Journal of King Saud University – Science 31 (2019) 83–88

Contents lists available at ScienceDirect

Journal of King Saud University – Science

journal homepage: www.sciencedirect.com

Review

Local fractional Sumudu decomposition method for linear partial differential equations with local fractional derivative

D. Ziane [a], D. Baleanu [b], K. Belghaba [a], M. Hamdi Cherif [a,*]

[a] Laboratory of Mathematics and Its Applications (LAMAP), University of Oran1 Ahmed Ben Bella, Oran, 31000, Algeria
[b] Institute of Space Sciences, Magurele, 077125 Bucharest, Romania

ARTICLE INFO

Article history:
Received 10 April 2017
Accepted 2 May 2017
Available online 6 May 2017

Keywords:
Local fractional derivative operator
Local fractional Sumudu decomposition method
Local fractional partial differential equations

ABSTRACT

In the paper, a combined form of the Sumudu transform method with the Adomian decomposition method in the sense of local fractional derivative, is proposed to solve fractional partial differential equations. This method is called the local fractional Sumudu decomposition method (LFSDM) and is used to describe the non-differentiable problems. It would be interesting to apply LFSDM to some well-known problems to see the benefits obtained.

Contents

1. Introduction

Historically, it has been established that the question of the numerical derivation of the fractional order of functions and its inverse integral operation has been discussed in various correspondences between Gottfried Leibniz (1646–1716), Guillaume de L'Hôpital (1661–1704). However, the question will remain confined to this fact and no major development was realized by these early precursors in this field of mathematics, it is only later, when studying certain phenomena in fluid mechanics, that it was observed the presence of an integral of order one-half in the equations of heat when it is desired, for example, to explain a lateral heat flux of a fluid flow as a function of the temporal evolution of the internal source. As a result, developments took place in different fields of study, in particular hydrodynamics, thermodynamics, diffusion theory, electrochemistry, to name but a few examples (Ziane et al., 2016).

In addition to the above, we find that the development of this branch has also led to the emergence of linear and nonlinear differential equations of fractional order, which became requires researchers to use conventional methods to solve them, for

* Corresponding author.
E-mail addresses: djeloulz@yahoo.com (D. Ziane), dumitru@cankaya.edu.tr (D. Baleanu), belghaba@yahoo.fr (K. Belghaba), mountassir27@yahoo.fr (M. Hamdi Cherif).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2017.05.002
1018-3647/

example, the Adomian decomposition method (Adomian et al., 1989; Adomian et al., 1990; Adomian et al., 1994), homotopy perturbation method (He et al., 1999; He et al., 2005; He et al., 2000) and variational iteration method (He et al., 1997; He et al., 1998; He et al., 1998). The reader wishing to learn more about the subject can consult (Singh et al., 2016; Kumar et al., 2017; Aslefallah and Shivanian, 2015; Hosseini et al., 2016; Abbasbandy et al., 2011; Soltani et al., 2016).

Recently, local fractional derivative and calculus theory has been introduced by the researcher in (Yang et al., 2011; Yang, 2012), which is set up on fractal geometry and which is the best method for describing the non-differentiable function defined on Cantor sets Yang and Hua, 2014. The fractional calculus is used in generalized Newtonian mechanics, the Maxwell's equations and the Hamiltonian mechanics Golmankhaneh et al., 2015. This then led to the emergence of ordinary differential equations or partial differential equations relating to this new concept, which became known as local fractional differential equations or local fractional partial differential equations, prompting some researchers to use the above-mentioned methods to solve this new type of equations, among them we find, local fractional Adomian decomposition method (Yang et al., 2015; Baleanu et al., 5350; Yang et al., 2013), local fractional homotopy perturbation method (Yang et al., 2015; Zhang et al., 2015), local fractional variational iteration method (Yang et al., 2014; Yang et al., 2026), local fractional variational iteration transform method Yang et al., 3659 and local fractional Laplace decomposition method Jassim et al., 2015. Other authors have also been interested in this area of research (see Kumar et al., 2017; Singh et al., 2016).

The objective of this study is coupling the Adomian decomposition method (ADM) with Sumudu transform in the sense of local fractional derivative. Then we apply this modified method to solve some examples related to linear local fractional partial differential equations.

2. Preliminaries

In this section, we present the basic theory of local fractional calculus and we focus specifically on the following concepts: local fractional derivative, local fractional integral, and local fractional Sumudu transform. Some important results are cited.

2.1. Local fractional derivative

Definition 2.1. The local fractional derivative of $\Phi(r)$ of order ν at $r = r_0$ is defined by (Yang et al., 2011; Yang, 2012)

$$\Phi^{(\nu)}(r) = \left.\frac{d^\nu \Phi}{dr^\nu}\right|_{r=r_0} = \frac{\Delta^\nu(\Phi(r) - \Phi(r_0))}{(r - r_0)^\nu}, \tag{2.1}$$

where

$$\Delta^\nu(\Phi(r) - \Phi(r_0)) \cong \Gamma(1+\nu)[(\Phi(r) - \Phi(r_0))]. \tag{2.2}$$

For any $r \in (\alpha, \beta)$, there exists

$$\Phi^{(\nu)}(r) = D_r^\nu \Phi(r),$$

denoted by

$$\Phi(r) \in D_r^\nu(\alpha, \beta).$$

Local fractional derivative of high order is written in the form

$$\Phi^{(m\nu)}(r) = \overbrace{D_r^{(\nu)} \cdots D_r^{(\nu)}}^{m\ \text{times}} \Phi(r), \tag{2.3}$$

and local fractional partial derivative of high order is

$$\frac{\partial^{m\nu}\Phi(r)}{\partial r^{m\nu}} = \overbrace{\frac{\partial^\nu}{\partial r^\nu} \cdots \frac{\partial^\nu}{\partial r^\nu}}^{m\ \text{times}} \Phi(r). \tag{2.4}$$

2.2. Local fractional integral

Definition 2.2. The local fractional integral of $\Phi(r)$ of order νin the interval $[\alpha, \beta]$ is defined as (Yang et al., 2011; Yang, 2012)

$$\begin{aligned} {}_\alpha I_\beta^{(\nu)} \Phi(r) &= \frac{1}{\Gamma(1+\nu)} \int_\alpha^\beta \Phi(\tau)(d\tau)^\nu \\ &= \frac{1}{\Gamma(1+\nu)} \lim_{\Delta\tau \to 0} \sum_{j=0}^{N-1} f(\tau_j)(\Delta\tau_j)^\nu, \end{aligned} \tag{2.5}$$

where $\Delta\tau_j = \tau_{j+1} - \tau_j, \Delta\tau = \max\{\Delta\tau_0, \Delta\tau_1, \Delta\tau_2, \cdots\}$ and $[\tau_j, \tau_{j+1}], \tau_0 = \alpha, \tau_N = \beta$, is a partition of the interval $[\alpha, \beta]$. For any $r \in (\alpha, \beta)$, there exists

$${}_\alpha I_r^{(\nu)} \Phi(r),$$

denoted by

$$\Phi(r) \in I_r^{(\nu)}(\alpha, \beta).$$

2.3. Some important results

Definition 2.3. The local fractional Laplace transform of $\Phi(r)$ of order $\nu$is

$$L_\nu\{\Phi(r)\} = \frac{1}{\Gamma(1+\nu)} \int_0^\infty E_\nu(-s^\nu r^\nu)\Phi(r)(dr)^\nu. \tag{2.6}$$

Definition 2.4. In fractal space, the Mittag–Leffler function, sine function and cosine function are defined as (Yang et al., 2011; Yang, 2012)

$$E_\nu(r^\nu) = \sum_{m=0}^{+\infty} \frac{r^{m\nu}}{\Gamma(1+m\nu)}, \quad 0 < \nu \leqslant 1, \tag{2.7}$$

$$\sin_\nu(r^\nu) = \sum_{m=0}^{+\infty} (-1)^m \frac{r^{(2m+1)\nu}}{\Gamma(1+(2m+1)\nu)}, \quad 0 < \nu \leqslant 1, \tag{2.8}$$

$$\cos_\nu(r^\nu) = \sum_{m=0}^{+\infty} (-1)^m \frac{r^{2m\nu}}{\Gamma(1+2m\nu)}, \quad 0 < \nu \leqslant 1. \tag{2.9}$$

The properties of local fractional derivatives and integral of non-differentiable functions are given by (Yang et al., 2011; Yang, 2012)

$$\frac{d^\nu}{dr^\nu} \frac{r^{m\nu}}{\Gamma(1+m\nu)} = \frac{r^{(m-1)\nu}}{\Gamma(1+(m-1)\nu)}, \tag{2.10}$$

$$\frac{d^\nu}{dr^\nu} E_\nu(r^\nu) = E_\nu(r^\nu), \tag{2.11}$$

$$\frac{d^\nu}{dr^\nu} \sin_\nu(r^\nu) = \cos_\nu(r^\nu), \tag{2.12}$$

$$\frac{d^\rho}{dr^\rho} \cos_\rho(r^\rho) = -\sin_\rho(r^\rho), \tag{2.13}$$

$${}_0 I_r^{(\nu)} \frac{r^{m\nu}}{\Gamma(1+m\nu)} = \frac{r^{(m+1)\nu}}{\Gamma(1+(m+1)\nu)}. \tag{2.14}$$

2.4. Local fractional Sumudu transform

We present here the definition of local fractional Sumudu transform (denoted in this paper by ${}^{LF}S_v$) and some properties concerning this transformation Srivastava et al., 1763. If there is a new transform operator ${}^{LF}S_v : \Phi(r) \longrightarrow \mathrm{F}(u)$, namely

$$ {}^{LF}S_v\left\{\sum_{m=0}^{\infty} a_m r^{mv}\right\} = \sum_{m=0}^{\infty} \Gamma(1+mv) a_m u^{mv}. \tag{2.15}$$

As typical examples, we have

$$ {}^{LF}S_v\{E_v(i^v r^v)\} = \sum_{m=0}^{\infty} i^{vm} u^{vm}. \tag{2.16}$$

$$ {}^{LF}S_v\left\{\frac{r^v}{\Gamma(1+v)}\right\} = u^v. \tag{2.17}$$

Definition 2.5. The local fractional Sumudu transform of $\Phi(r)$ of order $v$is

$$ {}^{LF}S_v\{\Phi(r)\} = \mathrm{F}_v(u) = \frac{1}{\Gamma(1+v)} \int_0^{\infty} E_v(-u^{-v} r^v) \frac{\Phi(r)}{u^v} (dr)^v, \quad 0 < v \leqslant 1 \tag{2.18}$$

Following (2.18), its inverse formula is defined by

$$ {}^{LF}S_v^{-1}\{\mathrm{F}_v(u)\} = \Phi(r), \quad 0 < v \leqslant 1. \tag{2.19}$$

Theorem 2.1. *(linearity). If ${}^{LF}S_v\{\Phi(r)\} = \mathrm{F}_v(u)$ and ${}^{LF}S_v\{\varphi(r)\} = \Psi_v(u)$, then one has*

$$ {}^{LF}S_v\{\Phi(r) + \varphi(r)\} = \mathrm{F}_v(u) + \Psi_v(u). \tag{2.20}$$

Proof. From the definition (2.18), we obtain the result. □

Theorem 2.2. *(local fractional Laplace-Sumudu duality). If $L_v\{\Phi(r)\} = \Phi_s^{L,v}(s)$ and ${}^{LF}S_v\{\Phi(r)\} = F_v(u)$, then one has*

$$ {}^{LF}S_v\{\Phi(r)\} = \frac{1}{u^v} L_v\left\{\Phi(\frac{1}{r})\right\}, \tag{2.21}$$

$$ L_v\{\Phi(r)\} = \frac{S_v\{\Phi(\frac{1}{s})\}}{s^v}. \tag{2.22}$$

Proof. By using the formulas (2.6) and (2.18), we get the results of this theorem. □

Theorem 2.3. *(1) (local fractional Sumudu transform of local fractional derivative). If ${}^{LF}S_v\{\Phi(r)\} = \mathrm{F}_v(u)$, then one has*

$$ {}^{LF}S_v\left\{\frac{d^v\Phi(r)}{dr^v}\right\} = \frac{\mathrm{F}_v(u) - \mathrm{F}(0)}{u^v}. \tag{2.23}$$

As the direct result of (2.23), we have the following results. If ${}^{LF}S_v\{\Phi(r)\} = \mathrm{F}_v(u)$, we obtain

$$ {}^{LF}S_v\left\{\frac{d^{nv}\Phi(r)}{dr^{nv}}\right\} = \frac{1}{u^{nv}}\left[\mathrm{F}_v(u) - \sum_{k=0}^{n-1} u^{kv}\Phi^{(kv)}(0)\right]. \tag{2.24}$$

When $n = 2$, from (2.24), we obtain

$$ {}^{LF}S_v\left\{\frac{d^{2v}\Phi(r)}{dr^{2v}}\right\} = \frac{1}{u^{2v}}\left[\mathrm{F}_v(u) - \Phi(0) - u^v\Phi^{(v)}(0)\right] \tag{2.25}$$

(2) (local fractional Sumudu transform of local fractional integral). If $S_v\{\Phi(r)\} = \mathrm{F}_v(u)$, then we have

$$ {}^{LF}S_v\left\{{}_0I_r^{(v)}\Phi(r)\right\} = u^v\mathrm{F}_v(u). \tag{2.26}$$

Proof. (see Srivastava et al. (1763)). □

Theorem 2.4. *(local fractional convolution). If ${}^{LF}S_v\{\Phi(r)\} = \mathrm{F}_v(u)$ and ${}^{LF}S_v\{\varphi(r)\} = \Psi_v(u)$, then one has*

$$ {}^{LF}S_v\{\Phi(r) * \varphi(r)\} = u^v\mathrm{F}_v(u)\Psi_v(u), \tag{2.27}$$

with

$$ \Phi(r) * \varphi(r) = \frac{1}{\Gamma(1+v)} \int_0^{\infty} \Phi(\tau)\varphi(r-\tau)(dr)^v. $$

Proof. (see Srivastava et al. (1763)). □

3. Local fractional Sumudu decomposition method

Let us consider the following linear operator with local fractional derivative

$$ L_v U(r,\tau) + R_v U(r,\tau) = g(r,\tau), \tag{3.1}$$

where $L_\alpha = \frac{\partial^{m\alpha}}{\partial t^{m\alpha}}$ $(m \in \mathbb{N}^*)$ denotes linear local fractional derivative operator of order mv, R_v denotes linear local fractional derivative operator of order less than L_v, and $g(r,\tau)$ is the non-differentiable source term.

Taking the local fractional Sumudu transform (denoted in this paper by ${}^{LF}S_v$) on both sides of (3.1), we obtain

$$ {}^{LF}S_v[L_v U(r,\tau)] + {}^{LF}S_v[R_v U(r,\tau)] = {}^{LF}S_v[g(r,\tau)]. \tag{3.2}$$

Using the property of the local fractional Sumudu transform, it follows

$$ {}^{LF}S_v[U(r,\tau)] = \sum_{k=0}^{m-1} u^{kv}\frac{\partial^{kv}U(r,0)}{\partial t^{kv}} + u^{mv}\left({}^{LF}S_v[g(r,\tau)]\right) - u^{mv}\left({}^{LF}S_v[R_v U(r,\tau)]\right). \tag{3.3}$$

In taking the inverse local fractional Sumudu transform of both sides of (3.3), it follows

$$ U(r,\tau) = \sum_{k=0}^{m-1} u^{kv}\frac{\partial^{kv}U(r,0)}{\partial t^{kv}}\frac{t^{kv}}{\Gamma(1+kv)} + {}^{LF}S_v^{-1}\left(u^{mv}\left({}^{LF}S_v[g(r,\tau)]\right)\right) - {}^{LF}S_v^{-1}\left(u^{mv}\left({}^{LF}S_v[R_v U(r,\tau)]\right)\right). \tag{3.4}$$

According to the Adomian decomposition method Adomian et al., 1989, we decompose the unknown function U as an infinite series given by

$$ U(r,\tau) = \sum_{n=0}^{\infty} U_n(r,\tau). \tag{3.5}$$

Substituting (3.5) in (3.4), we get

$$ \sum_{n=0}^{\infty} U_n(r,\tau) = \sum_{k=0}^{m-1}\left[\frac{\partial^{kv}U(r,0)}{\partial t^{kv}}\frac{t^{kv}}{\Gamma(1+kv)}\right] + {}^{LF}S_v^{-1}\left(u^{mv}\left({}^{LF}S_v[g(r,\tau)]\right)\right) - {}^{LF}S_v^{-1}\left(u^{mv}\left({}^{LF}S_v\left[R_v\sum_{n=0}^{\infty}U_n(r,\tau)\right]\right)\right). \tag{3.6}$$

On comparing both sides of (3.6), it then comes

$$
\begin{aligned}
U_0(r,\tau) &= \sum_{k=0}^{m-1}\left[\frac{\partial^{k\nu}U(r,0)}{\partial t^{k\nu}}\frac{t^{k\nu}}{\Gamma(1+k\nu)}\right] + {}^{LF}S_\nu^{-1}\left(u^{m\nu LF}S_\nu[g(r,\tau)]\right),\\
U_1(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^{m\nu}\left({}^{LF}S_\nu[R_\nu(U_0(r,\tau))]\right)\right),\\
U_2(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^{m\nu}\left({}^{LF}S_\nu[R_\nu(U_1(r,\tau))]\right)\right),\\
U_3(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^{m\nu}\left({}^{LF}S_\nu[R_\nu(U_2(r,\tau))]\right)\right),\\
&\vdots
\end{aligned}
\tag{3.7}
$$

The local fractional recursive relation in its general form is

$$
\begin{aligned}
U_0(r,\tau) &= \sum_{k=0}^{m-1}\frac{\partial^{k\nu}U(r,0)}{\partial t^{k\nu}}\frac{t^{k\nu}}{\Gamma(1+k\nu)} + {}^{LF}S_\nu^{-1}(u^{m\nu LF}S_\nu[g(r,\tau)]),\\
U_n(r,\tau) &= -{}^{LF}S_\nu^{-1}(u^{m\nu LF}S_\nu[R_\nu(U_{n-1}(r,\tau))]),
\end{aligned}
\tag{3.8}
$$

where $0<\nu\leqslant 1, n\in\mathbb{N}^*$ and $m=1,2,3,\ldots$

4. Applications

In this section, we will implement the proposed method local fractional Sumudu decomposition method (LFSDM) for solving some examples.

Example 4.1. First, we consider the following local fractional partial differential equation

$$
\begin{aligned}
&\frac{\partial^{\nu}U(r,\tau)}{\partial\tau^{\nu}} + \frac{\partial^{2\nu}U(r,\tau)}{\partial r^{2\nu}} - U(r,\tau) = 0,\\
&U(r,0) = \sin_\nu(r^\nu).
\end{aligned}
\tag{4.1}
$$

From (3.8) and (4.1), the successive approximations are

$$
\begin{aligned}
U_0(r,\tau) &= \sin_\nu(r^\nu),\\
U_n(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^\nu\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_{n-1}(r,\tau)}{\partial r^{2\nu}} - U_{n-1}(r,\tau)\right]\right)\right),\quad n\geqslant 1.
\end{aligned}
\tag{4.2}
$$

According to the successive formula (4.2), we have

$$
\begin{aligned}
U_0(r,\tau) &= \sin_\nu(r^\nu),\\
U_1(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^\nu\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_0(r,\tau)}{\partial r^{2\nu}} - U_0(r,\tau)\right]\right)\right),\\
U_2(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^\nu\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_1(r,\tau)}{\partial r^{2\nu}} - U_1(r,\tau)\right]\right)\right),\\
U_3(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^\nu\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_1(r,\tau)}{\partial r^{2\nu}} - U_1(r,\tau)\right]\right)\right),\\
&\vdots
\end{aligned}
\tag{4.3}
$$

and so on.

From the formulas (4.3), the first terms of local fractional Sumudu decomposition method are given by

$$
\begin{aligned}
U_0(r,\tau) &= \sin_\nu(r^\nu),\\
U_1(r,\tau) &= \sin_\nu(r^\nu)\frac{2\tau^\nu}{\Gamma(1+\nu)},\\
U_2(r,\tau) &= \sin_\nu(r^\nu)\frac{4\tau^{2\nu}}{\Gamma(1+2\nu)},\\
U_3(r,\tau) &= \sin_\nu(r^\nu)\frac{8\tau^{3\nu}}{\Gamma(1+3\nu)},\\
&\vdots\\
U_n(r,\tau) &= \sin_\nu(r^\nu)\frac{(2\tau^\nu)^n}{\Gamma(1+n\nu)}.
\end{aligned}
\tag{4.4}
$$

Then the local fractional series form is

$$
U(r,\tau) = \sin_\nu(r^\nu)\left(1+\frac{2\tau^\nu}{\Gamma(1+\nu)}+\frac{(2\tau^\nu)^2}{\Gamma(1+2\nu)}+\frac{(2\tau^\nu)^3}{\Gamma(1+3\nu)}+\cdots+\frac{(2\tau^\nu)^n}{\Gamma(1+n\nu)}+\cdots\right)
\tag{4.5}
$$

Hence the exact solution of (4.1) by local fractional Sumudu decomposition method has the form

$$
U(r,\tau) = \sin_\nu(r^\nu)\sum_{n=0}^{\infty}\frac{(2\tau^\nu)^n}{\Gamma(1+n\nu)} = \sin_\nu(r^\nu)E_\nu(2\tau^\nu).
\tag{4.6}
$$

Example 4.2. Second, we consider the local fractional Laplace equation as Yang et al., 2026

$$
\begin{aligned}
&\frac{\partial^{2\nu}U(r,\tau)}{\partial\tau^{2\nu}} + \frac{\partial^{2\nu}U(r,\tau)}{\partial r^{2\nu}} = 0,\\
&U(r,0) = 0,, \frac{\partial^{\nu}U(r,0)}{\partial\tau^{\nu}} = -E_\nu(r^\nu).
\end{aligned}
\tag{4.7}
$$

From (3.8) and (4.7), the formula of successive approximations is

$$
\begin{aligned}
U_0(r,\tau) &= -E_\nu(r^\nu)\frac{\tau^\nu}{\Gamma(1+\nu)},\\
U_n(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^{2\nu}\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_{n-1}(r,\tau)}{\partial r^{2\nu}}\right]\right)\right), n\geqslant 1.
\end{aligned}
\tag{4.8}
$$

According to the successive formula (4.8), we obtain

$$
\begin{aligned}
U_0(r,\tau) &= -E_\nu(r^\nu),\\
U_1(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^{2\nu}\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_0(r,\tau)}{\partial r^{2\nu}}\right]\right)\right),\\
U_2(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^{2\nu}\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_1(r,\tau)}{\partial r^{2\nu}}\right]\right)\right),\\
U_3(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^{2\nu}\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_2(r,\tau)}{\partial r^{2\nu}}\right]\right)\right),\\
&\vdots\\
U_n(r,\tau) &= -{}^{LF}S_\nu^{-1}\left(u^{2\nu}\left({}^{LF}S_\nu\left[\frac{\partial^{2\nu}U_{n-1}(r,\tau)}{\partial r^{2\nu}}\right]\right)\right).
\end{aligned}
\tag{4.9}
$$

From the formulas (4.9), the first terms of local fractional Sumudu decomposition method are given by

$$
\begin{aligned}
U_0(r,\tau) &= -E_\nu(r^\nu),\\
U_1(r,\tau) &= E_\nu(r^\nu)\frac{\tau^{3\nu}}{\Gamma(1+3\nu)},\\
U_2(r,\tau) &= -E_\nu(r^\nu)\frac{\tau^{5\nu}}{\Gamma(1+5\nu)},\\
U_3(r,\tau) &= E_\nu(r^\nu)\frac{\tau^{7\nu}}{\Gamma(1+7\nu)},\\
&\vdots\\
U_n(r,\tau) &= E_\nu(r^\nu)(-1)^{n+1}\frac{\tau^{(2n+1)\nu}}{\Gamma(1+(2n+1)\nu)}.
\end{aligned}
\tag{4.10}
$$

Then the local fractional series form is

$$
U(r,\tau) = -E_\nu(r^\nu)(1-\frac{\tau^{3\nu}}{\Gamma(1+3\nu)}+\frac{\tau^{5\nu}}{\Gamma(1+5\nu)}-\cdots+(-1)^n \times\frac{\tau^{(2n+1)\nu}}{\Gamma(1+(2n+1)\nu)}+\cdots)
\tag{4.11}
$$

Hence the exact solution of (4.7) by local fractional Sumudu decomposition method is given by

$$U(r,\tau) = -E_v(r^v)\sum_{n=0}^{\infty}(-1)^n \frac{\tau^{(2n+1)v}}{\Gamma(1+(2n+1)v)} = -E_v(r^v)\sin_v(\tau^v). \tag{4.12}$$

Example 4.3. Finally, we consider the following local fractional partial differential equation with the initial conditions

$$\frac{\partial^{4v}U(r,\tau)}{\partial\tau^{4\alpha}} - 2^v\frac{\partial^{3v}U(r,\tau)}{\partial r^{3v}} = 1, \tag{4.13}$$

$$\begin{aligned} &U(r,0) = E_v((2r)^v), \frac{\partial^v U(r,0)}{\partial\tau^v} = 0,\\ &\frac{\partial^{2v}U(r,0)}{\partial\tau^{2v}} = -2^{2v}E_v((2r)^v), \frac{\partial^{3v}U(r,0)}{\partial\tau^{3v}} = 0. \end{aligned} \tag{4.14}$$

From (3.8) and (4.13), we get the following formula

$$\begin{aligned} &U_0(r,\tau) = E_v((2r)^v) - 2^{2v}E_v((2r)^v)\frac{\tau^{2v}}{\Gamma(1+2v)} + \frac{\tau^{4v}}{\Gamma(1+4v)},\\ &U_n(r,\tau) = {}^{LF}S_v^{-1}\left(u^{4v}\left({}^{LF}S_v\left[2^v\frac{\partial^{3v}U_{n-1}(r,\tau)}{\partial r^{3v}}\right]\right)\right), n \geqslant 1. \end{aligned} \tag{4.15}$$

Using the formula (4.15), we obtain the following successive approximations

$$\begin{aligned} &U_0(r,\tau) = E_v((2r)^v)\left(1 - 2^{2v}\frac{\tau^{2v}}{\Gamma(1+2v)}\right) + \frac{\tau^{4v}}{\Gamma(1+4v)},\\ &U_1(r,\tau) = {}^{LF}S_v^{-1}\left(u^{4v}\left({}^{LF}S_v\left[2^v\frac{\partial^{3v}U_0(r,\tau)}{\partial r^{3v}}\right]\right)\right),\\ &U_2(r,\tau) = {}^{LF}S_v^{-1}\left(u^{4v}\left({}^{LF}S_v\left[2^v\frac{\partial^{3v}U_1(r,\tau)}{\partial r^{3v}}\right]\right)\right),\\ &U_3(r,\tau) = {}^{LF}S_v^{-1}\left(u^{4v}\left({}^{LF}S_v\left[2^v\frac{\partial^{3v}U_{2+*}(r,\tau)}{\partial r^{3v}}\right]\right)\right),\\ &\vdots\\ &U_n(r,\tau) = {}^{LF}S_v^{-1}\left(u^{4v}\left({}^{LF}S_v\left[2^v\frac{\partial^{3v}U_{n-1}(r,\tau)}{\partial r^{3v}}\right]\right)\right). \end{aligned} \tag{4.16}$$

According to the formulas (4.16), the first terms of local fractional Sumudu decomposition method have the form

$$\begin{aligned} &U_0(r,\tau) = E_v((2r)^v)\left(1 - 2^{2v}\frac{\tau^{2v}}{\Gamma(1+2v)}\right) + \frac{\tau^{4v}}{\Gamma(1+4v)},\\ &U_1(r,\tau) = E_v((2r)^v)\left(2^{4v}\frac{\tau^{4v}}{\Gamma(1+4v)} - 2^{6v}\frac{\tau^{6\alpha}}{\Gamma(1+6v)}\right),\\ &U_2(r,\tau) = E_v((2r)^v)\left(2^{8v}\frac{\tau^{8v}}{\Gamma(1+8v)} - 2^{10v}\frac{\tau^{10v}}{\Gamma(1+10v)}\right),\\ &U_3(r,\tau) = E_v((2r)^v)\left(2^{12v}\frac{\tau^{12v}}{\Gamma(1+12v)} - 2^{14v}\frac{\tau^{14v}}{\Gamma(1+14v)}\right)\\ &\vdots \end{aligned} \tag{4.17}$$

Then the local fractional series form is

$$\begin{aligned} U(r,\tau) = &\frac{\tau^{4v}}{\Gamma(1+4v)} + E_v((2r)^v)\left(1 - \frac{(2\tau)^{2v}}{\Gamma(1+2v)} + \frac{(2\tau)^{4v}}{\Gamma(1+4v)} - \frac{(2\tau)^{6v}}{\Gamma(1+6v)}\right.\\ &\left. + \frac{(2\tau)^{8v}}{\Gamma(1+8v)} - \frac{(2\tau)^{10v}}{\Gamma(1+10v)} + \cdots + (-1)^n\frac{(2\tau)^{2nv}}{\Gamma(1+2nv)} + \cdots\right) \end{aligned} \tag{4.18}$$

Hence the exact solution of (4.13) by local fractional Sumudu decomposition method take the form

$$\begin{aligned} U(r,\tau) &= \frac{\tau^{4v}}{\Gamma(1+4v)} + E_v((2r)^v)\sum_{n=0}^{\infty}(-1)^n\frac{(2\tau)^{2nv}}{\Gamma(1+2nv)}\\ &= \frac{\tau^{4v}}{\Gamma(1+4v)} + E_v((2r)^v)\cos((2\tau)^v). \end{aligned} \tag{4.19}$$

5. Conclusion

The coupling of Adomian decomposition method (ADM) and the Sumudu transform method in the sense of local fractional derivative, proved very effective to solve linear local fractional partial differential equations. The proposed algorithm provides the solution in a series form that converges rapidly to the exact solution if it exists. From the obtained results, it is clear that the LFSDM yields very accurate solutions using only a few iterates. As a result, the conclusion that comes through this work is that LFSDM can be applied to other linear local fractional partial differential equations of higher order, due to the efficiency and flexibility in the application as can be seen in the proposed examples.

References

Abbasbandy, S., Shivanian, E., Vajravelu, K., 2011. Mathematical properties of h-curve in the frame work of the homotopy analysis method. Commun. Nonlinear Sci. Numer. Simul. 16 (11), 4268–4275.

Adomian, G., 1989. Nonlinear Stochastic Systems Theory and Applications to Physics. Kluwer Academic Publishers, Netherlands.

Adomian, G., 1994. Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers, Boston.

Adomian, G., Rach, R., 1990. Equality of partial solutions in the decomposition method for linear or nonlinear partial differential equations. Comput. Math. Appl. 10, 9–12.

Aslefallah, M., Shivanian, E., 2015. Nonlinear fractional integro-differential reaction-diffusion equation via radial basis functions. Eur. Phys. J. Plus. 130 (47), 1–9.

Baleanu, D., Machado, J.A.T., Cattani, C., Baleanu, M.C., Yang, X.J., 5350. Local fractional variational iteration and decomposition methods for wave equation on cantor sets within local fractional operators. Abs. Appl. Anal., 1–6 A. ID 535048.

Golmankhaneh, A.K., Yang, X.J., Baleanu, D., 2015. Einstein field equations within local fractional calculus. Rom. J. Phys. 60 (1–2), 22–31. Bucharest.

He, J.H., 1997. A new approach to nonlinear partial differential equations. Comm. Nonlinear Sci. Numer. Simul. 2, 203–205.

He, J.H., 1998. A variational iteration approach to nonlinear problems and its applications. Mech. Appl. 20, 30–31.

He, J.H., 1998. Approximate analytical solution for seepage flow with fractional derivatives in porous media. Comput. Meth. Appl. Mech. Eng. 167, 57–68.

He, J.H., 1999. Homotopy perturbation technique. Comput. Meth. Appl. Mech. Eng. 178, 257–262.

He, J.H., 2000. A coupling method of homotopy technique and perturbation technique for nonlinear problems. Int. J. Nonlinear Mech. 35, 37–43.

He, J.H., 2005. Application of homotopy perturbation method to nonlinear wave equations. Chaos Solitons Fractals 26, 695–700.

Hosseini, V.R., Shivanian, E., Chen, W., 2016. Local radial point interpolation (MLRPI) method for solving time fractional diffusion-wave equation with damping. J. Comput. Phys. 312, 307–332.

Jassim, H.K., 2015. Local fractional Laplace decomposition method for nonhomogeneous heat equations arising in fractal heat flow with local fractional derivative. Int. J. Adv. Appl. Math. Mech. 2 (4), 1–7.

Kumar, D., Singh, J., Baleanu, D., 2017. A fractional model of convective radial fins with temperature-dependent thermal conductivity. Rom. Rep. Phys. 69 (103), 1–13.

Kumar, D., Singh, J., Baleanu, D., 2017. A hybrid computational approach for Klein-Gordon equations on Cantor sets. Nonlinear Dyn. 87, 511–517.

Singh, J., Kumar, D., Swroop, R., 2016. Numerical solution of time- and space-fractional coupled Burgers equations via homotopy algorithm. Alex. Eng. 55 (2), 1753–1763.

Singh, J., Kumar, D., Nieto, J.J., 2016. A reliable algorithm for local fractional Tricomi equation arising in fractal transonic flow. Entropy 18 (6), 1–8.

Soltani, L.A., Shivanian, E., Ezzati, R., 2016. Convection–radiation heat transfer in solar heat exchangers filled with a porous medium: exact and shooting homotopy analysis solution. Appl. Therm. Eng. 103, 537–542.

Srivastava, H.M., Golmankhaneh, A.K., Baleanu, D., Yang, X.J., 1763. Local fractional sumudu transform with application to IVPs on cantor sets. Abs. Appl. Anal., 1–7 A. ID 176395.

Yang, X.J., 2011. Fractional Functional Analysis and Its Applications. Asian Academic, Hong Kong.
Yang, X.J., 2012. Local Fractional Calculus and Its Applications. World Science Publisher, New York, NY, USA.
Yang, Y.J., Hua, L.Q., 2014. Variational iteration transform method for fractional differential equations with local fractional derivative. Abs. Appl. Anal., 1–9 Article ID 760957.
Yang, X.J., Baleanu, D., Zhong, W.P., 2013. Approximate solutions for diffusion equations on cantor space-time. Proc. Rom. Aca., Series A 14 (2), 127–133.
Yang, X.J., Baleanu, D., Khan, Y., Mohyud-din, S.T., 2014. Local fractional variational iteration method for diffusion and wave equation on cantor sets. Rom. J. Phys. 59 (1–2), 36–48. Bucharest.
Yang, X.J., Baleanu, D., Lazarevic, M.P., Cajic, M.S., 2015. Fractal boundary value problems for integral and differential equations fractional operators. Ther. Sci. 19 (3), 959–966.
Yang, X.J., Srivastava, H.M., Cattani, C., 2015. Local fractional homotopy perturbation method for solving fractal partial differential equations arising in mathematical physics. Rom. Rep. Phys. 67 (3), 752–761.
Yang, Y.J., Baleanu, D., Yang, X.J., 2026. A local fractional variational iteration method for laplace equation within local fractional operators. Abs. Appl. Anal., 1–6 A. ID 202650.
Yang, A.M., Li, J., Srivastava, H.M., Xie, G.N., Yang, X.J., 3659. Local fractional laplace variational iteration method for solving linear partial differential equations with local fractional derivative. Dis. Dyn. Nat. Soc., 1–8 A. ID 365981.
Zhang, Y., Cattani, C., Yang, X.J., 2015. Local fractional homotopy perturbation method for solving non-homogeneous heat conduction equations in fractal domains. Entropy 17, 6753–6764.
Ziane, D., 2016. Méthode Combinée des Perturbations HPM et VIM pour la Résolution des Equations Différentielles Ordinaires et EDP d'Ordre Fractionnaire (Doctoral Thesis). University of Oran1, Ahmed Ben Bella, Oran, 31000, Algeria.