*

$$\sigma_b(x, y) = \begin{cases} 4, & x = y = 0 \text{ or } 2 \text{ or } 3 \\ 0, & x = y = 1 \\ 1, & x \neq y. \end{cases}$$

Clearly, (X, σ_b, s) is a complete b-metric-like space with $s = 2$. Consider $T : X \to X$ defined by $T0 = 0, T1 = 1, T2 = 2$, and $T3 = 1$. Also, define $\alpha : X \times X \to \mathbb{R}_0^+$ as follows:

$$\alpha(x, y) = \begin{cases} \frac{1}{5}, & x + y = 1 \text{ or } 3 \\ 0, & x = y = 0 \\ 1, & x = y = 1 \\ \frac{1}{x+y+1}, & \text{otherwise} \end{cases}$$

It easily can be shown that T is triangular α-admissible. In order to check the condition (56), we choose $\delta = 4\varepsilon$ so that

$$4\varepsilon \leq \sigma_b(y, Ty)\frac{1 + \sigma_b(x, Tx)}{1 + M(x, y)} + N(x, y) < 8\varepsilon$$

which implies $\alpha(x, y)\sigma_b(Tx, Ty) < \varepsilon$.

Note that $\alpha(1, T1) \geq 1, \alpha(T1, 1) \geq 1$. Now, all conditions of Theorem 79 are satisfied and so T has a fixed point.

Also, let d_{σ_b} be the b-metric associated to b-metric-like σ_b defined by $d_{\sigma_b}(x, y) = 0$ if $x = y$ and $d_{\sigma_b}(x, y) = \sigma_b(x, y)$, elsewhere. Then condition (56) does not hold in b-metric space (X, d_{σ_b}). Let $\varepsilon = \frac{1}{4}, x = 0$, and $y = 2$. Then

$$1 = 4\varepsilon \leq d_{\sigma_b}(2, T2)\frac{1 + d_{\sigma_b}(0, T0)}{1 + M(0, 2)} + N(0, 2) = 1 < 4\varepsilon + 2\delta = 1 + 2\delta,$$

for each $\delta > 0$. But

$$\alpha(0, 2)d_{\sigma_b}(T0, T2) = \frac{1}{3} \not< \frac{1}{4}.$$

Theorem 80. *Let (X, σ_b, s) be a complete b-metric-like space and $T : X \to X$ be an α admissible mapping. Assume that there exists a function $\theta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions:*

(a) *$\theta(0) = 0$ and $\theta(t) > 0$ for every $t > 0$,*
(b) *θ is nondecreasing and right continuous,*
(c) *for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$2\varepsilon \leq \theta\left(\frac{1}{s}\sigma_b(y, Ty)\frac{1 + \sigma_b(x, Tx)}{1 + M(x, y)} + \frac{1}{S}N(x, y)\right) < 2\varepsilon + \delta \quad \text{implies}$$
$$\theta(2\alpha(x, y)\sigma_b(Tx, Ty)) < 2\varepsilon,$$

for all $x, y \in X$, then (56) is satisfied.

Theorem 81. *Let (X, σ_b, s) be a complete b-metric-like space with coefficient s and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(a) *T is an orbitally continuous generalized Meir–Keeler contraction of type (I),*
(b) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0) \geq 1$,*
(c) *if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geq 1$,*
(d) *$s > 1$ or β is a continuous function.*

Then T has a fixed point in X.

Theorem 82. *Let (X, σ_b, s) be a complete b-metric-like space, $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(a) *T is an orbitally continuous generalized Meir–Keeler contraction of type (II),*
(b) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0) \geq 1$,*
(c) *if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geq 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geq 1$,*
(d) *$s > 1$ or β is a continuous function.*

Then T has a fixed point in X.

In fact with the aid of α-admissibility of the contraction we will show that orbitally continuity assumption is not required whenever the following condition is satisfied.

(A) If $\{x_n\}$ is a sequence in X which converges to z with respect to τ_{σ_b} and satisfies $\alpha(x_{n+1}, x_n) \geq 1$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all n, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha\left(z, x_{n_k}\right) \geq 1$ and $\alpha\left(x_{n_k}, z\right) \geq 1$ for all k.

Theorem 83. *Let (X, σ_b, s) be a complete b-metric-like space with coefficient s and satisfies the condition (A). Also, let $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(a) *$T : X \to X$ is a generalized Meir–Keeler contraction of type (II),*
(b) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1, \alpha(Tx_0, x_0) \geq 1$,*
(c) *$s > 1$ or β is a continuous function.*

Then T has a fixed point in X.

Example 28. *Let (X, σ_b) and α be as in Example 27. Consider $T : X \to X$ defined by $T0 = T2 = 0$ and $T1 = 2$. Also, define $\beta : [0, +\infty) \to \left(0, \frac{1}{s}\right)$ as follows:*

$$\beta(x) = \begin{cases} \frac{1}{x}, & x = 3, 4, 8, \\ \frac{5}{9(x+1)}, & \text{otherwise} \end{cases}$$

In order to check the condition (53), we choose $\delta = \varepsilon$ so that

$$\varepsilon \leq \beta(\sigma_b(x, y))M(x, y) < \varepsilon + \delta = 2\varepsilon,$$

which implies $\alpha(x, y)\sigma_b(Tx, Ty) < \varepsilon$.

Therefore, the map T is a generalized Meir–Keeler contraction of type (I). Note that T is continuous with respect to τ_{σ_b} and $\alpha(0, T0) \geq 1, \alpha(T0, 0) \geq 1$. Now, all conditions of Theorem 81 are satisfied and so T has a fixed point.

Example 29. *Let $X = \mathbb{R}_0^+$ equipped with the b-metric-like $\sigma_b : X \times X \to \mathbb{R}_0^+$ defined by*

$$\sigma_b(x, y) = \left(x^2 + y^2\right)^2.$$

It is easy to see that (X, σ_b, s) is a complete b-metric-like space, with $s = 2$. Define the self mapping $T : X \to X$ and the functions $\alpha : X \times X \to [0, +\infty), \beta : [0, +\infty) \to \left(0, \frac{1}{s}\right)$ as follows:

$$T(x) = \begin{cases} \frac{x}{2}, & x \in [0,1], \\ \log(2x^5 + x^4 + 4x^2 + 4), & x \in (1, +\infty), \end{cases}$$

$$\alpha(x,y) = \begin{cases} 1, x, y \in [0,1], & \\ 0, & \text{otherwise} \end{cases}, \quad \beta(x) = \begin{cases} \frac{1}{4}, & x \in [0,1], \\ \frac{x}{3x+1}, & \text{otherwise.} \end{cases}$$

Then the mapping T is triangular α-admissible. On the other hand, the condition (A) holds on X. More precisely, if the sequence $\{x_n\} \subset X$ satisfies $\alpha(x_n, x_{n+1}) \geq 1, \alpha(x_{n+1}, x_n) \geq 1$, and $\lim_{n\to\infty} x_n = x$ with respect to τ_{σ_b}, for some $x \in X$, then $\{x_n\} \subset [0,1]$ and, moreover, $\lim_{n\to\infty}\left(x_n^2 + x^2\right)^2 = 4x^4$, which gives us $x = 0$. Hence $\alpha(x_n, x) \geq 1$ and $\alpha(x, x_n) \geq 1$.

Next, we prove that T is a generalized Meir–Keeler contraction. We show this in the three following steps.

Step 1. If $x \notin [0,1]$ or $y \notin [0,1]$.

In this case, $\alpha(x,y) = 0$ and evidently (52) holds.

Step 2. Let $x, y \in [0,1]$ with $\sigma_b(x,y) \in [0,1]$.

Let $\varepsilon > 0$ be given and choose $\delta = 3\varepsilon$. Now if $\varepsilon \leq \beta(\sigma_b(x,y))\sigma_b(x,y) = \frac{1}{4}\left(x^2 + y^2\right)^2 < \varepsilon + \delta = 4\varepsilon$, then

$$\alpha(x,y)\sigma_b(Tx, Ty) = \left(\frac{x^2}{4} + \frac{y^2}{4}\right)^2 = \frac{1}{16}\left(x^2 + y^2\right)^2 < \varepsilon$$

Step 3. Let $x, y \in [0,1]$ with $\sigma_b(x,y) \notin [0,1]$.

Take $\delta = 3\varepsilon$. Then the inequality

$$\varepsilon \leq \beta(\sigma_b(x,y))\sigma_b(x,y) = \frac{\left(x^2 + y^2\right)^4}{3\left(x^2 + y^2\right)^2 + 1} < 4\varepsilon,$$

implies that

$$\alpha(x,y)\sigma_b(Tx, Ty) = \frac{1}{16}\left(x^2 + y^2\right)^2 < \varepsilon.$$

Also, notice that $\alpha(0, T0) \geq 1$ and $\alpha(T0, 0) \geq 1$. Moreover, T has fixed points $x = 0$ and $x = 2$.

A remarkable fact concerning Example 29 is that the restriction of T to the interval $[0,1]$ is orbitally continuous and so by the definition of α that example fulfills all conditions of Theorem 81, too.

3.8. 2020, Gholamian, Fixed Points of Generalized α-Meir–Keeler Type Contractions and Meir–Keeler Contractions through Rational Expression in b-Metric-like Spaces, [116]

In this section, we present some fixed-point results for generalized α-Meir–Keeler contractions via rational expression in b-metric-like spaces.

Definition 75. *Let (X, σ_b, s) be a b-metric-like space with coefficient s. A triangular α admissible mapping $T : X \to X$ is said to be α-admissible Meir–Keeler contraction (or shortly α-Meir–Keeler contraction) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varepsilon \leqslant \sigma_b(x,y) < s(\varepsilon + \delta) \quad \text{implies } \alpha(x,y)\sigma_b(Tx, Ty) < \varepsilon$$

for all $x, y \in X$.

Applying definition of α-Meir–Keeler contraction, it is clear that

$$\alpha(x,y)\sigma_b(Tx, Ty) < \sigma_b(x,y)$$

for all $x, y \in X$ *when* $x \neq y$.

Remark 27. *Note that our definition of* α*-Meir–Keeler contraction is different from that of Definition 72. For this, take* $X = \{0,1,2,3\}$ *and* $\sigma_b : X \times X \to \mathbb{R}_0^+$ *defined by* $\sigma_b(x,y) = 1$, *if* $x \neq y$ *and* 0 , *otherwise. Then* (X, σ_b) *is a b-metric-like space with* $s = 2$. *Also, consider the mapping* $T : X \to X$ *defined by* $T0 = 0, T1 = T3 = 1$ *and* $T2 = 2$, *and functions* $\beta : [0,\infty) \to \left(0, \frac{1}{s}\right)$ *and* $\alpha : X \times X \to [0,\infty)$ *defined by*

$$\beta(t) = \frac{1}{t+1}, \quad \alpha(x,y) = \begin{cases} \frac{1}{5}, & x+y = 1 \text{ or } 3 \\ 0, & x = y = 0 \\ 1, & x = y = 1 \\ \frac{1}{2x+y+2}, & \text{otherwise} \end{cases}$$

It is easily can be checked that T *is an* α*-Meir–Keeler contraction. According to Definition 72, for* $x = 0, y = 3$ *and* $\varepsilon = \frac{1}{6}$ *we have* $\varepsilon \leqslant \beta(\sigma_b(0,3))\sigma_b(0,3) = \frac{1}{2} < \varepsilon + \delta$ *which does not imply that* $\alpha(0,3)\sigma_b(T0,T3) < \varepsilon$, *Since* $\alpha(0,3)\sigma_b(T0,T3) = \frac{1}{5}$.

From now on, for convenience, we denote by $\mathcal{B}_s$ the set of all functions $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \left(0, \frac{1}{s}\right)$ for a real number $s \geqslant 1$.

The definitions of two types (type I and II) of generalized α-Meir–Keeler contractions in b-metric-like spaces are as follows.

Definition 76. *Let* (X, σ_b, s) *be a b-metric-like space with coefficient s. A triangular* α *admissible mapping* $T : X \to X$ *is said to be a generalized* α*-Meir–Keeler contraction of type (I) if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leqslant M_\beta(x,y) < s(\varepsilon + \delta) \quad \text{implies } \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon, \tag{57}$$

where

$$M_\beta(x,y) = \max\{\sigma_b(x,y), \beta(x,Tx)\sigma_b(x,Tx), \beta(y,Ty)\sigma_b(y,Ty)\}$$

for all $x, y \in X$.

Definition 77. *Let* (X, σ_b, s) *be a b-metric-like space with coefficient s. A triangular* α *admissible mapping* $T : X \to X$ *is said to be a generalized* α*-Meir–Keeler contraction of type (II) if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon \leqslant N_\beta(x,y) < s(\varepsilon + \delta) \quad \text{implies } \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon,$$

where

$$N_\beta(x,y) = \max\left\{\sigma_b(x,y), \frac{1}{2}[\beta(x,Tx)\sigma_b(x,Tx) + \beta(y,Ty)\sigma_b(y,Ty)]\right\}$$

for all $x, y \in X$.

Here, we give fixed-point results for generalized α-Meir–Keeler contractions of type (I) in b-metric-like spaces..

Theorem 84. *Let* (X, σ_b, s) *be a complete b-metric like space and* $T : X \to X$ *be a mapping. Suppose that the following conditions hold:*

(a) *T is a continuous generalized* α*-Meir–Keeler contraction of type (I),*
(b) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geqslant 1$ *and* $\alpha(Tx_0, x_0) \geqslant 1$,
(c) *if* $\{x_n\}$ *is a sequence in X such that* $x_n \to z$ *as* $n \to \infty$ *and* $\alpha(x_n, x_m) \geqslant 1$ *for all* $n, m \in \mathbb{N}$, *then* $\alpha(z,z) \geqslant 1$.

Then T has a fixed point in X.

Theorem 85. *Let (X, σ_b, s) be a complete b-metric-like space and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*

(a) *T is a continuous generalized α-Meir–Keeler contraction of type (II),*
(b) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(Tx_0, x_0) \geqslant 1$,*
(c) *if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geqslant 1$ for all $n, m \in \mathbb{N}$, then $\alpha(z, z) \geqslant 1$.*

Then T has a fixed point in X.

There is an analogous result for α-Meir–Keeler contraction.

Proposition 7. *Consider a particular case of Theorem 84, whenever T is a generalized α-Meir–Keeler contraction, then T has a fixed point in X.*

Example 30. *Let $X = [0, 2]$ equipped with the b-metric-like $\sigma_b(x, y) = [\max\{x, y\}]^q$, where $q \geqslant 1$. Then (X, σ_b) is a complete b-metric-like space with $s = 2^{q-1}$. Consider the mapping $T : X \to X$ and the functions $\beta : X \times X \to \left(0, \frac{1}{2^{q-1}}\right)$ and $\alpha : X \times X \to [0, \infty)$ defined by*

$$T(x) = \frac{x}{2}, \quad \beta(x, y) = \begin{cases} \frac{1}{2^q}, & x, y \in [0, 1] \\ \frac{1}{2^{q+1}}, & \text{otherwise}, \end{cases} \quad \alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1], \\ \frac{1}{2^{2q}}, & \text{otherwise} \end{cases}$$

It easily can be shown that T is triangular α-admissible and continuous. In order to check the condition (57) without loss of generality, we may take $x \leqslant y$. Let $\varepsilon > 0$ be given. Consider the following two cases.

Case 1. If $0 \leqslant x \leqslant y \leqslant 1$, then we have $\sigma_b(Tx, Ty) = \left(\frac{y}{2}\right)^q$ and $M_\beta(x, y) = y^q$. We choose $\delta = \varepsilon$ so that $\varepsilon \leqslant M_\beta(x, y) = y^q < s(\varepsilon + \delta) = 2s\varepsilon$. It implies that $\alpha(x, y)\sigma_b(Tx, Ty) = \left(\frac{y}{2}\right)^q < \varepsilon$

Case 2 . If $0 \leqslant x \leqslant 1, 1 \leqslant y \leqslant 2$ or $1 < x \leqslant y \leqslant 2$, then we have

$$\sigma_b(Tx, Ty) = \left(\frac{y}{2}\right)^q, \quad M_\beta(x, y) = y^q.$$

We choose again $\delta = \varepsilon$ so that $\varepsilon \leqslant M_\beta(x, y) = y^q < s(\varepsilon + \delta) = 2s\varepsilon$. It follows that

$$\alpha(x, y)\sigma_b(Tx, Ty) < \left(\frac{y}{2}\right)^q < \varepsilon.$$

Therefore, the map T is a generalized α-Meir–Keeler contraction of type (I). Note that $\alpha(0, T0) \geqslant 1$ and $\alpha(T0, 0) \geqslant 1$. Now, all conditions of Theorem 84 are satisfied and so T has a fixed point.

Example 31. *Let $X = [0, \infty)$ equipped with the b-metric-like $\sigma_b : X \times X \to \mathbb{R}_0^+$ defined by*

$$\sigma_b(x, y) = \begin{cases} 0, & x = y \\ (x + y)^2, & x \neq y \end{cases}$$

It is easy to see that (X, σ_b) is a complete b-metric-like space with the coefficient $s = 2$. If we define the mapping $T : X \to X$ and the functions $\beta : X \times X \to \left(0, \frac{1}{2}\right)$ and $\alpha : X \times X \to [0, \infty)$ by

$$T(x) = \begin{cases} \frac{x}{4}, & x \in [0, 1], \\ \ln(x^2 + 1), & x \in (1, \infty), \end{cases} \quad \alpha(x, y) = \begin{cases} 1, & x, y \in [0, 1] \\ 0, & \text{otherwise} \end{cases}$$

and

$$\beta(x, y) = \begin{cases} \frac{1}{4}, & x, y \in [0, 1] \\ \frac{1}{x+y+2}, & \text{otherwise} \end{cases}$$

then the mapping T is triangular α-admissible, which is not continuous. On the other hand, the condition (A) holds. Indeed, if the sequence $\{x_n\} \subseteq X$ satisfies $\alpha(x_n, x_{n+1}) \geqslant 1$ or $\alpha(x_{n+1}, x_n) \geqslant 1$, and $\lim_{n\to\infty} x_n = x$, then $\{x_n\} \subseteq [0,1]$, and $x = 0$. Hence $\alpha(x_n, 0) \geqslant 1$ and $\alpha(0, x_n) \geqslant 1$. Next, assume that $x, y \in [0,1]$ with $x < y$. Then, for $\varepsilon > 0$, we choose $\delta = \varepsilon$ so that $\varepsilon \leqslant \beta(x,y)\sigma_b(x,y) = \frac{1}{4}(x+y)^2 < 2(\varepsilon+\delta)$. It implies that

$$\alpha(x,y)\sigma_b(Tx,Ty) = \left(\frac{x}{4} + \frac{y}{4}\right)^2 = \frac{1}{16}(x+y)^2 < \varepsilon$$

Other cases are obvious by the definition of α. Therefore, the mapping T is a generalized α-Meir–Keeler contraction. Also, notice that $\alpha(0, T0) \geqslant 1$ and $\alpha(T0, 0) \geqslant 1$. Then, we conclude that all of the assumptions of Proposition 7 are satisfied. Moreover, T has a fixed point $x = 0$.

Now, we introduce fixed-point results via rational expression in b-metric-like space.

Theorem 86. *Let (X, σ_b, s) be a complete b-metric-like space, $T : X \to X$ be a triangular α-admissible mapping and $\beta \in \mathcal{B}_s$. Suppose that the following conditions hold:*

(a) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geqslant 1$ and $\alpha(Tx_0, x_0) \geqslant 1$,
(b) if $\{x_n\}$ is a sequence in X such that $x_n \to z$ as $n \to \infty$ and $\alpha(x_n, x_m) \geqslant 1$ for all $n, m \in \mathbb{N}$, then $\alpha(x_n, z) \geqslant 1$ for all $n \in \mathbb{N}$,
(c) for each $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following condition

$$\begin{aligned} 4s\varepsilon &\leqslant \sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)} + \sigma_b(x,Tx) + \sigma_b(y,Ty) + N_\beta(x,y) < s(4\varepsilon+\delta) \\ &\Rightarrow \alpha(x,y)\sigma_b(Tx,Ty) < \varepsilon \end{aligned} \quad (58)$$

Then T has a fixed point in X.

Theorem 87. *Let (X, σ_b, s) be a b-metric-like space, $T : X \to X$ be an α-admissible mapping and $\beta \in \mathcal{B}_s$. Assume that there exists a function $\theta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions:*

(a) $\theta(0) = 0$ and $\theta(t) > 0$ for every $t > 0$,
(b) θ is nondecreasing and right continuous,
(c) for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} 4\varepsilon &\leqslant \theta\left(\frac{1}{s}\sigma_b(y,Ty)\frac{1+\sigma_b(x,Tx)}{1+M_\beta(x,y)} + \frac{1}{s}\sigma_b(x,Tx) + \frac{1}{s}\sigma_b(y,Ty) + \frac{1}{s}N_\beta(x,y)\right) < 4\varepsilon+\delta \\ &\Rightarrow \theta(4\alpha(x,y)\sigma_b(Tx,Ty)) < 4\varepsilon \end{aligned}$$

for all $x, y \in X$. Then (58) is satisfied.

4. Meir–Keeler Contractions on *M*-Metric SPACES

4.1. 2015, Asadi, Fixed-Point Theorems for Meir–Keeler Type Mappings in M-Metric Spaces with Applications, [117]

In this section, we study new fixed-point results for Meir–Keeler contraction in M-metric spaces. We also use Gupta-Saxena type contraction to obtain these results.

Firstly, we give a new version of Meir–Keeler type contraction for an M-metric space.

Definition 78. *A Meir–Keeler mapping is a mapping $T : X \to X$ on an M-metric space (X, m) such that*

$\forall \varepsilon > 0 \exists \delta > 0$ *such that* $\forall x, y \in X$ *and* $\varepsilon \leq m(x,y) < \varepsilon + \delta \Rightarrow m(Tx, Ty) < \varepsilon$.

Theorem 88. *Let (X, m) be a complete M-metric space and let T be a mapping from X into itself satisfying the following condition:*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in X \varepsilon \leq m(x,y) < \varepsilon + \delta \Rightarrow m(Tx, Ty) < \varepsilon$$

Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to x^*.

Put

$$C(x,y) = m(x,y) + \frac{(1+m(x,Tx))m(y,Ty)}{1+m(x,y)} + \frac{m(x,Tx)m(y,Ty)}{m(x,y)}.$$

Theorem 89. *Let (X,m) be a complete M-metric space and let T be a continuous mapping from X into itself satisfying the following condition:*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x,y \in X \varepsilon \leq kC(x,y) < \varepsilon + \delta \Rightarrow m(Tx,Ty) < \varepsilon, \tag{59}$$

for some $0 < k < \frac{1}{3}$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to x^*.

Here, we give a integral version of the Gupta-Saxena result.

Theorem 90. *Let (X,m) be an M-metric space and let T be a self-mapping defined on X.*

Assume that there exists a function $\varphi : [0,\infty) \to [0,\infty)$ satisfying the following:

(1) $\varphi(0) = 0$ and $t > 0 \Rightarrow \varphi(t) > 0$;
(2) φ is nondecreasing and right continuous;
(3) for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \varphi(kC(x,y)) < \varepsilon + \delta \Rightarrow \varphi(m(Tx,Ty)) < \varepsilon$$

for some $0 < k < \frac{1}{3}$ and for all $x,y \in X$ with $x \neq y$.

Then (59) is satisfied.

Corollary 14. *Let (X,m) be an M-metric space and let T be a self-mapping defined on X. Let $h : [0,\infty) \to [0,\infty)$ be a locally integrable function such that*

(1)

$$t > 0 \Rightarrow \int_0^t h(s)ds > 0;$$

(2) for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{1}{k}\varepsilon \leq \int_0^{C(x,y)} h(s)ds < \frac{1}{k}\varepsilon + \delta \Rightarrow \int_0^{\frac{1}{k}m(Tx,Ty)} h(s)ds < \frac{1}{k}\varepsilon$$

for some $0 < k < \frac{1}{3}$ and for all $x,y \in X$ with $x \neq y$.

Then (59) is satisfied.

4.2. 2021, Asim, Mujahid and Uddin, Meir–Keeler Contraction in Rectangular M-Metric Space, [118]

In this section, we investigate some results for a Meir–Keeler type contraction in rectangular M-metric space.

Definition 79 ([119]). *If X be a non-empty set. A function $r : X \times X \to \mathbb{R}_0^+$ is said to be a rectangular metric on X if it satisfies the following (for all $x,y \in X$ and for all distinct point $u,v \in X \setminus \{x,y\}$):*

(i) $r(x,y) = 0$, if and only if $x = y$,
(ii) $r(x,y) = r(y,x)$,
(iii) $r(x,y) \leq r(x,u) + r(u,v) + r(v,y)$.

Then, the pair (X,r) is called a rectangular metric space. Also, called Branciari distance space or generalized metric space.

Definition 80 ([120]). *If X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}_0^+$ is said to be a partial rectangular metric on X, if for any $x, y \in X$ and for all distinct point $u, v \in X \backslash \{x, y\}$ it satisfies the following conditions:*

(i) $x = y$ *if and only if* $\rho(x, y) = \rho(x, x) = \rho(y, y)$,

(ii) $\rho(x, x) \leq \rho(x, y)$,

(ii) $\rho(x, y) = \rho(y, x)$,

(iv) $\rho(x, y) \leq \rho(x, u) + \rho(u, v) + \rho(v, y) - \rho(u, u) - \rho(v, v)$.

Then, the pair (X, ρ) is called a partial rectangular metric space.

Notation: ref. [44] The following notations are useful in the sequel:

(i) $m_{xy} := m(x, x) \vee m(y, y) = \min\{m(x, x), m(y, y)\}$,

(ii) $M_{xy} := m(x, x) \wedge m(y, y) = \max\{m(x, x), m(y, y)\}$.

Notation: ref. [121] The following notations are useful in the sequel:

(i) $m_{r_{xy}} := \min\{m_r(x, x), m_r(y, y)\}$,

(ii) $M_{r_{xy}} := \max\{m_r(x, x), m_r(y, y)\}$.

Definition 81 ([121]). *If X be a non-empty set and $m_r : X \times X \to \mathbb{R}_0^+$ is a mapping. If it satisfying the following conditions for all $x, y \in X$:*

(i) $m_r(x, y) = m_{r_{xy}} = M_{r_{xy}} \Longleftrightarrow x = y$,

(ii) $m_{r_{xy}} \leq m_r(x, y)$,

(iii) $m_r(x, y) = m_r(y, x)$,

(iv) $\Big(m_r(x, y) - m_{r_{xy}}\Big) \leq (m_r(x, u) - m_{r_{xu}}) + (m_r(u, v) - m_{r_{uv}}) + \Big(m_r(v, y) - m_{r_{vy}}\Big)$ *for all* $u, v \in X \backslash \{x, y\}$.

Then, the pair (X, m_r) is called a rectangular M-metric space.

The following definition is new version of the definition in Meir–Keeler for a rectangular M-metric space.

Definition 82. *Let (X, m_r) be a rectangular M-metric space. A mapping $T : X \to X$ is said to be Meir–Keeler contraction if for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y \in X \quad \varepsilon \leq m_r(x, y) < \varepsilon + \delta \Rightarrow m_r(Tx, Ty) < \varepsilon$$

Theorem 91. *Let (X, m_r) be a M-complete rectangular metric space and let T a Meir–Keeler contraction. Then, T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to x^*.*

Put $C_r(x, y) = m_r(x, y) + \frac{(1+m_r(x,Tx))m_r(y,Ty)}{1+m_r(x,y)} + \frac{m_r(x,Tx)m_r(y,Ty)}{m_r(x,y)}$.

Theorem 92. *Let (X, m_r) be a complete rectangular M-metric space and let T be a continuous mapping from X into itself satisfying the following condition:*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in X \quad \varepsilon \leq KC_r(x, y) < \varepsilon + \delta \Rightarrow m_r(Tx, Ty) < \varepsilon, \tag{60}$$

for some $K \in [0, \frac{1}{3}]$. Then, T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T_n(x)\}$ converges to x^*.

Idea of Samet et al. [115], we shall state an integral version of the Gupta-Saxena result.

Theorem 93. *Let (X, m_r) be a rectangular M-metric space and T be a self mapping defined on X. Assume that there exists a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following:*

(i) $\varphi(0) = 0$ *and* $t > 0 \Rightarrow \varphi(t) > 0$;

(ii) φ is nondecreasing and right continuous;
(iii) for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \le \varphi(KC_r(x,y) < \varepsilon + \delta \Rightarrow \varphi(m_r(Tx,Ty)) < \varepsilon,$$

for some $K \in]0, \frac{1}{3}[$ and for all $x, y \in X$ with $x \neq y$.

Then (60) is satisfied.

4.3. 2022, Asim and Meenu, Fixed-Point Theorem via Meir–Keeler Contraction in Rectangular Mb-Metric Space, [122]

In this section, we give some fixed-point results for Meir–Keeler contraction in rectangular M_b-metric space.

Asim et al. [123] established rectangular M_b-metric space, as follows.

Definition 83 ([123])**.** *Let X be a non-empty set and $m_{rb} : X \times X \to [0,\infty)$ be a mapping then m_{rb} is a rectangular M_b-metric if it satisfies the following conditions:*

(1) $m_{rb}(x,x) = m_{rb}(x,y) = m_{rb}(y,y)$ *if and only if* $x = y$,
(2) $m_{rb_{x,y}} \le m_{rb}(x,y)$,
(3) $m_{rb}(x,y) = m_{rb}(y,x)$,
(4) *there exists a real number $s \ge 1$ such that*

$$\begin{aligned} m_{rb}(x,y) - m_{rb_{x,y}} \le & s\left[\left(m_{rb}(x,u) - m_{rb_{x,u}}\right) + \left(m_{rb}(u,v) - m_{rb_{u,v}}\right)\right. \\ & \left.+\left(m_{rb}(v,y) - m_{rb_{v,y}}\right)\right] - m_{rb}(u,u) - m_{rb}(v,v) \end{aligned}$$

for all $x, y \in X$ and all distinct $u, v \in X \backslash \{x,y\}$. The pair (X, m_{rb}, s) is called rectangular M_b-metric space.

Here, we give main theorem in rectangular M_b-metric space.

Theorem 94. *Let (X, m_{rb}, s) is a rectangular M_b-metric space with coefficient s. Define a triangular α-admissible mapping $T : X \to X$ such that it satisfies the following conditions:*

(1) *there exists $x_0 \in X$ for which $\alpha(x_0, Tx_0) \ge 1$ and $\alpha(Tx_0, x_0) \ge 1$.*
(2) *Let x_n be a m_{rb} convergent sequence in X, i.e., $\{x_n\} \to z$ as $n \to \infty$. Also, $\alpha(x_n, x_m) \ge 1$ and $\alpha(x_n, x) \ge 1$ for all $n, m \in \mathbb{N}$.*
(3) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that the following condition hold:*

$$2s\varepsilon \le m_{rb}(y,Ty)\frac{1 + m_{rb}(x,Tx)}{1 + M(x,y)} + N(x,y) < s(2\varepsilon + \delta)$$

then $\alpha(x,y)m_{rb}(Tx,Ty) < \varepsilon$.

Then T has a unique fixed point x^* in X.

Example 32. *Let $X = \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ and a rectangular M_b-metric is defined on X by*

$$m_{rb}(x,y) = \left(\frac{x+y}{2}\right)^2.$$

Hence (X, m_{rb}, s) is rectangular M_b-metric space with $s = 3$. Define a mapping $T : X \to X$ is defined by

$$Tx = \frac{x}{3}$$

and $\alpha : X \times X \to [0,\infty)$ by $\alpha(x,y) = \max\{x,y\}$. One can easily see that conditions (1) and (2) of Theorem 94 are satisfied. Now for condition (3), we have the following cases $(\text{ for } \delta > 0):$

Case 1: If $x = 0$ and $y = 1$, then we have

$$
\begin{aligned}
2s\varepsilon &\le m_{rb}(y, Ty)\frac{1 + m_{rb}(x, Tx)}{1 + M(x, y)} + N(x, y) < s(2\varepsilon + \delta) \\
2 \times 3\varepsilon &\le m_{rb}(1, T1)\frac{1 + m_{rb}(0, T0)}{1 + M(0, 1)} + N(0, 1) < 3(2\varepsilon + \delta) \\
&\Longrightarrow \varepsilon \le \frac{29}{312} < \varepsilon + \frac{\delta}{2} \\
&\Longrightarrow \frac{1}{36} < \varepsilon \Longrightarrow \alpha(x, y)m_{rb}(Tx, Ty) < \varepsilon.
\end{aligned}
$$

Case 2: If $x = 0$ and $y = \frac{1}{3}$, then we have

$$
\begin{aligned}
6\varepsilon &\le m_{rb}\left(\frac{1}{3}, T\frac{1}{3}\right)\frac{1 + m_{rb}(0, T0)}{1 + M\left(0, \frac{1}{3}\right)} + N\left(0, \frac{1}{3}\right) < 3(2\varepsilon + \delta) \\
&\Longrightarrow \varepsilon \le \frac{229}{18360} < \varepsilon + \frac{\delta}{2} \\
&\Longrightarrow \frac{1}{972} < \varepsilon \Longrightarrow \alpha(x, y)m_{rb}(Tx, Ty) < \varepsilon.
\end{aligned}
$$

Case 3: If $x = 0$ and $y = \frac{2}{3}$, then we have

$$
\begin{aligned}
6\varepsilon &\le m_{rb}\left(\frac{2}{3}, T\frac{2}{3}\right)\frac{1 + m_{rb}(0, T0)}{1 + M(0, \frac{2}{3})} + N\left(0, \frac{2}{3}\right) < 3(2\varepsilon + \delta) \\
&\Longrightarrow \varepsilon \le \frac{242}{5238} < \varepsilon + \frac{\delta}{2} \\
&\Longrightarrow \frac{2}{243} < \varepsilon \Longrightarrow \alpha(x, y)m_{rb}(Tx, Ty) < \varepsilon.
\end{aligned}
$$

Case 4: If $x = \frac{1}{3}$ and $y = \frac{2}{3}$, then we have

$$
\begin{aligned}
6\varepsilon \quad &\le m_{rb}\left(\frac{2}{3}, T\frac{2}{3}\right)\frac{1 + m_{rb}\left(\frac{1}{3}, T\frac{1}{3}\right)}{1 + M\left(\frac{1}{3}, \frac{2}{3}\right)} + N\left(\frac{1}{3}, \frac{2}{3}\right) < 3(2\varepsilon + \delta) \\
&\Longrightarrow \varepsilon \le \frac{10913}{157464} < \varepsilon + \frac{\delta}{2} \\
&\Longrightarrow \frac{1}{54} < \varepsilon \Longrightarrow \alpha(x, y)m_{rb}(Tx, Ty) < \varepsilon.
\end{aligned}
$$

Case 5: If $x = \frac{1}{3}$ and $y = 1$, then we have

$$
\begin{aligned}
6\varepsilon &\le m_{rb}(1, T1)\frac{1 + m_{rb}\left(\frac{1}{3}, T\frac{1}{3}\right)}{1 + M\left(\frac{1}{3}, 1\right)} + N\left(\frac{1}{3}, 1\right) < 3(2\varepsilon + \delta) \\
&\Longrightarrow \varepsilon \le \frac{100}{1053} < \varepsilon + \frac{\delta}{2} \\
&\Longrightarrow \frac{4}{81} < \varepsilon \Longrightarrow \alpha(x, y)m_{rb}(Tx, Ty) < \varepsilon.
\end{aligned}
$$

Case 6: If $x = \frac{2}{3}$ and $y = 1$, then we have

$$\begin{aligned}
6\varepsilon &\leq m_{rb}(1, T1)\frac{1 + m_{rb}(\frac{2}{3}, T\frac{2}{3})}{1 + M(\frac{2}{3}, 1)} + N\left(\frac{2}{3}, 1\right) < 3(2\varepsilon + \delta) \\
&\Longrightarrow \varepsilon \leq \frac{19309}{118584} < \varepsilon + \frac{\delta}{2} \\
&\Longrightarrow \frac{25}{324} < \varepsilon \Longrightarrow \alpha(x, y)m_{rb}(Tx, Ty) < \varepsilon.
\end{aligned}$$

Therefore, the condition (3) is also satisfied for some $\delta > 0$. Thus, the example meets all the hypothesis of Theorem 94. Hence $x^* = 0$ is a unique fixed point of the mapping T.

The following corollary is a sharpened version of Corollary 3.3 of Samet et al. [115].

Corollary 15. *In Theorem 94, if we replace condition (3) by*

(1) Assume that for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$\begin{aligned}
2s\varepsilon &\leq \int_0^{m_{rb}(y,Ty)\frac{1+m_{rb}(x,Tx)}{1+M(x,y)}+N(x,y)} \psi(t)dt < s(2\varepsilon + \delta) \\
&\Longrightarrow \int_0^{\alpha(x,y)m_{rb}(Tx,Ty)} \psi(t)dt < \varepsilon
\end{aligned}$$

where $\psi : [0, \infty) \to [0, \infty)$ be a locally integrable function such that

$$\int_0^t \psi(u)du > 0 \ \ \forall t > 0.$$

Then T has a unique fixed point x^* in X.

5. Conclusions

In this paper, we aim to present some results for Meir–Keeler contractions on partial metric spaces and on their generalizations metric-like spaces and M-metric spaces. Meir–Keeler contraction is an attractive research topic of fixed-point theory. We prove the existence and uniqueness of the fixed point for this contraction. Such fixed-point theorems can be applied both in mathematics and in the natural and applied sciences. In future studies, it seems that the analog of the presented Meir–Keeler contraction results can be studied in some other abstract spaces or/and different type control functions.

Author Contributions: All authors contributed equally and significantly in writing this paper. All authors have read and agreed to publish the present version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors wish to thank the referees for their careful reading of the manuscript and valuable suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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