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MODIFIED ATANGANA-BALEANU FRACTIONAL DIFFERENTIAL OPERATORS

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Abstract. Fractional differential operators are mostly investigated for functions of real variables. In this paper, we present two fractional differential operators for a class of normalized analytic functions in the open unit disk. The suggested operators are investigated according to concepts in geometric function theory, using the concepts of convexity and starlikeness. Therefore, we reformulate the new operators in the Ma-Minda class of analytic functions, in order to act on normalized analytic functions. Our method is based on subordination, superordination, and majorization theory. As an application, we employ these operators to generalize Bernoulli's equation and a special class of Briot-Bouquet equations. The solution of the generalized equation is formulated by a hypergeometric function.

1. Introduction

Complex fractional differential and integral operators are suggested by Srivastava and Owa [22]. These operators are generalized by Ibrahim using two parameters for a variable in a complex domain [11]. These operators are utilized to formulate different classes of analytic functions, which are called fractional analytic functions [23]. Moreover, they are formulated in the general class of algebraic fractional differential equations to study the stability [12, 13]. Recent applications of the fractional calculus in science and engineering, including numerical solutions for different types of fractional differential equations can be located in [17]-[25].

We continue our study concerning the topic: fractional differential operators of a complex variable. In this study, we extend the well known AB-fractional differential operators to a complex domain. We desire to investigate the geometric properties of these operators. Therefore, we act them on the class of normalized analytic functions. As a result, we indicate that these operators are normalized under some conditions. Consequently, we formulate them in the Ma-Minda class of analytic functions using the subordination concept. This class will bring the geometric properties of the suggested operators such as starlikeness and convexity

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in the open unit disk. As an application, we employ these operators to generalize Bernoulli and Briot-Bouquet differential equations. We discover that the solutions can be formulated by hypergeometric functions.

2. Methods

We utilize the following notions.

2.1. Geometric concepts. In this part, we illustrate some concepts in the geometric function theory, which are located in [15]-[18]

Definition 2.1. Let $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Two analytic functions f_1, f_2 in $\mathbb{U}$ are subordinated ($f_1 \prec f_2$ or $f_1(z) \prec f_2(z), z \in \mathbb{U}$) if there exists an analytic function $\omega, |\omega| \leq |z| < 1$ satisfying

$$f_1(z) = f_2(\omega(z)), \quad z \in \mathbb{U}.$$

And f_1 is majorized by f_2 ($f_1 \ll f_2$) if ω satisfies

$$f_(z) = \omega(z) f_2(z), \quad z \in \mathbb{U}.$$

There is a connection between subordination and majorization [5] in the open unit disk for some special classes including the convex class ($\mathcal{C}$)

$$1 + \Re\left(\frac{zv''(z)}{v'(z)}\right) > 0, \quad z \in \mathbb{U}$$

and starlike functions ($\mathcal{S}^*$)

$$\Re\left(\frac{zv'(z)}{v(z)}\right) > 0, \quad z \in \mathbb{U}.$$

Definition 2.2. We assume the following class of normalized analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, z \in \mathbb{U}.$$

This class is symbolized by Λ and it represents the class of univalent functions when $f(0) = f'(0) - 1 = 0$.

Two functions $f, g \in \Lambda$ are called convoluted ($f * g$) when

$$\begin{aligned}(f * g)(z) &= \left(z + \sum_{n=2}^{\infty} a_n\, z^n\right) * \left(z + \sum_{n=2}^{\infty} g_n\, z^n\right) \\ &= z + \sum_{n=2}^{\infty} a_n g_n z^n.\end{aligned}$$

Definition 2.3. The generalized Mittag-Leffler function is formulated by [24]

$$E^{\vartheta}_{\nu,\mu}(z) = \sum_{n=0}^{\infty} \frac{(\vartheta)_n}{\Gamma(\nu n + \mu)} \frac{z^n}{n!},$$

where $(\vartheta)_n$ indicates the Pochhammer symbol and

$$E^{1}_{\nu,\mu}(z) = E_{\nu,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + \mu)}.$$

2.2. Complex fractional differential operator. A fractional differential operator for the complex Atangana and Baleanu is defined as follows [7] (AB fractional operator for a real variable can be located in [4]): the Hankel contour $\mathbb{D}$ can be defined as the union of the following three subcontours:

- $[D1] := \{z + re^{-i\pi}(z-c) : 1 > r > \varepsilon\}$,
- $[D2] := \{z + \varepsilon\, e^{i\theta}(z-c) : -\pi < \theta < \pi\}$,
- $[D3] := \{z + r\, e^{i\pi}(z-c) : \varepsilon < r < 1\}$

where ε is a small positive constant; the following fractional differential operators are defined

$$^{ABC}\Delta^{\nu} h(z) = \frac{\alpha(\nu)}{2\pi i(1-\nu)} \int_{\mathbb{D}} h'(\zeta) E^{\nu}(-\mu_{\nu}(z-\zeta)^{\nu}) d\zeta, \tag{2.1}$$

where $\alpha(\nu)$ is normalized by $\alpha(0) = \alpha(1) = 1$ and $E_{\nu}(\omega)$ is the Mittag-Leffler function taking the modified formula

$$E^{\nu}(\chi) = \sum_{n=0}^{\infty} \Gamma(-\nu n)\chi^{n}.$$

Moreover, they introduced the following fractional differential operator

$$^{ABR}\Delta^{\nu} h(z) = \frac{\alpha(\nu)}{2\pi i(1-\nu)} \frac{d}{dz} \int_{\mathbb{D}} h(\zeta) E^{\nu}(-\mu_{\nu}(z-\zeta)^{\nu}) d\zeta, \tag{2.2}$$

$$\Big(\mu_{\nu} = \frac{\nu}{1-\nu},\ 0 \le \nu \le 1\}\Big).$$

To present the AB-modified fractional differential operators of a complex variable, we shall use double Mittag-Leffler functions in Definition 2.3. The justification for this modification is that in the geometric function theory, we deal with different classes of analytic functions, such as normalized, mutivalent, harmonic, meromorphic, multivalent meromorphic, multivalent harmonic, ...etc. Therefore, one of the suggested Mittag-Leffler functions should concern about the power of the variable z in $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This technique is important to satisfy the existence of the AB-fractional differential operator in the same class of the function f.

Definition 2.4. For $f \in \Lambda$, the above operators (2.1) and (2.2) are extended to the complex plane as follows:

$$^{ABC}\Delta_z^{\nu} f(z) = \frac{\alpha(\nu)}{1-\nu} \int_0^z f'(\zeta) E_{\nu,\omega}(-\mu_{\nu}\zeta^{\nu}) E_{\nu}(-\mu_{\nu}(z-\zeta)^{\nu}) d\zeta, \tag{2.3}$$

and

$$^{ABR}\Delta_z^{\nu} f(z) = \frac{\alpha(\nu)}{1-\nu} \frac{d}{dz} \int_0^z f(\zeta) E_{\nu,\omega}(-\mu_{\nu}\zeta^{\nu}) E_{\nu}(-\mu_{\nu}(z-\zeta)^{\nu}) d\zeta, \tag{2.4}$$

where ω indicates the power of z in the power series of $f(z)$.

Example 2.1. Assume that $f(z) = z$, According to [21]-Theorem 2.4 or [10]-Theorem 11.2, we have

$$\begin{aligned} {}^{ABC}\Delta_z^\nu(z) &= \frac{\alpha(\nu)}{1-\nu}\int_0^z E_\nu(-\mu_\nu \zeta^\nu)E_\nu(-\mu_\nu(z-\zeta)^\nu)d\zeta \\ &= \frac{\alpha(\nu)}{1-\nu}\left(zE^2_{\nu,2}(-\mu_\nu(z)^\nu)\right) \\ &= \frac{\alpha(\nu)}{1-\nu}\left(z\sum_{k=0}^{\infty}\frac{(2)_k z^k}{k!\Gamma(k\nu+2)}\right), \end{aligned}$$

where $(t)_n = t(t+1)...(t+n-1)$. Now by virtue of [21]-Theorem 2.2, we attain

$$\begin{aligned} {}^{ABR}\Delta_z^\nu(z) &= \frac{\alpha(\nu)}{1-\nu}\frac{d}{dz}\int_0^z E_\nu(-\mu_\nu \zeta^\nu)E_\nu(-\mu_\nu(z-\zeta)^\nu)\zeta\, d\zeta \\ &= \frac{\alpha(\nu)}{1-\nu}\left(z^2E^2_{\nu,3}(-\mu_\nu(z)^\nu)\right)' \\ &= \frac{\alpha(\nu)}{1-\nu}\left(zE^2_{\nu,2}(-\mu_\nu(z)^\nu)\right). \end{aligned}$$

Consequently, we receive

$$ {}^{ABC}\Delta_z^\nu(z) = {}^{ABR}\Delta_z^\nu(z). $$

In general, we get

$$ {}^{ABC}\Delta_z^\nu(z^n) = \left(\frac{\alpha(\nu)}{1-\nu}\right) nz^n\left(E^2_{\nu,1+n}(-\mu_\nu(z)^\nu)\right), \quad n \geq 1, $$

$$ {}^{ABR}\Delta_z^\nu(z^n) = \left(\frac{\alpha(\nu)}{1-\nu}\right) z^n\left(E^2_{\nu,1+n}(-\mu_\nu(z)^\nu)\right). $$

We then look at some of the above operators' geometric behaviors.

Proposition 2.1. *Let $f \in \Lambda$ and $b(\nu) := \dfrac{\alpha(\nu)}{1-\nu}$. Then*

(A)

$$ {}^{\mathfrak{ABC}}\Delta_z^\nu f(z) := \frac{{}^{ABC}\Delta_z^\nu f(z)}{b(\nu)E^2_{\nu,2}(-\mu_\nu(z)^\nu)} \in \Lambda $$

and

$$ {}^{\mathfrak{ABR}}\Delta_z^\nu f(z) := \frac{{}^{ABR}\Delta_z^\nu f(z)}{b(\nu)E^2_{\nu,2}(-\mu_\nu(z)^\nu)} \in \Lambda; $$

(B) ${}^{\mathfrak{ABR}}\Delta_z^\nu f(z) \ll {}^{\mathfrak{ABC}}\Delta_z^\nu f(z)$;

(C) ${}^{\mathfrak{ABR}}\Delta_z^\nu f(z) \prec {}^{\mathfrak{ABC}}\Delta_z^\nu f(z)$, *whenever* ${}^{\mathfrak{ABC}}\Delta_z^\nu f(z)$ *is convex* [8] *for* $0.28 < |z| := \varrho \leq \sqrt{2}-1$ *or a starlike function* [8] *for* $0.21 < \varrho < 0.3$;

(D) $\left({}^{\mathfrak{ABR}}\Delta_z^\nu f(z)\right)' \prec \left({}^{\mathfrak{ABC}}\Delta_z^\nu f(z)\right)'$, *whenever* ${}^{\mathfrak{ABC}}\Delta_z^\nu f(z)$ *is locally univalent in* $\varrho \leq 3-\sqrt{8}$;

(E) ${}^{\mathfrak{ABC}}\Delta_z^\nu f(z) = z\left({}^{\mathfrak{ABR}}\Delta_z^\nu f(z)\right)', \quad \nu \to 0.$

Proof. Let $f \in \Lambda$. Then a computation leads to

$$\begin{aligned}
&{}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\\
&= \frac{{}^{ABC}\Delta_z^{\nu} f(z)}{b(\nu)E^2_{\nu,2}(-\mu_\nu(z)^\nu)}\\
&= \frac{z^{-\nu n}\, {}^{ABC}\Delta_z^{\nu} f(z)}{z^{-\nu n}\, b(\nu)E^2_{\nu,2}(-\mu_\nu(z)^\nu)}\\
&= \frac{z^{-\nu n} b(\nu)E^2_{\nu,2}(-\mu_\nu(z)^\nu)\, z + \sum_{n=2}^{\infty} z^{-\nu n} a_n b(\nu) n \left(E^2_{\nu,1+n}(-\mu_\nu(z)^\nu)\right) z^n}{z^{-\nu n} b(\nu)E^2_{\nu,2}(-\mu_\nu(z)^\nu)}\\
&= z + \sum_{n=2}^{\infty} a_n n \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^n\\
&\Rightarrow\ {}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) \in \Lambda.
\end{aligned}$$

In the same way, we get ${}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z) \in \Lambda$.
It is sufficient to prove [5]

$$|{}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)| \leq |{}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)|.$$

A calculation yields

$$\begin{aligned}
|{}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)| &= \left|z + \sum_{n=2}^{\infty} a_n \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^n\right|\\
&\leq \left|z + \sum_{n=2}^{\infty} a_n n \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^n\right|\\
&= |{}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)|.
\end{aligned}$$

The third part immediately follows by [5]-Corollary 1 and 2 respectively. For (D), since ${}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z) \prec {}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)$ when $|z| \leq 3 - \sqrt{8}$ then in view of [5]-Theorem 3, we have $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)' \prec \left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right)'$. Lastly, a computation implies the following conclusion

$$\begin{aligned}
z\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)' &= z\left(1 + \sum_{n=2}^{\infty} a_n n \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^{n-1}\right)\\
&= z + \sum_{n=2}^{\infty} a_n n \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^n\\
&={}^{\mathfrak{ABC}}\ \Delta_z^{\nu} f(z), \nu \to 0.
\end{aligned}$$

□

Definition 2.5. Assume that $f, g \in \Lambda$. Then f is in the class $\mathcal{A}_\nu(\alpha, g(z)), \alpha \in [0,1]$ if it achieves one of the following subordination inequalities [14] :

$$(1-\alpha)^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) + \alpha z[{}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)]' \prec g(z),$$

or

$$(1-\alpha)^{\mathfrak{ABR}}\Delta_z^{\nu} f(z) + \alpha z[{}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)]' \prec g(z),$$

which is equivalent to (Proposition 2.1-[(E)])

$$(1-\alpha)^{\mathfrak{ABR}}\Delta_z^{\nu} f(z) + \alpha[^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)] \prec g(z).$$

3. Results

We start with the following result:

Theorem 3.1. *Consider the modified AB-operators* $^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)$ *and* $^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)$. *If* $f \in \mathcal{C}$ *then* $^{\mathfrak{ABR}}\Delta_z^{\nu} f(z) \in \mathcal{C}$ *and* $^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) \in \mathcal{S}^*$, *whenever* $\nu \to 0$.

Proof. By the definition of the operators $^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)$, we have

$$\begin{aligned}
^{\mathfrak{ABR}}\Delta_z^{\nu} f(z) &= z + \sum_{n=2}^{\infty} a_n \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^n \\
&= \left(z + \sum_{n=2}^{\infty} \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^n\right) * \left(z + \sum_{n=2}^{\infty} a_n z^n\right) \\
&= \left(z + \sum_{n=2}^{\infty} z^n\right) * \left(z + \sum_{n=2}^{\infty} a_n z^n\right) \\
&= \left(\frac{z}{(1-z)}\right) * f(z) \\
&:= \mathbb{k}(z) * f(z),
\end{aligned}$$

where $\mathbb{k}(z) \in \mathcal{C}$ where $\mathcal{C}$ is the subclass of convex functions in $\mathbb{U}$. Thus, in view of the convolution properties [20], we have $^{\mathfrak{ABR}}\Delta_z^{\nu} f(z) \in \mathcal{C}$. Similarly, for the operator $^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)$, we obtain

$$\begin{aligned}
^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) &= z + \sum_{n=2}^{\infty} a_n n \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^n \\
&= \left(z + \sum_{n=2}^{\infty} n \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right) z^n\right) * \left(z + \sum_{n=2}^{\infty} a_n z^n\right) \\
&\approx \left(z + \sum_{n=2}^{\infty} n z^n\right) * \left(z + \sum_{n=2}^{\infty} a_n z^n\right) \\
&= \left(\frac{z}{(1-z)^2}\right) * f(z) \\
&:= K(z) * f(z),
\end{aligned}$$

where $K(z)$ is the Koebe function. Thus, in view of [20], the convolution properties imply that $^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) \in \mathcal{S}^*$. □

In the sequel, we request the next lemma [18](P139-140).

Lemma 3.1. *Let* $v \in \Lambda$. *Then*

(a) $v(z) + \alpha\xi\, v'(z) \prec (1+\alpha)z + \alpha z^2 \Rightarrow v(z) \prec z,$ *when* $\alpha \in (0, 1/3]$;
(b) $zv'(z)[1+v(z)] + \alpha v^2(z) \prec z + (1+\alpha)z^2 \Rightarrow v(z) \prec z,$ *when* $|1+\alpha| \le 1/4$;
(c) $[zv'(z) - v(z)]e^{\alpha(v(z))} + e^{v(z)} \prec e^z \Rightarrow v(z) \prec z,$ *when* $|\alpha - 1| \le \pi/2$;
(d) $zv'(z)(1+\alpha v(z)) + v(z) \prec 2z + \alpha z^2 \Rightarrow v(z) \prec z,$ *when* $|\alpha| \le 1/2$;

(e) $zv'(z)e^{\alpha\, v(z)} + v(z) \prec z(1+\alpha z e^{\alpha z}) \Rightarrow v(z) \prec z,$ *when* $|\alpha| \le 1;$

(f) $v(z) + \dfrac{zv'(z)}{1+\alpha\, v(z)} \prec z \Rightarrow v(z) \prec z,$ *when* $|\alpha| \le 1;$

and the solution is sharp.

Theorem 3.2. *Consider the operator* ${}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)$ *and* $\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right)$, $\nu \to 0.$ *Then*

(a) $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) + a\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right) \prec (1+a)z + az^2 \Rightarrow \left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec z,$ *when* $a \in (0, 1/3];$

(b) $\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right)[1+\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)] + a\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)^2 \prec z + (1+a)z^2 \Rightarrow$ $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec z,$ *when* $|1+a| \le 1/4;$

(c) $[\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right) - \left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)]e^{a\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)} + e^{\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)} \prec e^z \Rightarrow$ $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec z,$ *when* $|a-1| \le \pi/2;$

(d) $\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right)\left(1 + a\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)\right) + \left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec 2z + az^2 \Rightarrow$ $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec z,$ *when* $|a| \le 1/2;$

(e) $\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right)e^{a\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)} + \left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec z(1+aze^{az}) \Rightarrow$ $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec z,$ *when* $|a| \le 1;$

(f) $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) + \dfrac{\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right)}{1 + a\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)} \prec z \Rightarrow$ $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec z,$ *when* $|a| \le 1;$

and the solution is sharp.

Proof. According to Proposition 2.1, we obtain $\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \in \Lambda$; similarly, we have $\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right) \in \Lambda$. Let

$$v(z) := \left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right)$$

and

$$zv'(z) := \left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right).$$

Then in view of Proposition 2.1-[(E)], we have all the above desire assertions. □

Corollary 3.1. *If one of the subordination of Theorem 3.2 holds, then*

$$\Big|\left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right)\Big| \le \frac{n}{|a_n|}, \quad n \ge 2, a_n \ne 0. \tag{3.1}$$

Proof. In virtue of Theorem 3.2, we have

$$\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec z, \quad z \in \mathbb{U}.$$

But the function $h(z) = z$ is starlike, where $\Re(zh'(z)/h(z)) > 0$. Thus, in view of results in [19] or [6], the inequality $F \prec G, G \in \mathcal{S}$ (the class of univalent functions) implies $|\phi_n| \le n$, where ϕ_n is the coefficients of F. Hence, we obtain the desired inequality, by letting

$$\phi_n := a_n\left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)}\right).$$

□

Remark 3.1. Corollary 3.1 can be generalized for a starlike function $\Psi(z)$ such that

$$\left({}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)\right) \prec \Psi(z) := z + \sum_{n=2} \psi_n z^n, \quad z \in \mathbb{U},$$

then we have (3.1). Moreover, if

$$\left({}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)\right) \prec \Psi(z), \quad z \in \mathbb{U},$$

then

$$\Big| \left(\frac{E^2_{\nu,1+n}(-\mu_\nu)}{E^2_{\nu,2}(-\mu_\nu)} \right) \Big| \leq \frac{1}{|a_n|}, \quad n \geq 2, a_n \neq 0. \tag{3.2}$$

Theorem 3.3. *Suppose that $g \in \mathcal{C}$ and $f_1 \in \Lambda$ is the univalent solution of the fractional D'Alembert's equation*

$$(1-\alpha)^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) + \alpha z[^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)]' = g(z). \tag{3.3}$$

If f and $f_1 \in \mathcal{A}_\nu(\alpha, g)$ then $f(z) \prec f_1(z)$.

Proof. Suppose that

$$\Theta[f(z)] = (1-\alpha)^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) + \alpha z[^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)]'.$$

Obviously, $\Theta[f(0)] = g(0) = 0$. Since $f, f_1 \in \Lambda$ then $f(0) = f_1(0) = 0$. Furthermore, we indicate that

$$\Theta[f(z)] = (1-\alpha)^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) + \alpha z[^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)]' \prec g(z)$$

and

$$\Theta[f_1(z)] = (1-\alpha)^{\mathfrak{ABC}}\Delta_z^{\nu} f_1(z) + \alpha z[^{\mathfrak{ABC}}\Delta_z^{\nu} f_1(z)]' \prec g(z).$$

Therefore, according to [18]-Theorem 3.4.c, we get $f(z) \prec f_1(z)$, where f_1 is the best dominant. □

Remark 3.2. The D'Alembert's equation (3.3) describes the linear combination of two functions formulating by

$$\Upsilon = \chi\, F(d\chi/d\Upsilon) + G(d\chi/d\Upsilon).$$

This equation is rearranged in many formulas depending on its applications [2]. Theorem 3.2 showed that every solution of (3.3) is bounded by a univalent solution (if it exists).

We continue to study the results of fractional differential equation. Next outcome displays that a resolution of the fractional differential equation can be deliberated as a resolution of the Briot-Bouquet equation. The most motivating consequence is that the equation has a univalent solution with a positive real part.

Theorem 3.4. *Assume that g is an analytic function and λ is a starlike function in $\mathbb{U}$. If $f \in \Lambda$ is a solution of the Bernoulli's equation*

$$(1-\alpha)^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) + \alpha z[^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)]' = g(z),$$

such that

$$\Re\left((1-\alpha)^{\mathfrak{ABC}}\Delta_z^{\nu} f(z) + \alpha z[^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)]'\right) > 0. \tag{3.4}$$

then f *is a solution of the Briot-Bouquet equation*

$$f(z)+\frac{f'(z)\lambda(z)}{f(z)\lambda'(\xi)}=g(z)$$

with $\Re(f(z))>0$.

Moreover, if $\Theta[f(z)]\in\mathcal{S}^*(\alpha)$ *then*

$$f\in\mathcal{A}_\nu\left(\alpha,\,\frac{z}{(1-z)^{2-2\alpha}}\right),\quad \alpha\in[0,1],\ |z|\in(0.21,0.3)$$

and

$$(\Theta[\sigma(z)])'\prec\left(\frac{z}{(1-z)^{2-2\alpha}}\right)'.$$

Proof. By the starlikeness of λ, we have

$$\Re\left(\frac{z\lambda'(z)}{\lambda(z)}\right)>0,\quad z\in\mathbb{U}.$$

Formulate a function $\Omega:\mathbb{U}\to\mathbb{U}$ as follows:

$$\Omega(z):=\left(\frac{z\lambda'(z)}{\lambda(z)}\right)\Theta[f(z)].$$

Consequently, we get $\Re\left(\Omega(z)\right)>0$. By using [18]-Theorem 3.4j, the Briot-Bouquet equation

$$f(z)+\frac{f'(z)\lambda(z)}{\phi(z)\lambda'(z)}=g(z)$$

has a solution with the real positive part: $\Re(f(z))>0$.

By the starlikeness of $\Theta[f(z)]$ and [20]-Corollary 2.2, we find that a probability measure $\omega\in\partial\mathbb{U}$ can be occurred such that

$$\Theta[f(z)]=\int_{\partial\mathbb{U}}\frac{z}{(1-tz)^{2-2\alpha}}d\omega(z).$$

Which means that $\Theta[f(z)]$ achieves the majority relation

$$\Theta[f(z)]\ll\frac{z}{(1-z)^{2-2\alpha}}.$$

Since $\dfrac{z}{(1-z)^{2-2\alpha}}$ is starlike in $\mathbb{U}$ then according to [5]-Corollary 2, we obtain

$$\Theta[f(z)]\prec\frac{z}{(1-z)^{2-2\alpha}},\quad |z|\in(0.21,0.3),$$

which implies that $f\in\mathcal{A}_\nu\left(\alpha,\,\dfrac{z}{(1-z)^{2-2\alpha}}\right),\alpha\in[0,1],\ |z|\in(0.21,0.3)$. Lastly, the assertion is indicated by [5]-Theorem 3. □

Remark 3.3. The fractional operator ${}^{\mathfrak{ABC}}\Delta_z^\nu f(z)$ in Theorems 3.3 and 3.4 can be replaced by ${}^{\mathfrak{ABR}}\Delta_z^\nu f(z)$.

4. Conclusion

- Modified AB-Fractional differential operators are defined in a complex domain (${}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)$ and ${}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)$) acting in the class of normalized analytic functions. Proposition 2.1 showed the normalization formulas of these fractional operators. Also, it indicated the relation between these operators using the subordination and majorization concepts.
- Geometric interpolation is a difficult task in fractional calculus. As a result, the abc-fractional differential operator is studied geometrically in terms of a complex variable in this manner. This method reveals a variety of qualities and opens the door for future generations to consider or change it in order to obtain additional geometric data. Theorem 3.1, for example, stated the necessary conditions for the abc-differential operator to be starlike. Other properties talked about convexity.
- Geometric properties are illustrated. Differential inequalities are formulated to include them. We explored that when f is convex then ${}^{\mathfrak{ABR}}\Delta_z^{\nu} f(z)$ is also convex, while ${}^{\mathfrak{ABC}}\Delta_z^{\nu} f(z)$ is starlike (see Theorem 3.1). Theorem 3.2 illustrated differential inequalities for formulas involving the suggested fractional differential operators. The results are sharp.
- Applications presented the action of solutions. We indicated that the solution can be formulated by a special function type generalized hypergeometric function.
- For future works, the formulated operators can be used to generalize some classes of analytic functions or to define other kinds of differential operators.

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