

Fractional Hamiltonian analysis of higher order derivatives systems

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The fractional Hamiltonian analysis of 1+1 dimensional field theory is investigated and the fractional Ostrogradski's formulation is obtained. The fractional path integral of both simple harmonic oscillator with an acceleration-squares part and a damped oscillator are analyzed. The classical results are obtained when fractional derivatives are replaced with the integer order derivatives. © 2006 American Institute of Physics. [DOI: [10.1063/1.2356797](https://doi.org/10.1063/1.2356797)]

I. INTRODUCTION

Fractional calculus deals with the generalization of differentiation and integration to noninteger orders. Fractional calculus has gained importance, especially during the last three decades.¹⁻⁵ A large body of mathematical knowledge on fractional integrals and derivatives has been constructed. Fractional calculus, as a natural generalization of classical calculus, has played a significant role in engineering, science, and pure and applied mathematics in recent years. The fractional derivatives are the infinitesimal generators of a class of translation invariant convolution semigroups that appear universally as attractors.

Various applications of fractional calculus are based on replacing the time derivative in an evolution equation with a derivative of fractional order. The results of several recent researchers confirm that fractional derivatives seem to arise for important mathematical reasons.⁵⁻²¹

The fractional variational principles represents an important part of fractional calculus and it is deeply related to the fractional quantization procedure. There are several proposed methods to obtain the fractional Euler-Lagrange equations and the corresponding fractional Hamiltonians. However, this issue is not yet completely clarified and it requires a more further detailed analysis.

The quantization of systems with fractional derivatives is a novel area in the theory of application of fractional differential and integral calculus. *Schrödinger* equation was considered with the first order time derivative modified to Caputo fractional ones in Ref. 22. In this case the obtained Hamiltonian was found to be non-Hermitian and nonlocal in time. In addition, the obtained wave functions are not invariant under the time reversal. The quantization of fractional Klein-Gordon field and fractional electromagnetic potential in the Coulomb gauge and the temporal gauge were investigated very recently in Ref. 23.

Recently, the fractional variational principles and the fractional Euler-Lagrange were obtained.^{24,25}

Even more recently, the fractional constrained Lagrangian and Hamiltonian were analyzed.^{26,27} The notion of the fractional Hessian²⁷ was introduced and the Euler-Lagrange equations were obtained for a Lagrangian linear in velocities.²⁶ Besides, the Hamiltonian equations

have been obtained for systems with linear velocities.²⁸ The classical fields with fractional derivatives were investigated by using the fractional Lagrangian formulation and the fractional Euler-Lagrange equations were obtained in Ref. 29.

Nonlocal theories have been investigated in several physical problems.³⁰ During the last decade, the nonlocal theories were subjected to an intense debate.³¹⁻³⁵ A Hamilton formalism for nonlocal Lagrangians was developed in Refs. 34 and 35, an equivalent singular first order Lagrangian was obtained and the corresponding Hamiltonian was pulled back on the phase space by using the corresponding constraints.³⁴ It was shown the space-time noncommutative field theories are acausal and the unitarity is lost.^{36,37} The fractional Lagrangians and Hamiltonians are typical examples of nonlocal theories.

For these reasons the fractional quantization of field theory is an interesting issue to be investigated.

In this paper we analyze the fractional Hamiltonian quantization of nonsingular systems possessing higher order derivatives.

The plan of the paper is as follows.

In Sec. II the 1+1 classical dimensional field theory analysis of nonlocal theories is briefly reviewed and the fractional generalization of Ostrogradski's formulation is presented. In Sec. III the path integral quantization of the simple harmonic oscillator with an acceleration-squares part is analyzed. Section IV is dedicated to the fractional path integral formulation of the damped oscillator. Finally, Sec. V is dedicated to our conclusions.

II. FRACTIONAL FIELD THEORY

A. Classical nonlocal theory

Let us start with an ordinary local Lagrangian depending on a finite number of derivatives at a given time, namely,

$$L(q(t), \dot{q}(t), \dots, q^{(n)}(t)). \quad (1)$$

The next step is to consider a Lagrangian depending on a piece of the trajectory $q(t, \lambda)$ for $\forall \lambda$ belonging to an interval $[a, b]$,

$$L^{\text{non}}(t) = L(q(t + \lambda)), \quad (2)$$

where a, b are real numbers. Therefore a nonlocal Lagrangian was introduced. In this case the action function corresponding to (2) is given by

$$S(q) = \int dt L^{\text{non}}(t) \quad (3)$$

and the Euler-Lagrange equation corresponding to (3) are given by

$$\int dt \frac{\delta L^{\text{non}}(t)}{\delta(q(t))} = 0. \quad (4)$$

Equations (4) should be understood as a functional relation to be satisfied by physical trajectories, i.e., a Lagrangian constraint. These functional relations define a subspace J_R of physical trajectories $J_R \subset J$, in the space of all possible trajectories.^{32,34} The crucial point is that there is no dynamics except the displacement inside the trajectory, namely,

$$q(t) \rightarrow q(t + \lambda). \quad (5)$$

Let us introduce now the dynamical variable $Q(t, \lambda)$ as follows:

$$Q(t, \lambda) = q(t + \lambda). \quad (6)$$

If we consider a field $Q(t, \lambda)$ instead of a trajectory $q(t)$, such that

$$\dot{Q}(t, \lambda) = Q'(t, \lambda), \quad (7)$$

where $\dot{Q} = \partial Q(t, \lambda) / \partial t$ and $Q'(t, \lambda) = \partial Q(t, \lambda) / \partial \lambda$ we obtain a field theory in one spatial and one time dimension, namely a 1+1 dimensional formulation of nonlocal Lagrangians.^{32,34}

The coordinates and momenta are suppose to have the following forms:

$$Q(t, \lambda) = \sum_{m=0}^{\infty} e_m(\lambda) q^{(m)}(t), \quad P(t, \lambda) = \sum_{m=0}^{\infty} e^m(\lambda) p_{(m)}(t), \quad (8)$$

where

$$\{q^{(n)}(t), p_{(m)}(t)\} = \delta_m^n \quad (9)$$

and

$$e_m(\lambda) = \frac{\lambda^m}{m!}, \quad e^m(\lambda) = (-\partial_\lambda)^m \delta(\lambda). \quad (10)$$

Therefore, the Hamiltonian for the 1+1 dimensional field becomes

$$H(t, [Q, P]) = \int d\lambda P(t, \lambda) Q'(t, \lambda) - \tilde{L}(t, [Q]), \quad (11)$$

where P denotes the canonical momentum of Q . The phase space is T^*J together with the fundamental Poisson brackets,

$$\{Q(t, \lambda), P(t, \lambda')\} = \delta(\lambda - \lambda'). \quad (12)$$

The functional $\tilde{L}(t, [Q])$ is defined as follows:

$$\tilde{L}(t, [Q]) = \int d\lambda \delta(\lambda) \mathcal{L}(t, \lambda). \quad (13)$$

By using (13), the primary constraint arises as given below:

$$\phi(t, \lambda, [q, P]) = P(t, \lambda) - \int d\sigma \chi(\lambda, -\sigma) \varepsilon(t; \sigma, \lambda) \approx 0. \quad (14)$$

Here $\varepsilon(t; \sigma, \lambda)$ and $\chi(\lambda, -\sigma)$ have the following definition:

$$\varepsilon(t; \sigma, \lambda) = \frac{\partial \mathcal{L}(t, \sigma)}{\partial Q(t, \lambda)}, \quad \chi(\lambda, -\sigma) = \frac{\varepsilon(\lambda) - \varepsilon(\sigma)}{2}, \quad (15)$$

where $\varepsilon(\lambda)$ is the sigma distribution. The Euler-Lagrange equation is guaranteed by itself,

$$\dot{\phi} \sim \psi = \int d\sigma \xi(t; \sigma, \lambda). \quad (16)$$

B. Fractional Ostrogradski's construction

Higher-derivatives theories^{38,39} appear naturally as corrections to general relativity and cosmic strings.⁴⁰ Unconstrained higher order derivatives possess specific features, namely they have more degrees of freedom than lower-derivative theories and they lack a lower-energy bound. A method how to remove all these problems was presented in Ref. 41. It was observed that the nonlocal formulation translates into infinite order Ostrogradski's formulation.^{34,35}

In this section, we would like to derive both the Lagrangian and the Hamiltonian formalisms for nonsingular Lagrangians with fractional order derivatives starting from the Hamiltonian formalism of nonlocal theories.³² Let us consider the following Lagrangian to start with:

$$L(q, t) = L(t, q^{\alpha_m}), \quad (17)$$

where the generalized coordinates are defined as

$$q^{\alpha_m} = {}_a \mathbf{D}_t^{\alpha_m} x(t), \quad (18)$$

where m is a natural number.

To obtain the reduced phase space quantization, we start with the infinite dimensional phase space $T^*J(t) = \{Q(t, \lambda), P(t, \lambda)\}$.

The key issue is to find an appropriate generalization of (10) for the fractional case. As it was pointed out in Refs. 32 and 34, the coordinates and the momenta are considered as a Taylor series. Therefore, the first step is to generalize the classical series to the fractional case. A natural extension is to use instead of the factorial the Gamma function. In this way we introduce naturally the generalized functions⁴² instead of $e_m(\lambda)$ and $e^m(\lambda)$ given by (10).

As it is already known, several fractional Taylor's series expansions were developed,^{3,43} therefore we have to decide which one is appropriate for our generalization. Since we are dealing with fractional Riemann-Liouville derivatives we choose the generalization proposed in Ref. 44, namely,

$$\begin{aligned} Q(t, \lambda) &= \sum_{m=-\infty}^{\infty} e_{\alpha_m}(\lambda) q^{(\alpha_m)}(t), \\ P(t, \lambda) &= \sum_{m=-\infty}^{\infty} e^{\alpha_m}(\lambda) p_{(\alpha_m)}(t), \end{aligned} \quad (19)$$

where

$$e_{\alpha_m}(\lambda) = \frac{(\lambda - \lambda_0)^{\alpha_m}}{\Gamma(\alpha_m + 1)}, \quad e^{\alpha_m}(\lambda) = \mathbf{D}_{\lambda}^{\alpha_m} \delta(\lambda - \lambda_0), \quad (20)$$

and $\alpha_m = m + \alpha$, with $0 \leq \alpha < 1$. Here λ_0 is a constant. The coefficients in (19) are new canonical variables,

$$\{q^{(\alpha_m)}, p_{(\alpha_m)}\} = \delta_{\alpha_m}^{\alpha_{m'}}. \quad (21)$$

By using (21), we obtain that

$$\sum_{m=-\infty}^{\infty} e^{\alpha_m}(\lambda) e_{\alpha_m}(\lambda') = \delta(\lambda - \lambda'), \quad (22)$$

and

$$\int_{-\infty}^{+\infty} d\lambda e^{\alpha_m}(\lambda) e_{\alpha_m'}(\lambda) = \delta_{\alpha_m}^{\alpha_{m'}}. \quad (23)$$

Therefore, $e^{\alpha_m}(\lambda)$ and $e_{\alpha_m}(\lambda)$ form an orthonormal basis.

We stress the fact that (22) and (23) involve the generalized functions and the relations have the meaning in the sense of generalized functions approach.^{42,44}

The fractional Hamiltonian is now given by

$$H = \sum_{m=-\infty}^{\infty} p^{\alpha_m} q^{\alpha_{m+1}} - L(q^0, q^{\alpha_m}). \quad (24)$$

The momenta constraints become an infinite set of constraints,

$$\phi_n = p_{\alpha_n}(t) - \sum_{m=n}^{\infty} {}_t\mathbf{D}_b^{\alpha_{m-n}} \frac{\partial L}{\partial q^{(\alpha_{m+1})}(t)} = 0. \quad (25)$$

The fractional Euler-Lagrange equations are as follows;

$$\sum_{l=-\infty}^{\infty} {}_t\mathbf{D}_b^{\alpha_l} \frac{\partial L(t)}{\partial q^{\alpha_l}(t)} = 0. \quad (26)$$

An interesting property of the fractional series proposed by Riemann and discussed by Hardy in Ref. 44 is that when α_m become integers, the usual form of Taylor series is obtained. Therefore one should notice that for integer values of α_m , we have

$$p_{\alpha_m}(t) - \sum_{l=0}^{n-m-1} \left(-\frac{d}{dt}\right)^l \frac{\partial L(t)}{\partial (\partial_t^{l+m+1} q(t))} = 0, \quad (27)$$

which is the definition of Ostrogradski's momenta.³⁸

In this case the Euler-Lagrange equation for the original fractional derivative Lagrangian²⁶⁻³⁰ is given below,

$$\sum_{l=0}^n {}_t\mathbf{D}_b^{\alpha_l} \frac{\partial L(t)}{\partial q^{\alpha_l}(t)} = 0. \quad (28)$$

Now, from this equation, for integer values of α_m we obtain the Euler-Lagrange equation for a higher derivative Lagrangian,^{32,34,38} namely

$$\sum_{l=0}^n \left(-\frac{d}{dt}\right)^l \frac{\partial L(t)}{\partial (\partial_t^l q(t))} = 0. \quad (29)$$

The constraints (27) and (29) lead us to eliminate canonical pairs $\{q^{\alpha_l}, p_{\alpha_l}\} (l \geq n)$.

In this case the infinite dimensional phase space is reduced to a finite dimensional one. The reduced space is coordinated by $T^*J^n = \{q^{\alpha_l}, p_{\alpha_l}\}$ with $l=0, 1, \dots, n-1$. The Hamiltonian in the reduced space is given by

$$H = \sum_{m=0}^{n-1} p^{\alpha_m} q^{\alpha_{m+1}} - L(q^0, q^{\alpha_m}). \quad (30)$$

One should notice that the canonical reduced phase space Hamiltonian (30) is obtained in terms of the reduce canonical phase space coordinates $\{q^{\alpha_l}, p_{\alpha_l}\}$ with $l=0, 1, \dots, n-1$. In this case the path integral quantization of a field system is given by

$$K = \int \prod_{m=0}^{n-1} dq^{\alpha_m} dp^{\alpha_m} e^{i\int dt (\sum_{m=0}^{n-1} p^{\alpha_m} q^{\alpha_{m+1}} - H)}. \quad (31)$$

We observe that when α are integers, we obtain the path integral for systems with higher order Lagrangians.^{32,45,46}

III. FRACTIONAL PATH INTEGRAL QUANTIZATION OF A SIMPLE HARMONIC OSCILLATOR POSSESSING AN ACCELERATION-SQUARES PART

The classical Lagrangian to start with is given by⁴¹

$$L_c = \frac{1}{2}(1 + \epsilon^2 \omega^2)x^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\epsilon^2 \dot{x}^2. \quad (32)$$

The fractional generalization of (32) has the following form:

$$L = \frac{1}{2}(1 + \epsilon^2 \omega^2)({}_t\mathbf{D}_a^\alpha x(t))^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\epsilon^2 [{}_t\mathbf{D}_a^\alpha ({}_t\mathbf{D}_a^\alpha x(t))]^2. \quad (33)$$

The independent coordinates are $x(t)$ and ${}_t\mathbf{D}_a^\alpha x(t)$, respectively. Let us denote their corresponding momenta as $p_1^\alpha = p_x$ and $p_2^\alpha = p_{({}_t\mathbf{D}_a^\alpha x(t))}$. The fractional canonical momenta are³⁸

$$p_1^\alpha = \frac{\partial L}{\partial {}_t\mathbf{D}_a^\alpha x(t)} - {}_t\mathbf{D}_a^\alpha \left(\frac{\partial L}{\partial {}_t\mathbf{D}_a^{2\alpha} x(t)} \right), \quad p_2^\alpha = \frac{\partial L}{\partial {}_t\mathbf{D}_a^{2\alpha} x(t)}. \quad (34)$$

By making use of (33), we obtain the forms of the fractional canonical momenta, as given below:

$$p_1^\alpha = (1 + \epsilon^2 \omega^2) {}_t\mathbf{D}_a^\alpha x(t) + \epsilon^2 {}_t\mathbf{D}_a^{3\alpha} x(t), \quad (35)$$

$$p_2^\alpha = -\epsilon^2 {}_t\mathbf{D}_a^{2\alpha} x(t). \quad (36)$$

The fractional canonical Hamiltonian becomes

$$H = p_1^\alpha {}_t\mathbf{D}_a^\alpha x(t) + p_2^\alpha {}_t\mathbf{D}_a^{2\alpha} x(t) - L. \quad (37)$$

Taking into account (33), (35), and (36) the form of (37) is given by

$$H = \frac{1}{2} \left[2p_1^\alpha {}_t\mathbf{D}_a^\alpha x(t) - \frac{(p_2^\alpha)^2}{\epsilon^2} + \omega^2 x^2(t) - (1 + \epsilon^2 \omega^2) ({}_t\mathbf{D}_a^\alpha x(t))^2 \right]. \quad (38)$$

By making use of (38), the fractional path integral is written as

$$K = \int dx d({}_t\mathbf{D}_a^\alpha x(t)) dp_1^\alpha dp_2^\alpha e^{i \int dt (p_1^\alpha x(t) + p_2^\alpha {}_t\mathbf{D}_a^\alpha x(t) - H)}. \quad (39)$$

IV. FRACTIONAL PATH INTEGRAL QUANTIZATION OF DAMPED HARMONIC OSCILLATOR

The Lagrangian for this system in Ostrogradski's notations³⁸ takes the form⁹

$$L = \frac{1}{2} m q_1^2 + i \frac{\gamma}{2} q_{1/2}^2 - V(q_0), \quad (40)$$

where

$$q^{\alpha_n} = {}_t\mathbf{D}_b^{\alpha_n} x, \quad n = 0, 1, 2. \quad (41)$$

Here $\alpha_0=0$, $\alpha_1=1/2$, $\alpha_2=1$, and $q_0=x$, $q_1=\dot{x}$, $q_{1/2} = {}_t\mathbf{D}_b^{1/2} x$, $q_2=\ddot{x}$.

The expressions for canonical momenta are

$$p_0 = i \gamma x_{(1/2)} + i m x_{(3/2)}, \quad (42)$$

$$p_{1/2} = m \dot{x}. \quad (43)$$

By using (40) the classical Euler-Lagrange equation of motion read as⁹

$$m\ddot{x} + \gamma\dot{x} + \frac{\partial V}{\partial x} = 0. \quad (44)$$

The canonical reduced Hamiltonian has the following expression:

$$H = \frac{p_{1/2}^2}{2m} + q_{1/2}p_0 - i\frac{\gamma}{2}q_{1/2}^2 + V(q_0). \quad (45)$$

As a result the corresponding fractional path integral representation is given by

$$K = \int d\mu \exp i \left[\int \left(q_{1/2}p_{1/2} - \frac{p_{1/2}^2}{2m} + i\frac{\gamma}{2}q_{1/2}^2 - V(q_0) \right) dt \right], \quad (46)$$

where $d\mu = dq_0 dp_0 dq_{1/2} dp_{1/2}$.

The path integral representation for (46) is an integration over the canonical phase space coordinates (q_0, p_0) and $(q_{1/2}, p_{1/2})$. Integrating over $p_{1/2}$ and p_0 , we obtain

$$K = \int dq_0 dq_{1/2} \exp i \int \left(\frac{1}{2}mq_1^2 - V(q_0) + i\frac{\gamma}{2}q_{1/2}^2 \right) dt. \quad (47)$$

Equation (47) can be put in a compact form as follows:

$$K = \int dq_0 e^{if((1/2)mq_1^2 - V(q_0))dt} dq_{1/2} e^{if(i(\gamma/2)q_{1/2}^2)dt}. \quad (48)$$

After performing an integration over $q_{1/2}$, (48) becomes

$$K = C \int dq_0 e^{if((1/2)mq_1^2 - V(q_0))dt},$$

where C represents a constant.

V. CONCLUSIONS

The interest in fractional quantization appears because it describes both conservative systems and nonconservative systems as well. The fractional quantization of field theory is not an easy task, especially when the fractional Hamiltonian is involved. The fractional derivatives represent the generalization of the classical ones, and therefore some of the classical properties are lost, e.g., the fractional Leibniz rule or the chain rule that becomes more complicated than the classical counterparts. The fractional path integral formulation deserves further investigations, mainly because the fractional generalization of the classical case is not yet completely understood. Namely, for a system possessing second class constraints in Dirac's classification it is difficult to find the corresponding fractional generalization. In addition, there are no fractional formulations of the classical secondary or tertiary constraints due to the fact that the fractional Hamiltonian is not a constant of motion.

In this paper we generalize to the fractional case the nonlocal theories in one space and one time dimension via the infinite Ostrogradski's formalism. The classical Taylor series involved in this problem are convergent because of the properties of the Dirac's delta function. Namely, the coordinates and the corresponding momenta are defined as Taylor series and the Ostrogradski's canonical pairs fulfill the classical Poisson's brackets commutation relations. The generalization to the fractional case of all the above mentioned results is not straightforward because there exist many formulations for the fractional Taylor series. However, a powerful tool in fractional field theory is to work to the Riemann-Liouville derivatives because of their important property of integration by parts. Therefore, in this paper we focus on the fractional Taylor series involving the Riemann-Liouville derivatives. We assumed that the fractional Lagrangian density has a compact

support in the x directions. In this work we have obtained the path integral quantization for fractional generalization of a 1+1 dimensional nonlocal field theory. The path integral formulation for the simple harmonic oscillator with an acceleration-squares part as well as for the damped oscillator are obtained. It is worthwhile to mention that the general expression for the path integral leads to the path integral representation for systems with higher order Lagrangians.

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