



Research article

Further studies on ordinary differential equations involving the M -fractional derivative

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Abstract: In the current paper, the power series based on the M -fractional derivative is formally introduced. More peciesely, the Taylor and Maclaurin expansions are generalized for fractional-order differentiable functions in accordance with the M -fractional derivative. Some new definitions, theorems, and corollaries regarding the power series in the M sense are presented and formally proved. Several ordinary differential equations (ODEs) involving the M -fractional derivative are solved to examine the validity of the results presented in the current study.

Keywords: M -fractional derivative; power series; new definitions, theorems and corollaries; ordinary differential equations

Mathematics Subject Classification: 34A08, 35C10

1. Introduction

Fractional differential equations (FDEs) are precision tools to describe many nonlinear phenomena from porous media to other areas of scientific disciplines. Researchers have used different local and nonlocal fractional derivatives to model the phenomena around them. For example, Yang *et al.* [1]

considered an advection-dispersion equation with the conformable derivative and obtained its analytical solutions using the Fourier transform. Hosseini *et al.* [2] studied the Biswas-Arshed model involving the beta derivative and derived its soliton waves through the Jacobi and Kudryashov techniques. In [3], the authors steered an analytical study on a Caputo time-fractional equation using a capable analytic scheme. In another paper, Sulaiman *et al.* [4] explored the coupled Burgers system involving the Mittag-Leffler kernel through the Laplace homotopy perturbation method. Generally, the most widely used fractional derivatives that have been adopted by many authors are the conformable derivative [5–8], the beta derivative [9–12], the Caputo derivative [13–16], and the Atangana-Baleanu derivative [17–20]. For more information about the fractional derivatives, see [21–30].

The M -fractional derivative is another type of fractional derivatives that lies in the class of the local fractional derivatives (Compared to the nonlocal fractional derivatives such as the Caputo fractional derivative). This local fractional derivative is a generalization of other local fractional derivatives like the conformable fractional derivative. The M -fractional derivative was first proposed by Sousa and Oliveira in [31] that encompasses a number of ordinary derivative properties such as linearity, product rule, etc. Sousa and Oliveira [31] also developed a series of classical results from the Rolle's theorem to other theorems in the M sense. Such results led to the use of this well-behaved derivative in the studies of many researchers. In this respect, Yusuf *et al.* [32] gained solitons of the Ginzburg-Landau equation involving the M -fractional derivative using the generalized Bernoulli method. Özkan [33] used the simplest equation approach to derive exact solutions of Biswas-Arshed equation with the M -fractional derivatives. Tariq *et al.* [34] found optical solitons of Schrödinger-Hirota equation involving the M -fractional derivative through the Fan's method. Zafar *et al.* [35] tried to acquire optical solitons of Biswas-Arshed model with the M -fractional derivative using the sinh-Gordon method.

For $f : [0, \infty) \rightarrow \mathbb{R}$, the M -fractional derivative of f of order α is given by [31]

$${}_iD_M^{\alpha,\beta} f(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(x_i E_\beta(\varepsilon x^{-\alpha})) - f(x)}{\varepsilon}, \quad (1.1)$$

where $x > 0$ and $\alpha \in (0, 1)$. Here, ${}_iE_\beta(\cdot)$, $\beta > 0$ is the Mittag-Leffler function [36]. If the M -fractional derivative of f of order α exists, then it is said that f is α -differentiable. Note that for the α -differentiable function f , one can define

$${}_iD_M^{\alpha,\beta} f(0) = \lim_{x \rightarrow 0^+} {}_iD_M^{\alpha,\beta} f(x),$$

provided that

$$\lim_{x \rightarrow 0^+} {}_iD_M^{\alpha,\beta} f(x),$$

exists.

It can be readily shown that for the α -differentiable functions, the M -fractional derivative satisfies the following features [31]:

- A. ${}_iD_M^{\alpha,\beta} (af + bg) = a({}_iD_M^{\alpha,\beta} f) + b({}_iD_M^{\alpha,\beta} g)$, $a, b \in \mathbb{R}$.
- B. ${}_iD_M^{\alpha,\beta} x^p = \frac{p}{\Gamma(\beta+1)} x^{p-\alpha}$, $p \in \mathbb{R}$.
- C. If $f(x) = c$, then ${}_iD_M^{\alpha,\beta} f = 0$.
- D. ${}_iD_M^{\alpha,\beta} (fg) = g({}_iD_M^{\alpha,\beta} f) + f({}_iD_M^{\alpha,\beta} g)$.

$$E. {}_iD_M^{\alpha,\beta} \left(\frac{f}{g} \right) = \frac{g({}_iD_M^{\alpha,\beta} f) - f({}_iD_M^{\alpha,\beta} g)}{g^2}.$$

$$F. {}_iD_M^{\alpha,\beta} (f \circ g)(x) = f'(g(x)) {}_iD_M^{\alpha,\beta} g(x), \text{ where } f \text{ is differentiable at } g(x).$$

$$G. {}_iD_M^{\alpha,\beta} f(x) = \frac{x^{1-\alpha}}{\Gamma(\beta+1)} \frac{df}{dx}.$$

H. If $a > 0$ and $f : [a, b] \rightarrow \mathbb{R}$ is continuous and α -differentiable for some $\alpha \in (0, 1)$, then there is $c \in (a, b)$ such that

$${}_iD_M^{\alpha,\beta} f(c) = \alpha \left(\frac{f(b) - f(a)}{b^\alpha - a^\alpha} \right), \quad \beta > 0.$$

Abdeljawad [37] introduced the conformable power series and applied such a representation for a group of certain functions. The main aim of the current paper is to introduce the power series based on the M -fractional derivative and prove some new theorems and corollaries regarding it.

The next sections of the present paper are as follows: In Section 2, the power series based on the M -fractional derivative is introduced. More precisely, the Taylor and Maclaurin expansions are generalized for fractional-order differentiable functions in accordance with the M -fractional derivative. Furthermore, some new definitions, theorems, and corollaries regarding the power series in the M sense are presented and formally proved, in this section. In Section 3, a number of ODEs with the M -fractional derivative are solved to examine the validity of the results presented. The paper totalizes the outcomes in Section 4.

2. New definitions, theorems and corollaries

In the current section, some new definitions, theorems, and corollaries regarding the power series in the M sense are presented and formally proved.

Definition 2.1. *An infinite series*

$$a_0 + \sum_{n=1}^{\infty} a_n x^{n\alpha},$$

is called an α -power series in x^α . Additionally, the series

$$a_0 + \sum_{n=1}^{\infty} a_n (x^\alpha - x_0^\alpha)^n,$$

is known as an α -power series in $x^\alpha - x_0^\alpha$ which is more general than the previous one.

Definition 2.2. *An infinite α -power series*

$$f(x_0) + \sum_{n=1}^{\infty} \left(\frac{\Gamma(\beta+1)}{\alpha} \right)^n \frac{{}_iD_M^{\alpha,\beta} f(x_0)}{n!} (x^\alpha - x_0^\alpha)^n,$$

is referred to as the α -Taylor expansion of the function f at x_0 provided that f is infinitely α -differentiable at x_0 .

Definition 2.3. *An infinite α -power series*

$$f(0) + \sum_{n=1}^{\infty} \left(\frac{\Gamma(\beta+1)}{\alpha} \right)^n \frac{{}_iD_M^{\alpha,\beta} f(0)}{n!} x^{n\alpha},$$

is known as the α -Maclaurin expansion of the function f provided that f is infinitely α -differentiable at $x_0 = 0$.

Definition 2.4. A sequence $\{f_n\}$ is called convergent uniformly to f on the set $E \subseteq \mathbb{R}$, if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in E$, $|f_n(x) - f(x)| < \varepsilon$.

Theorem 2.5. Assume $\{f_n\}$ converges uniformly to f on the set $E \subseteq \mathbb{R}$. Let x be a limit point of E and let $\lim_{t \rightarrow x} f_n(t) = l_n$. Then, $\{l_n\}$ converges and $\lim_{t \rightarrow x} f(t) = \lim_{n \rightarrow \infty} l_n$. Particularly

$$\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t).$$

For the proof, see [38, 39].

Theorem 2.6. (Uniform convergence and the truncated M -fractional derivative) Let $0 < \alpha < 1$ and $a \geq 0$. Suppose $\{f_n\}$ is M -fractional differentiable on (a, b) such that $\{f_n(x_0)\}$ converges for some point x_0 on (a, b) . If $\{{}_iD_M^{\alpha, \beta} f_n\}$ converges uniformly on (a, b) , then $\{f_n\}$ converges uniformly on (a, b) to a function f and for every $x \in (a, b)$, we have

$${}_iD_M^{\alpha, \beta} f(x) = \lim_{n \rightarrow \infty} {}_iD_M^{\alpha, \beta} f_n(x).$$

Proof. Suppose $\varepsilon > 0$ and consider $N_1 \in \mathbb{N}$ such that $m, n \geq N_1$. Now, $t \in (a, b)$ implies

$$|f_m(x_0) - f_n(x_0)| < \frac{\varepsilon}{2},$$

and

$$|{}_iD_M^{\alpha, \beta} f_m(t) - {}_iD_M^{\alpha, \beta} f_n(t)| < \frac{\alpha \varepsilon}{2(b^\alpha - a^\alpha)}. \quad (2.1)$$

If we apply the mean value theorem (H) to the function $f_m - f_n$ where $m, n \geq N_1$, from (2.1), we find

$$\begin{aligned} |f_m(x) - f_n(x) - f_m(t) + f_n(t)| &= |(f_m(x) - f_n(x)) - (f_m(t) - f_n(t))| \\ &\leq \frac{1}{\alpha} |{}_iD_M^{\alpha, \beta} f_m(z) - {}_iD_M^{\alpha, \beta} f_n(z)| |x^\alpha - t^\alpha| \\ &\leq \frac{|x^\alpha - t^\alpha| \alpha \varepsilon}{2\alpha(b^\alpha - a^\alpha)} \\ &\leq \frac{\varepsilon}{2}, \end{aligned}$$

for every $x, t \in (a, b)$, where z is a point between x and t . Thus, for every $x \in (a, b)$ and $m, n \geq N_1$, the following

$$\begin{aligned} |f_m(x) - f_n(x)| &= |(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0)) + (f_m(x_0) - f_n(x_0))| \\ &\leq |(f_m(x) - f_n(x)) - (f_m(x_0) - f_n(x_0))| + |f_m(x_0) - f_n(x_0)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

implies that $\{f_n\}$ converges uniformly on (a, b) . Let $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ and $x \in (a, b)$.

Let us fix a point c on (a, b) . Suppose that

$$h(\lambda) = \left({}_i E_\beta(\lambda c^{-\alpha})\right)^\alpha,$$

then

$$\begin{aligned} h(0) &= 1, \\ \frac{dh}{d\lambda}(0) &= \frac{\alpha}{\Gamma(\beta + 1)} c^{-\alpha}. \end{aligned}$$

Thus, there exists a positive number γ such that

$$\left| \left({}_i E_\beta(\lambda c^{-\alpha})\right)^\alpha - c^\alpha \right| = c^\alpha \left| {}_i E_\beta(\lambda c^{-\alpha})^\alpha - 1 \right| = c^\alpha |h(\lambda) - h(0)| < c^\alpha \frac{2\alpha|\lambda|}{\Gamma(\beta + 1)} c^{-\alpha} = \frac{2\alpha|\lambda|}{\Gamma(\beta + 1)},$$

for $0 < |\lambda| < \gamma$. Furthermore, there exists $N_2 \in \mathbb{N}$ such that $m, n \geq N_2$. Now, $x \in (a, b)$ implies

$$\left| {}_i D_M^{\alpha, \beta} f_m(x) - {}_i D_M^{\alpha, \beta} f_n(x) \right| < \frac{\Gamma(\beta + 1)}{2} \varepsilon.$$

Now, for $0 < |\lambda| < \gamma$, one can define

$$\begin{aligned} g_n(\lambda) &= \frac{f_n\left({}_i E_\beta(\lambda c^{-\alpha})\right) - f_n(c)}{\lambda}, \\ g(\lambda) &= \frac{f\left({}_i E_\beta(\lambda c^{-\alpha})\right) - f(c)}{\lambda}. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} g_n(\lambda) = g(\lambda)$ and $\lim_{\lambda \rightarrow 0} g_n(\lambda) = {}_i D_M^{\alpha, \beta} f_n(c)$, for $m, n \geq N_2$, we have

$$\begin{aligned} |g_m(\lambda) - g_n(\lambda)| &= \left| \frac{f_m\left({}_i E_\beta(\lambda c^{-\alpha})\right) - f_m(c)}{\lambda} - \frac{f_n\left({}_i E_\beta(\lambda c^{-\alpha})\right) - f_n(c)}{\lambda} \right| \\ &= \frac{1}{|\lambda|} \left| \left(f_m\left({}_i E_\beta(\lambda c^{-\alpha})\right) - f_n\left({}_i E_\beta(\lambda c^{-\alpha})\right) \right) - (f_m(c) - f_n(c)) \right| \\ &= \frac{1}{\alpha|\lambda|} \left| \left({}_i E_\beta(\lambda c^{-\alpha})\right)^\alpha - c^\alpha \right| \left| {}_i D_M^{\alpha, \beta} (f_m - f_n)(z) \right| \\ &< \frac{1}{\alpha|\lambda|} \frac{2\alpha|\lambda|}{\Gamma(\beta + 1)} \frac{\Gamma(\beta + 1)}{2} \varepsilon \\ &= \varepsilon, \end{aligned}$$

where z is a point between 0 and λ . This shows $\{g_n\}$ converges uniformly to g on $0 < |\lambda| < \gamma$. Theorem 1 implies that $\lim_{\lambda \rightarrow 0} g(\lambda)$ exists and $\lim_{\lambda \rightarrow 0} g(\lambda) = \lim_{n \rightarrow \infty} \lim_{\lambda \rightarrow 0} g_n(\lambda)$. This means that ${}_i D_M^{\alpha, \beta} f(c)$ exists and

$${}_i D_M^{\alpha, \beta} f(c) = \lim_{\lambda \rightarrow 0} g(\lambda) = \lim_{\lambda \rightarrow 0} \lim_{n \rightarrow \infty} g_n(\lambda) = \lim_{n \rightarrow \infty} \lim_{\lambda \rightarrow 0} g_n(\lambda) = \lim_{n \rightarrow \infty} {}_i D_M^{\alpha, \beta} f_n(c).$$

□

Corollary 2.7. In Theorem 2, if for every $n \in \mathbb{N}$, f_n is differentiable in the usual context, then property (G) implies

$${}_iD_M^{\alpha,\beta} f(x) = \frac{x^{1-\alpha}}{\Gamma(\beta+1)} \lim_{n \rightarrow \infty} f_n'(x),$$

for all $x \in (a, b)$.

Theorem 2.8. Suppose that $0 < \alpha < 1$, $0 < R \leq x_0^\alpha$ and the α -power series

$$\sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n,$$

converges on $I = \left((x_0^\alpha - R)^{\frac{1}{\alpha}}, (x_0^\alpha + R)^{\frac{1}{\alpha}} \right)$, and $f(x) = \sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n$ for $x \in I$. Then the series

$$\sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n,$$

converges uniformly on every closed interval of I . The function f is continuous and α -differentiable in I , and

$${}_iD_M^{\alpha,\beta} f(x) = \frac{\alpha}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1}.$$

Proof. Suppose $[a, b] \subseteq I$ and p is a point in $[a, b]$ such that for every $x \in [a, b]$, $|x^\alpha - x_0^\alpha| \leq |p^\alpha - x_0^\alpha|$. Then,

$$|a_n (x^\alpha - x_0^\alpha)^n| < |a_n (p^\alpha - x_0^\alpha)^n|,$$

for all $x \in [a, b]$. Since

$$\sum_{n=0}^{\infty} a_n (p^\alpha - x_0^\alpha)^n,$$

converges absolutely, the Weierstrass M -test yields the uniform convergence of the series

$$\sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n,$$

on $[a, b]$. Since

$$\lim_{n \rightarrow \infty} \sup \sqrt[n]{\frac{\alpha}{\Gamma(\beta+1)} n |a_n|} = \lim_{n \rightarrow \infty} \sup \sqrt[n]{|a_n|},$$

the series

$$\sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n$$

and

$$\frac{\alpha}{\Gamma(\beta+1)} \sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1}$$

have a similar interval of convergence. Accordingly

$$\frac{\alpha}{\Gamma(\beta + 1)} \sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1},$$

converges uniformly on every $[a, b] \subseteq I$. Now, if

$$s_n(x) = \sum_{k=0}^n a_k (x^\alpha - x_0^\alpha)^k,$$

then

$${}_i D_M^{\alpha, \beta} s_n(x) = \frac{\alpha}{\Gamma(\beta + 1)} \sum_{k=1}^n k a_k (x^\alpha - x_0^\alpha)^{k-1}.$$

Since the sequences $\{s_n\}$ and $\{{}_i D_M^{\alpha, \beta} s_n\}$ converge uniformly on $[a, b]$, they also converge uniformly on (a, b) . Therefore, Theorem 2 and its corollary imply that ${}_i D_M^{\alpha, \beta} f(x)$ exists on (a, b) and

$${}_i D_M^{\alpha, \beta} f(x) = \lim_{n \rightarrow \infty} {}_i D_M^{\alpha, \beta} s_n(x) = \frac{\alpha}{\Gamma(\beta + 1)} \sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1}.$$

But, for any x which $x \in I$, there exists a closed interval $[a, b]$ such that $x \in (a, b) \subseteq [a, b] \subseteq I$. This reveals that ${}_i D_M^{\alpha, \beta} f(x)$ exists for any $x \in I$ and

$${}_i D_M^{\alpha, \beta} f(x) = \frac{\alpha}{\Gamma(\beta + 1)} \sum_{n=1}^{\infty} n a_n (x^\alpha - x_0^\alpha)^{n-1}.$$

The continuity of f is yielded from the existence of ${}_i D_M^{\alpha, \beta} f$. □

Corollary 2.9. *Under the hypotheses of Theorem 3, f has M -fractional derivatives of all orders in*

$$\left((x_0^\alpha - R)^{\frac{1}{\alpha}}, (x_0^\alpha + R)^{\frac{1}{\alpha}} \right),$$

which are given by

$${}_i^k D_M^{\alpha, \beta} f(x) = \left(\frac{\alpha}{\Gamma(\beta + 1)} \right)^k \sum_{n=k}^{\infty} n(n-1) \times \dots \times (n-k+1) a_n (x^\alpha - x_0^\alpha)^{n-k}.$$

In particular

$${}_i^k D_M^{\alpha, \beta} f(x_0) = \left(\frac{\alpha}{\Gamma(\beta + 1)} \right)^k k! a_k.$$

Corollary 2.10. *Suppose $0 < \alpha < 1$, $R > 0$, and the α -power series*

$$\sum_{n=0}^{\infty} a_n x^{n\alpha},$$

converges on $(0, R^{\frac{1}{\alpha}})$, and $f(x) = \sum_{n=0}^{\infty} a_n x^{n\alpha}$, where $0 < x < R^{\frac{1}{\alpha}}$. Then, the series

$$\sum_{n=0}^{\infty} a_n x^{n\alpha},$$

converges uniformly on every closed interval of $(0, R^{\frac{1}{\alpha}})$. The function f is continuous and α -differentiable in $(0, R^{\frac{1}{\alpha}})$, and

$${}_i D_M^{\alpha, \beta} f(x) = \frac{\alpha}{\Gamma(\beta + 1)} \sum_{n=1}^{\infty} n a_n (x^\alpha)^{n-1}.$$

It is easy to show that

$${}_i^k D_M^{\alpha, \beta} f(x) = \left(\frac{\alpha}{\Gamma(\beta + 1)} \right)^k \sum_{n=k}^{\infty} n(n-1) \times \cdots \times (n-k+1) a_n (x^\alpha)^{n-k}.$$

Since $\lim_{x \rightarrow 0^+} {}_i^k D_M^{\alpha, \beta} f(x)$ exists for $k = 0, 1, \dots$,

$${}_i^k D_M^{\alpha, \beta} f(0) = \left(\frac{\alpha}{\Gamma(\beta + 1)} \right)^k k! a_k.$$

Corollary 2.11. *If two α -power series*

$$\sum_{n=0}^{\infty} a_n (x^\alpha - x_0^\alpha)^n, \text{ and } \sum_{n=0}^{\infty} b_n (x^\alpha - x_0^\alpha)^n,$$

represent the same function in a neighborhood, then $a_n = b_n$ for all n . This means that the α -power series expansion of a function about a given point is uniquely determined.

3. Applications

In this section, by using the α -power series method, several linear and nonlinear ODEs with the M -fractional derivative are solved to examine the validity of the results presented in the current study.

Example 3.1. *Firstly, we deal with a problem involving the M -fractional derivative as [40]*

$${}_i D_M^{\alpha, \beta} y(t) = y(t) + 1, \quad \lim_{t \rightarrow 0^+} y(t) = 0, \quad (3.1)$$

where the exact solution of Eq (3.1) is

$$y(t) = \exp\left(\frac{\Gamma(\beta + 1)}{\alpha} t^\alpha\right) - 1.$$

According to Section 2, we adopt a solution for Eq (3.1) as

$$y(t) = \sum_{n=0}^{\infty} a_n (t^\alpha)^n. \quad (3.2)$$

By substituting Eq (3.2) into (3.1) and simplifying the resulting expression, we get

$$\frac{\alpha}{\Gamma(\beta + 1)} \sum_{n=1}^{\infty} n a_n t^{(n-1)\alpha} = \sum_{n=0}^{\infty} a_n t^{n\alpha} + 1,$$

and so

$$\left(\frac{\alpha}{\Gamma(\beta + 1)} a_1 - a_0 - 1 \right) + \sum_{n=2}^{\infty} \left(\frac{\alpha(n+1)}{\Gamma(\beta + 1)} a_{n+1} - a_n \right) t^{n\alpha} = 0.$$

By performing some simple operations, we achieve

$$\lim_{t \rightarrow 0^+} y(t) = 0 \rightarrow a_0 = 0,$$

$$\begin{aligned} \frac{\alpha}{\Gamma(\beta + 1)} a_1 - 1 = 0 &\rightarrow a_1 = \frac{\Gamma(\beta + 1)}{\alpha}, \\ \frac{\alpha(n+1)}{\Gamma(\beta + 1)} a_{n+1} - a_n = 0 &\rightarrow a_{n+1} = \frac{\Gamma(\beta + 1)}{\alpha(n+1)} a_n, \quad n = 1, 2, \dots \end{aligned} \quad (3.3)$$

From (3.3), it is clear that

$$\begin{aligned} a_2 &= \frac{\Gamma(\beta + 1)}{2\alpha} a_1 = \frac{1}{2} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^2, \\ a_3 &= \frac{\Gamma(\beta + 1)}{3\alpha} a_2 = \frac{1}{3!} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^3, \\ &\vdots \\ a_n &= \frac{1}{n!} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^n, \quad n = 2, 3, \dots \end{aligned}$$

By applying the above coefficients in Eq (3.2), the solution of Eq (3.1) is derived as

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^n t^{n\alpha},$$

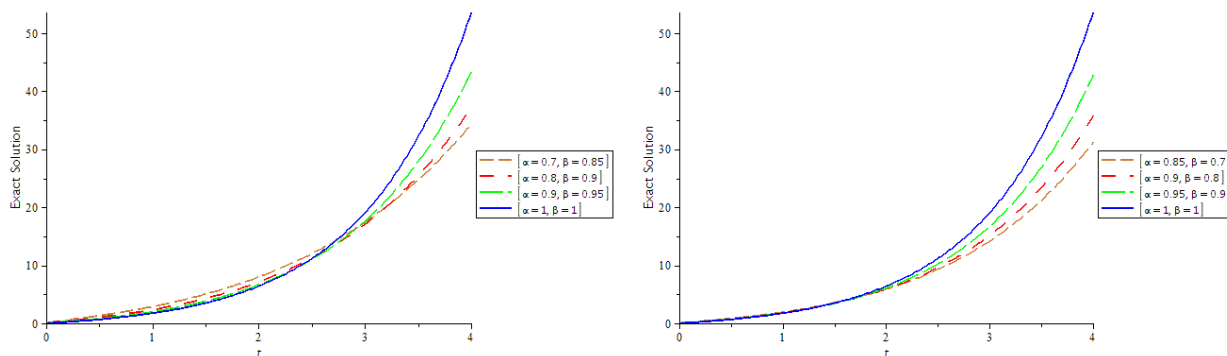
or

$$y(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^n t^{n\alpha} - 1, \quad t > 0,$$

converging to

$$y(t) = \exp\left(\frac{\Gamma(\beta + 1)}{\alpha} t^\alpha\right) - 1.$$

The exact solution of Eq (3.1) for different sets of α and β has been plotted in Figure 1.



(a) Gold ($\alpha = 0.7, \beta = 0.85$), Red ($\alpha = 0.8, \beta = 0.9$), Green ($\alpha = 0.9, \beta = 0.95$), and Blue ($\alpha = 1, \beta = 1$),
 (b) Gold ($\alpha = 0.85, \beta = 0.7$), Red ($\alpha = 0.9, \beta = 0.8$), Green ($\alpha = 0.95, \beta = 0.9$), and Blue ($\alpha = 1, \beta = 1$),

Figure 1. The exact solution of Example 1 for different sets of α and β .

Example 3.2. Secondly, we want to deal with a problem with the M -fractional derivative as

$${}_i D_M^{0.5, \beta} y(t) = y(t) + t^{\frac{1}{2}}, \quad y(2) = 0, \quad (3.4)$$

where the exact solution of Eq (3.4) is

$$y(t) = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta + 1)} \right) e^{2\Gamma(\beta + 1)(t^{\frac{1}{2}} - 2^{\frac{1}{2}})} - t^{\frac{1}{2}} - \frac{1}{2\Gamma(\beta + 1)}.$$

Based on Section 2, the solution of Eq (3.4) is assumed to be

$$y(t) = \sum_{n=0}^{\infty} a_n (t^{\frac{1}{2}} - 2^{\frac{1}{2}})^n. \quad (3.5)$$

By setting Eq (3.5) in (3.4) and simplifying the resulting expression, we find

$$\frac{1}{2\Gamma(\beta + 1)} \sum_{n=1}^{\infty} n a_n (t^{\frac{1}{2}} - 2^{\frac{1}{2}})^{n-1} = \sum_{n=0}^{\infty} a_n (t^{\frac{1}{2}} - 2^{\frac{1}{2}})^n + (t^{\frac{1}{2}} - 2^{\frac{1}{2}}) + 2^{\frac{1}{2}},$$

or

$$\begin{aligned} & \left(\frac{1}{2\Gamma(\beta + 1)} a_1 - a_0 - 2^{\frac{1}{2}} \right) + \left(\frac{1}{\Gamma(\beta + 1)} a_2 - a_1 - 1 \right) (t^{\frac{1}{2}} - 2^{\frac{1}{2}}) \\ & + \sum_{n=2}^{\infty} \left(\frac{(n+1)}{2\Gamma(\beta + 1)} a_{n+1} - a_n \right) (t^{\frac{1}{2}} - 2^{\frac{1}{2}})^n \\ & = 0. \end{aligned}$$

Applying some simple operations, we obtain

$$\begin{aligned} y(2) = 0 & \rightarrow a_0 = 0, \\ \frac{1}{2\Gamma(\beta + 1)} a_1 - a_0 - 2^{\frac{1}{2}} & = 0 \rightarrow a_1 = 2^{\frac{3}{2}} \Gamma(\beta + 1), \end{aligned}$$

$$\begin{aligned} \frac{1}{\Gamma(\beta+1)}a_2 - a_1 - 1 = 0 &\rightarrow a_2 = \Gamma(\beta+1)\left(2^{\frac{3}{2}}\Gamma(\beta+1) + 1\right), \\ \frac{(n+1)}{2\Gamma(\beta+1)}a_{n+1} - a_n = 0 &\rightarrow a_{n+1} = \frac{2\Gamma(\beta+1)}{n+1}a_n, \quad n = 2, 3, \dots \end{aligned} \quad (3.6)$$

From (3.6), it is found that

$$\begin{aligned} a_3 &= \frac{2\Gamma(\beta+1)}{3}a_2 = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right) \frac{2^3\Gamma^3(\beta+1)}{3!}, \\ a_4 &= \frac{2\Gamma(\beta+1)}{4}a_3 = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right) \frac{2^4\Gamma^4(\beta+1)}{4!}, \\ &\vdots \\ a_n &= \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right) \frac{2^n\Gamma^n(\beta+1)}{n!}, \quad n = 3, 4, \dots \end{aligned}$$

Inserting the above coefficients into Eq (3.5) leads to

$$y(t) = 2^{\frac{3}{2}}\Gamma(\beta+1)\left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right) + \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right) \sum_{n=2}^{\infty} \frac{2^n(\Gamma(\beta+1))^n}{n!} \left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)^n, \quad t > 0,$$

or

$$y(t) = \left(2^{\frac{1}{2}} + \frac{1}{2\Gamma(\beta+1)}\right) e^{2\Gamma(\beta+1)\left(t^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)} - t^{\frac{1}{2}} - \frac{1}{2\Gamma(\beta+1)}.$$

The exact solution of Eq (3.4) for different values of β has been portrayed in Figure 2.

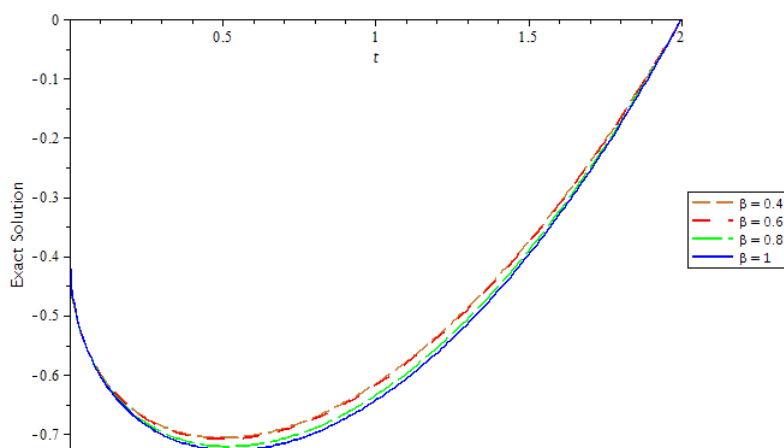


Figure 2. The exact solution of Example 2 for different values of β .

Example 3.3. Thirdly, we deal with a problem involving the M -fractional derivative as

$${}_i D_M^{\alpha, \beta} y(t) = 1 + (y(t))^2, \quad \lim_{t \rightarrow 0^+} y(t) = 0, \quad (3.7)$$

which has the following exact solution

$$y(t) = \tan\left(\frac{\Gamma(\beta+1)}{\alpha} t^\alpha\right).$$

According to Section 2, we adopt a solution for Eq (3.7) as

$$y(t) = \sum_{n=0}^{\infty} a_n (t^\alpha)^n. \quad (3.8)$$

By substituting Eq (3.8) into (3.7) and simplifying the resulting expression, we get

$$\frac{\alpha}{\Gamma(\beta + 1)} \sum_{n=1}^{\infty} n a_n t^{(n-1)\alpha} = 1 + \left(\sum_{n=0}^{\infty} a_n t^{n\alpha} \right)^2,$$

or

$$\left(\frac{\alpha a_1}{\Gamma(\beta + 1)} - a_0^2 - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{(n+1)\alpha}{\Gamma(\beta + 1)} a_{n+1} - \sum_{i=0}^n a_i a_{n-i} \right) t^{n\alpha} = 0.$$

By performing some simple operations, we achieve

$$\begin{aligned} \lim_{t \rightarrow 0^+} y(0) = 0 &\rightarrow a_0 = 0, \\ \frac{\alpha}{\Gamma(\beta + 1)} a_1 - a_0^2 - 1 = 0 &\rightarrow a_1 = \frac{\Gamma(\beta + 1)}{\alpha}, \\ \frac{(n+1)\alpha}{\Gamma(\beta + 1)} a_{n+1} - \sum_{i=0}^n a_i a_{n-i} = 0 &\rightarrow a_{n+1} = \frac{\Gamma(\beta + 1)}{(n+1)\alpha} \sum_{i=0}^n a_i a_{n-i}. \end{aligned} \quad (3.9)$$

From (3.9), it is clear that

$$\begin{aligned} n = 1 &\rightarrow a_2 = \frac{\Gamma(\beta + 1)}{2\alpha} a_0 a_1 = 0, \\ n = 2 &\rightarrow a_3 = \frac{\Gamma(\beta + 1)}{3\alpha} (2a_0 a_2 + a_1^2) = \frac{1}{3} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^3, \\ n = 3 &\rightarrow a_4 = \frac{\Gamma(\beta + 1)}{4\alpha} (2a_0 a_3 + 2a_1 a_2) = 0, \\ n = 4 &\rightarrow a_5 = \frac{\Gamma(\beta + 1)}{5\alpha} (2a_0 a_4 + 2a_2^2 + 2a_1 a_3) = \frac{2}{15} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^5, \\ &\vdots \end{aligned}$$

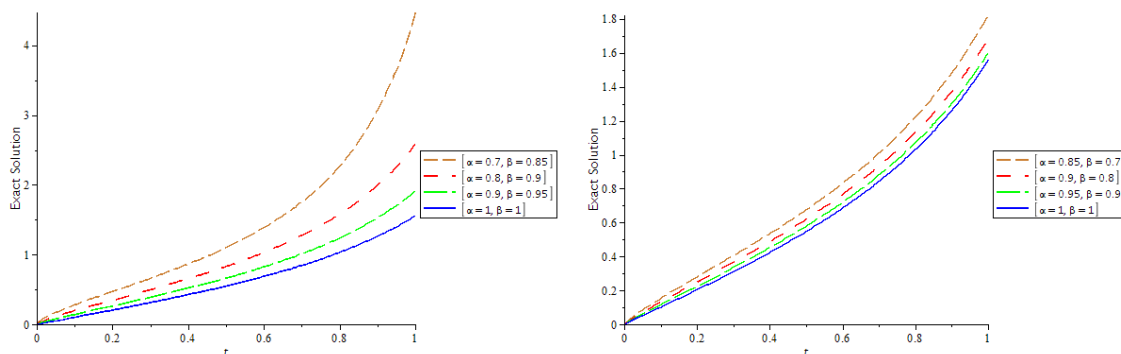
By applying the above coefficients in Eq (3.8), the solution of Eq (3.7) is derived as

$$y(t) = \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha + \frac{1}{3} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^3 t^{3\alpha} + \frac{2}{15} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^5 t^{5\alpha} + \dots, \quad 0 < t < \left(\frac{\alpha\pi}{2\Gamma(\beta + 1)} \right)^2,$$

converging to

$$y(t) = \tan \left(\frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right).$$

The exact solution of Eq (3.7) for different sets of α and β has been plotted in Figure 3.



(a) Gold ($\alpha = 0.7, \beta = 0.85$), Red ($\alpha = 0.8, \beta = 0.9$), Green ($\alpha = 0.9, \beta = 0.95$), and Blue ($\alpha = 1, \beta = 1$), (b) Gold ($\alpha = 0.85, \beta = 0.7$), Red ($\alpha = 0.9, \beta = 0.8$), Green ($\alpha = 0.95, \beta = 0.9$), and Blue ($\alpha = 1, \beta = 1$),

Figure 3. The exact solution of Example 3 for different sets of α and β .

Example 3.4. In the end, we will deal with a problem with the M -fractional derivative as

$${}_i D_M^{\alpha, \beta} y(t) = 1 - (y(t))^2, \quad \lim_{t \rightarrow 0^+} y(0) = 0, \quad (3.10)$$

which has the following exact solution

$$y(t) = \tanh\left(\frac{\Gamma(\beta + 1)}{\alpha} t^\alpha\right).$$

Based on Section 2, the solution of Eq (3.10) is supposed to be

$$y(t) = \sum_{n=0}^{\infty} a_n (t^\alpha)^n. \quad (3.11)$$

By setting Eq (3.11) in (3.10) and simplifying the resulting expression, we find

$$\frac{\alpha}{\Gamma(\beta + 1)} \sum_{n=1}^{\infty} n a_n t^{(n-1)\alpha} = 1 - \left(\sum_{n=0}^{\infty} a_n t^{n\alpha} \right)^2,$$

or

$$\left(\frac{\alpha a_1}{\Gamma(\beta + 1)} + a_0^2 - 1 \right) + \sum_{n=1}^{\infty} \left(\frac{(n+1)\alpha}{\Gamma(\beta + 1)} a_{n+1} + \sum_{i=0}^n a_i a_{n-i} \right) t^{n\alpha} = 0.$$

Through applying some simple operations, we obtain

$$\begin{aligned} \lim_{t \rightarrow 0^+} y(0) = 0 &\rightarrow a_0 = 0, \\ \frac{\alpha}{\Gamma(\beta + 1)} a_1 + a_0^2 - 1 &= 0 \rightarrow a_1 = \frac{\Gamma(\beta + 1)}{\alpha}, \\ \frac{(n+1)\alpha}{\Gamma(\beta + 1)} a_{n+1} + \sum_{i=0}^n a_i a_{n-i} &= 0 \rightarrow a_{n+1} = -\frac{\Gamma(\beta + 1)}{(n+1)\alpha} \sum_{i=0}^n a_i a_{n-i}. \end{aligned} \quad (3.12)$$

From (3.12), it is found that

$$\begin{aligned} n = 1 &\rightarrow a_2 = -\frac{\Gamma(\beta + 1)}{2\alpha} a_0 a_1 = 0, \\ n = 2 &\rightarrow a_3 = -\frac{\Gamma(\beta + 1)}{3\alpha} (2a_0 a_2 + a_1^2) = -\frac{1}{3} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^3, \\ n = 3 &\rightarrow a_4 = -\frac{\Gamma(\beta + 1)}{4\alpha} (2a_0 a_3 + 2a_1 a_2) = 0, \\ n = 4 &\rightarrow a_5 = -\frac{\Gamma(\beta + 1)}{5\alpha} (2a_0 a_4 + 2a_2^2 + 2a_1 a_3) = \frac{2}{15} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^5, \\ &\vdots \end{aligned}$$

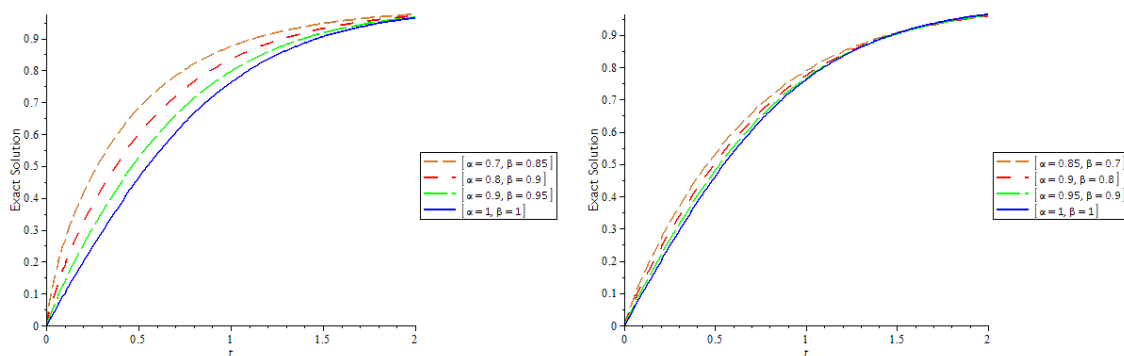
Inserting the above coefficients into Eq (3.11) leads to

$$y(t) = \frac{\Gamma(\beta + 1)}{\alpha} t^\alpha - \frac{1}{3} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^3 t^{3\alpha} + \frac{2}{15} \left(\frac{\Gamma(\beta + 1)}{\alpha} \right)^5 t^{5\alpha} - \dots,$$

or

$$y(t) = \tanh \left(\frac{\Gamma(\beta + 1)}{\alpha} t^\alpha \right).$$

The exact solution of Eq (3.10) for different sets of α and β has been portrayed in Figure 4.



(a) Gold ($\alpha = 0.7, \beta = 0.85$), Red ($\alpha = 0.8, \beta = 0.9$), Green ($\alpha = 0.9, \beta = 0.95$), and Blue ($\alpha = 1, \beta = 1$),
 (b) Gold ($\alpha = 0.85, \beta = 0.7$), Red ($\alpha = 0.9, \beta = 0.8$), Green ($\alpha = 0.95, \beta = 0.9$), and Blue ($\alpha = 1, \beta = 1$).

Figure 4. The exact solution of Example 4 for different sets of α and β .

4. Conclusions

The key goal of the current paper was to conduct a new investigation on ordinary differential equations involving the M -fractional derivative. In this respect, first, the α -Taylor expansion and the α -Maclaurin expansion were established based on the M -fractional derivative. Then, several definitions, theorems, and corollaries regarding the power series in the M sense were given and successfully proved. To examine the effectiveness of the results provided in the present work, some ordinary differential equations involving the M -fractional derivative were solved. The Maple package

as a worthwhile tool was formally adopted to deal with symbolic computations. As a possible future work, the authors will apply the power series in the M sense to solve other well-known ODEs involving the M -fractional derivative.

Conflict of interest

All authors declare no conflicts of interest in this paper.

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