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The (k,s)-fractional calculus of k-Mittag-Leffler function

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Abstract

In this paper, we introduce the (k,s)-fractional integral and differential operators involving k-Mittag-Leffler function $E^{\delta}_{k,\rho,\beta}(z)$ as its kernel. Also, we establish various properties of these operators. Further, we consider a number of certain consequences of the main results.

Keywords: fractional integral; k-fractional integral operator; (k,s)-fractional integral; (k,s)-fractional differential; k-Mittag-Leffler function

1 Introduction

Applications and importance of fractional calculus have recently been paid attention to to an ever increasing extent. In mathematical analysis, the fractional calculus is a very useful tool to carry out differentiations and integrations with the real numbers or with the complex numbers powers of the fractional calculus (for example, differential or integral operators). Miller and Ross [1] and Kiryakova [2] described a complete description of fractional calculus operators along with some of their properties and applications can be found in the research of monographs. It is quite well known that there are a number of different definitions of fractional integrals and their applications. Each definition has its own advantages and is appropriate for applications to a different type of problems. Lately, Atangana and Baleanu [3] have introduced one more dimension to this study by proposing a derivative that is based upon the generalized Mittag-Leffler function, since the Mittag-Leffler function is more appropriate in expressing nature than a power function. For the more recent improvements of fractional calculus, the reader may refer to [4–6]. Integral inequalities are taken up to be significant as these are helpful in the study of various courses of differential and integral equations (see [7]). During the past several years, several researchers have obtained various fractional integral inequalities comprising the different fractional differential and integral operators. This subject has received attention of various researchers and mathematicians during the last few decades. The k-symbols are well known from many references related to finite difference calculus (see, [8–11]). Recently, k-fractional integral operators have been considered in the literature by various authors. For this purpose, we start with the following properties in the literature. Díaz and Pariguan (see [12]) have introduced the Pochhammer k-symbols and k-gamma function, which are defined as

$$(\delta)_{n,k} = \begin{cases} 1 & (n = 0, \delta \in \mathbb{C}), \\ \delta(\delta + k)\cdots(\delta + (n-1)k) & (n \in \mathbb{N}, \delta \in \mathbb{C}, k > 0), \end{cases} \tag{1}$$

and

$$\Gamma_k(\eta) = \int_0^\infty t^{\eta-1} e^{-\frac{t^k}{k}}\, dt, \quad \eta \in \mathbb{C}, k > 0, \Re(z) > 0. \tag{2}$$

In the same paper, they defined the relations

$$\Gamma_k(\eta + k) = \eta \Gamma_k(\eta) \tag{3}$$

and

$$\Gamma_k(\eta) = (k)^{\frac{\eta}{k}-1} \Gamma\left(\frac{\eta}{k}\right). \tag{4}$$

Mubeen and Habibullah [13] introduced a variant of fractional integrals which was based on the k-gamma function, called the k-fractional integral, and gave its applications. The k-fractional integral defined is as

$$I_k^\mu\big(f(x)\big) = \frac{1}{k\Gamma_k(\mu)} \int_0^x (x-\tau)^{\frac{\mu}{k}-1} f(\tau)\, d\tau. \tag{5}$$

Clearly, when $k = 1$ then $I_k^\mu(f(x))$ leads to the result of the Riemann-Liouville (R-L) fractional integration formula (see [14]); we have

$$I^\mu\big(f(x)\big) = \frac{1}{\Gamma(\mu)} \int_0^x (x-\tau)^{\mu-1} f(\tau)\, d\tau. \tag{6}$$

Also, they defined the following formulas of the k-fractional integral:

$$I_k^\rho\big(x^{\frac{\beta}{k}-1}\big) = \frac{\Gamma_k(\beta)}{\Gamma_k(\rho+\beta)} x^{\frac{\rho}{k}+\frac{\beta}{k}-1} \tag{7}$$

and

$$I_k^\rho\big((x-u)^{\frac{\beta}{k}-1}\big) = \frac{\Gamma_k(\beta)}{\Gamma_k(\rho+\beta)} (x-u)^{\frac{\rho}{k}+\frac{\beta}{k}-1}. \tag{8}$$

Recently Sarikaya *et al.* [15] have introduced the Riemann-Liouville (k, s)-fractional integral of order $\mu > 0$ is defined as

$${}_k^s I_a^\mu f(x) = \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_a^x \big(x^{s+1} - t^{s+1}\big)^{\frac{\mu}{k}-1} t^s f(t)\, dt, \tag{9}$$

where $x \in [a, b]$, $k > 0$ and $s \in \mathbb{R}\backslash\{-1\}$. In the same paper, they defined the following result:

$${}_k^s I_a^\mu \big[\big(t^{s+1} - a^{s+1}\big)^{\frac{\lambda}{k}-1}\big] = \frac{\Gamma_k(\lambda)}{(s+1)^{\frac{\mu}{k}} \Gamma_k(\lambda+\mu)} \big(x^{s+1} - a^{s+1}\big)^{\frac{\lambda+\mu}{k}-1}. \tag{10}$$

The applications of fractional calculus found in many recent papers (see [16–19]). Recently, the researchers established certain Hermite-Hadamard type inequalities via generalized k-fractional integrals, Grüss type integral inequalities for generalized Riemann-Liouville k-fractional integrals and (k,s)-Riemann-Liouville fractional integral inequalities for continuous random variables by using the idea of (k,s)-fractional integrals [20–23].

The Swedish mathematician Mittag-Leffler [24] has defined the Mittag-Leffler function, which is denoted and defined by the following series:

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}, \quad z \in \mathbb{C}; \Re(\rho) > 0. \tag{11}$$

Wiman [25] introduced a generalized form of the Mittag-Leffler function, which is defined as

$$E_{\rho,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \beta)}, \quad z, \beta \in \mathbb{C}; \Re(\rho) > 0. \tag{12}$$

For more details of Mittag-Leffler functions defined in (11) and (12) such as their various generalizations and applications in different fields, the reader may refer to [4, 14, 26, 27] and in particular the work of Saigo and Kilbas [28]. In recent years, the Mittag-Leffler function (11) and some of its different generalizations and applications have been considered numerically in the complex plane $\mathbb{C}$ (see [29, 30]). Prabhakar [31] have introduced a new generalization of the Mittag-Leffler function $E_{\rho,\beta}(z)$.

Recently many researchers have investigated the importance and great consideration of Mittag-Leffler function in the theory of special functions for exploring some of their generalizations and applications. Extensions for these functions are found in [32]. Srivastava and Tomovski [6] have defined further the generalized form of the Mittag-Leffler function $E^{\delta}_{\rho,\beta}(z)$.

Recently Dorrego [33] have introduced the k-Mittag-Leffler function $E^{\delta}_{k,\rho,\beta}(z)$ (where $k > 0$) defined as

$$E^{\delta}_{k,\rho,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\rho n + \beta)} \frac{z^n}{n!}, \tag{13}$$

where $\rho, \beta, \delta \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$, $k > 0$ and $(\delta)_{n,k}$ is the Pochhammer k-symbol defined in (1).

2 (k,s)-fractional integrals and differentials of k-Mittag-Leffler functions

In this section, we introduce (k,s)-fractional integral and differential operators which involve k-Mittag-Leffler function $E^{\delta}_{k,\rho,\beta}(z)$ as its kernel. In this continuation of the study of (k,s)-fractional calculus, we define integral operators in terms of (k,s) as follows.

Definition 1 If $k > 0$ and $\rho, \delta, \omega \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$, then

$$\left({}^{s}_{k}\varepsilon^{\omega;\delta}_{a+;\rho,\beta} f\right)(x) = \frac{1}{k} \int_a^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\rho}{k}}\right) \tau^s f(\tau)\, d\tau, \tag{14}$$

where $x > \rho$. Substituting $s = 0$, then (14) reduces to the operator

$$\left({}_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f\right)(x) = \int_a^x (x-\tau)^{\frac{\beta}{k}-1}E^{\delta}_{\rho,\beta}\left(\omega(x-\tau)^{\frac{\rho}{k}}\right)f(\tau)\,d\tau; \tag{15}$$

see [34]. In fact, when $\omega = 0$ and $k = 1$ then the integral operator in (15) reduces to the well-known Riemann-Liouville fractional integral operator defined as

$$\left(I^{\mu}_{a+}f\right)(x) = \frac{1}{\Gamma(\mu)}\int_a^x \frac{f(\tau)}{(x-\tau)^{1-\beta}}\,d\tau \quad \left(\mathbb{R}(\mu) > 0\right). \tag{16}$$

Here, we introduce (k,s)-fractional order integrations and differentiations which are defined by the integral operators ${}^s_kI^{\mu}_{a+}$ and ${}^s_kI^{\mu}_{\beta-}$ and (k,s)-fractional differential operators $D^{\mu}_{\rho+,k}$ and $D^{\mu}_{\rho-,k}$. Also, we called these integral operators ${}^s_kI^{\mu}_{a+}$ and ${}^s_kI^{\mu}_{\beta-}$, they are the left and right-sided Riemann-Liouville (k,s)-fractional integral operators, respectively. Similarly, the operators ${}^s_kD^{\mu}_{a+,k}$ and ${}^s_kD^{\mu}_{a-}$ are, respectively, the left- and right-sided Riemann-Liouville (k,s)-fractional differential operators. To define the left- and right-sided Riemann-Liouville (k,s)-fractional integral operators, first we define the well-known Lebesgue measurable integral of a real or complex valued function, which is denoted and defined as

$$L(\rho,\beta) = \left\{f : \|\phi\|_1 = \int_\rho^\beta \left|\phi(\tau)\right|d\tau < \infty; \phi \in L(\rho,\beta)\right\}. \tag{17}$$

Definition 2 For $\phi(x) \in L(\rho,\beta)$; $\mu \in \mathbb{C}$; $\Re(\mu) > 0$ and $k > 0$, then we define the R-L left-sided (k,s)-fractional integral operator of order μ as

$$\begin{aligned} {}^s_{a,k}D^{-\mu}_x f(x) &= {}^s_{a,k}I^{\mu}_x f(x) = {}^s_kI^{\mu}_{a+}f(x) = \left({}^s_kI^{\mu}_{a+}f\right)(x) \\ &= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^x \frac{f(t)}{(x^{s+1}-t^{s+1})^{1-\frac{\mu}{k}}}t^s\,dt \quad (x > a). \end{aligned} \tag{18}$$

Similarly, we can define the R-L right-sided (k,s)-fractional integral operator of order μ as

$$\begin{aligned} {}^s_{\rho,k}D^{-\mu}_\beta f(x) &= {}^s_{\rho,k}I^{\mu}_\beta f(x) = {}^s_kI^{\mu}_{a-}f(x) = \left({}^s_kI^{\mu}_{a-}f\right)(x) \\ &= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_x^\beta \frac{f(t)}{(x^{s+1}-t^{s+1})^{1-\frac{\mu}{k}}}t^s\,dt \quad (x < \beta). \end{aligned} \tag{19}$$

Definition 3 For $k > 0$; $s \in \mathbb{R}\backslash\{-1\}$; $\mu \in \mathbb{C}$, $\Re(\mu) > 0$ and $n = [\Re(\mu)] + 1$, then the Riemann-Liouville left- and right-sided (k,s)-fractional differential operators are defined as

$$\left({}^s_kD^{\mu}_{a+}f\right)(x) = \left[\frac{1}{x^s}\left(\frac{d}{dx}\right)^n\right]\left(k^n\,{}^s_kI^{nk-\mu}_{a+}f\right)(x), \tag{20}$$

$$\left({}^s_kD^{\mu}_{a-}f\right)(x) = \left[\frac{1}{x^s}\left(-\frac{d}{dx}\right)^n\right]\left(k^n\,{}^s_kI^{nk-\mu}_{a-}f\right)(x), \tag{21}$$

respectively. Substituting $k = 1$ and $s = 0$, then the Riemann-Liouville left- and right-sided (k,s)-fractional integrals and derivatives will reduce to the well-known Riemann-Liouville left-sided and right-sided fractional integrals and derivatives see ([14, 35]).

Definition 4 The Riemann-Liouville (k,s)-fractional derivative operator ${}_k^sD_{a+}^{\mu}$ defined in (18) is generalized by the (k,s)-fractional derivative operator is denoted by ${}_k^sD_{a+}^{\mu,\nu}$ where μ is the order such that $0 < \mu < 1$ and ν is the type of this generalized (k,s)-fractional derivative operator such that $0 < \nu < 1$, we define the generalized (k,s)-fractional derivative operator with respect to x as follows:

$$\left({}_k^sD_{a+}^{\mu,\nu}f\right)(x) = \left[{}_k^sI_{\rho+}^{\nu(k-\mu)}\left(\frac{1}{x^s}\frac{d}{dx}\right)\left(k\,{}_k^sI_{a+}^{(1-\nu)(k-\mu)}f\right)\right](x). \tag{22}$$

Obviously, when $\nu = 0$ then (22) reduces to the Riemann-Liouville (k,s)-fractional derivative operator ${}_k^sD_{a+}^{\mu}$ (18).

Lemma 1 *For $k > 0$, the following result for (k,s)-fractional derivative operator $D_{\rho+,k}^{\mu,\nu}$ defined in* (22) *holds true*:

$$\left({}_k^sD_{a+}^{\mu,\nu}\left[\left(\tau^{s+1} - a^{s+1}\right)^{\frac{\lambda}{k}-1}\right]\right)(x) = \frac{\Gamma_k(\lambda)}{(s+1)^{-\frac{\mu}{k}}\Gamma_k(\lambda-\mu)}\left(x^{s+1} - a^{s+1}\right)^{\frac{\lambda-\mu}{k}-1}, \tag{23}$$

with $x > \rho$, $0 < \mu < 1$, $0 < \nu < 1$ and $\Re(\lambda) > 0$.

Proof We obtain from equation (10)

$$\begin{aligned}
&\left({}_k^sI_{a+}^{(1-\nu)(k-\mu)}\left[\left(\tau^{s+1} - a^{s+1}\right)^{\frac{\lambda}{k}-1}\right]\right)(x)\\
&\quad= \frac{\Gamma_k(\lambda)}{(s+1)^{\frac{(1-\nu)(k-\mu)}{k}}\Gamma_k((1-\nu)(k-\mu)+\lambda)}\left(x^{s+1} - a^{s+1}\right)^{\frac{(1-\nu)(k-\mu)+\lambda}{k}-1}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{x^s}\frac{d}{dx}\left({}_k^sI_{a+}^{(1-\nu)(k-\mu)}\left[\left(t^{s+1} - a^{s+1}\right)^{\frac{\lambda}{k}-1}\right]\right)(x)\\
&\quad= \frac{[(1-\nu)(1-\mu)+\lambda-k]\Gamma_k(\lambda)}{k(s+1)^{\frac{(1-\nu)(k-\mu)}{k}-1}\Gamma_k((1-\nu)(k-\mu)+\lambda)}\left(x^{s+1} - a^{s+1}\right)^{\frac{(1-\nu)(k-\mu)+\lambda}{k}-2},
\end{aligned}$$

which, by applying the relation given in (3), yields

$$\begin{aligned}
&\left({}_k^sD_{a+}^{\mu,\nu}\left[\left(\tau^{s+1} - a^{s+1}\right)^{\frac{\lambda}{k}-1}\right]\right)(x)\\
&\quad= \frac{\Gamma_k(\lambda)}{\Gamma_k((1-\nu)(k-\mu)+\lambda-k)}\\
&\qquad\times \left[{}_k^sI_{a+}^{\nu(k-\mu)}\left(x^{s+1} - a^{s+1}\right)^{\frac{(1-\nu)(k-\mu)+\lambda}{k}-2}\right](x)\\
&\quad= \frac{\Gamma_k(\lambda)}{(s+1)^{\frac{\nu(k-\mu)+(1-\nu)(k-\mu)}{k}-1}\Gamma_k((1-\nu)(k-\mu)+\lambda-k)}\\
&\qquad\times \frac{\Gamma_k((1-\nu)(k-\mu)+\lambda-k)}{\Gamma_k(\nu(k-\mu)+(1-\nu)(k-\mu)+\lambda-k)}\left(x^{s+1} - a^{s+1}\right)^{\frac{\lambda-\mu}{k}-1}\\
&\quad= \frac{\Gamma_k(\lambda)}{(s+1)^{-\frac{\mu}{k}}\Gamma_k(\lambda-\mu)}\left(x^{s+1} - a^{s+1}\right)^{\frac{\lambda-\mu}{k}-1},
\end{aligned}$$

which is the desired proof. □

Theorem 1 *For $k > 0$, the following result always holds true*:

$$\left(\frac{1}{x^{\frac{s}{m}}}\frac{d}{dx}\right)^m \left[\left(x^{s+1}-a^{s+1}\right)^{\frac{c}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right] \tag{24}$$

$$= \frac{(s+1)^m(x^{s+1}-a^{s+1})^{\frac{c}{k}-m-1}}{k^n} E^{\delta}_{k,\rho,\beta-mk}\left[\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right], \tag{25}$$

where $s \in \mathbb{R}\backslash\{-1\}$, $\mu, \rho, \beta, \delta \in \mathbb{C}$, $\Re(\mu) > 0$ *and* $\Re(\beta) > 0$, $\Re(\rho) > 0$, $\Re(\delta) > 0$.

Proof The proof is obvious by applying $(\frac{1}{x^{\frac{s}{m}}}\frac{d}{dx})^m$ where $m = 1, 2, \ldots$. □

Theorem 2 *Suppose* $k > 0$, $x > a$ ($a \in \mathbb{R}_+ = [0,\infty)$) *and* $\rho, \beta, \delta, \omega \in \mathbb{C}$, $\Re(\beta) > 0$, $\Re(\rho) > 0$, $\Re(\delta) > 0$, $\Re(\mu) > 0$, *then*

$$ {}^s_k I^{\mu}_{a+}\left[\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-a^{s+1}\right)^{\rho}\right)\right](x)$$

$$= \frac{\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta+\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}} E^{\delta}_{k,\rho,\beta+\mu}\left[\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right], \tag{26}$$

$$ {}^s_k D^{\mu}_{a+}\left[\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right](x)$$

$$= \frac{\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta-\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}} E^{\delta}_{k,\rho,\beta-\mu}\left[\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right] \tag{27}$$

and

$$ {}^s_k D^{\mu,\nu}_{a+}\left[\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right](x)$$

$$= \frac{\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta-\mu}{k}-1}}{(s+1)^{-\frac{\mu}{k}}} E^{\delta}_{k,\rho,\beta-\mu}\left[\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right]. \tag{28}$$

Proof

$$ {}^s_k I^{\mu}_{a+}\left[\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right]$$

$$= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \int_{\rho}^{x} \frac{(\tau^{s+1}-a^{s+1})^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}(\omega(\tau^{s+1}-a^{s+1})^{\frac{\rho}{k}})\tau^s}{(x^{s+1}-\tau^{s+1})^{1-\frac{\mu}{k}}}\,d\tau$$

$$= \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}\omega^n}{\Gamma_k(\rho n+\beta)n!}$$

$$\times \int_0^x \left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta+an}{k}-1}\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\mu}{k}-1}\tau^s\,d\tau.$$

Substituting $\tau^{s+1} = a^{s+1} + y(x^{s+1}-a^{s+1})$, this implies $\tau^s\,d\tau = (\frac{x^{s+1}-a^{s+1}}{s+1})\,dy$, we have

$$ {}^s_k I^{\mu}_{a+}\left[\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}\omega^n}{\Gamma_k(\rho n+\beta)n!}\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta+\mu+\rho n}{k}-1} \frac{(s+1)^{\frac{-\mu}{k}}}{k\Gamma_k(\mu)} \int_0^1 (1-y)^{\frac{\beta+\rho n}{k}-1} y^{\frac{\mu}{k}-1}\,dy$$

$$= \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}\omega^n}{\Gamma_k(\rho n+\beta)n!} \left(x^{s+1}-a^{s+1}\right)^{\frac{\beta+\mu+\rho n}{k}-1} \cdot \frac{\Gamma_k(\rho n+\beta)\Gamma_k(\mu)}{\Gamma_k(\mu)\Gamma_k(\rho n+\beta+\mu)}$$
$$= \frac{(x^{s+1}-a^{s+1})^{\frac{\beta+\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}} \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}\omega^n (x^{s+1}-a^{s+1})^{\frac{\rho n}{k}-1}}{\Gamma_k(\rho n+\beta+\mu)n!}$$
$$= \frac{(x^{s+1}-a^{s+1})^{\frac{\beta+\mu}{k}-1}}{(s+1)^{\frac{\mu}{k}}} E^{\delta}_{k,\rho,\beta+\mu}\left(\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right).$$

This completes the proof of (26).

Now, we have

$${}^s_kD^{\mu}_{a+}\left[\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right]$$
$$= \left(\frac{1}{x^{\frac{s}{n}}}\frac{d}{dx}\right)^n \left\{k^n \, {}^s_kI^{nk-\mu}_{a+}\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right\}$$

and using (26) this takes the following form:

$${}^s_kD^{\mu}_{a+}\left[\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right]$$
$$= k^n \left(\frac{1}{x^{\frac{s}{n}}}\frac{d}{dx}\right)^n \left\{(s+1)^{-\frac{\mu}{k}-n}\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta-\mu}{k}+n-1} E^{\delta}_{k,\rho,\beta-\mu+nk}\left(\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right\}.$$

Applying (24), we have

$${}^s_kD^{\mu}_{a+}\left[\left(\tau^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right](x)$$
$$= \left\{(s+1)^{-\frac{\mu}{k}}\left(x^{s+1}-a^{s+1}\right)^{\frac{\beta-\mu}{k}-1} E^{\delta}_{k,\rho,\beta-\mu}\left(\omega\left(x^{s+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right\}.$$

This completes the desired proof.

Now to prove (28), we have

$$\left({}^s_kD^{\mu,\nu}_{a+,k}\left[\left(\tau^{a+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{a+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right]\right)(x)$$
$$= \left({}^s_kD^{\mu,\nu}_{a+}\left[\sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\rho n+\beta)}\frac{\omega^n}{n!}\left(t^{a+1}-a^{s+1}\right)^{\frac{\rho n+\beta}{k}-1}\right]\right)(x).$$

This can be written as

$$= \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\rho n+\beta)}\frac{\omega^n}{n!}\left({}^s_kD^{\mu,\nu}_{a+}\left[\left(\tau^{a+1}-a^{s+1}\right)^{\frac{\rho n+\beta}{k}-1}\right]\right)(x).$$

By applying (23), we get

$$\left({}^s_kD^{\mu,\nu}_{a+}\left[\left(\tau^{a+1}-a^{s+1}\right)^{\frac{\beta}{k}-1} E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{a+1}-a^{s+1}\right)^{\frac{\rho}{k}}\right)\right]\right)(x)$$
$$= \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(\rho n+\beta)}\frac{\omega^n}{n!} \cdot \frac{\Gamma_k(\rho n+\beta)}{(s+1)^{-\frac{\mu}{k}}\Gamma_k(\rho n+\beta-\mu)}\left(x^{a+1}-a^{s+1}\right)^{\frac{\rho n+\beta-\mu}{k}-1}$$

$$= \frac{(x^{a+1} - a^{s+1})^{\frac{\beta-\mu}{k}-1}}{(s+1)^{-\frac{\mu}{k}}} \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\rho n + \beta - \mu)} \frac{[\omega(x^{a+1} - a^{s+1})^{\frac{\rho}{k}}]^n}{n!}$$

$$= \frac{(x^{a+1} - a^{s+1})^{\frac{\beta-\mu}{k}-1}}{(s+1)^{-\frac{\mu}{k}}} E^{\delta}_{k,\rho,\beta-\mu}\big(\omega\big(x^{a+1} - a^{s+1}\big)^{\frac{\rho}{k}}\big),$$

which completes the desired proof. □

Remark 1 If we substitute $s = 0$ in (26), (27) and (28), then we have the results of the k-Mittag-Leffler function (see [34]). Similarly if $s = 0$ and $k = 1$, then from the above equations we get the integral and differential operators of the classical Mittag-Leffler function (see [6]).

3 Some properties of the operator $({}^{s}_{k}\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f)(x)$

Theorem 3 *For $k > 0$, $\rho, \beta, \delta \in \mathbb{C}$, $\omega \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$ and $\Re(\mu) > 0$, we have*

$$\big({}^{s}_{k}\varepsilon^{\omega;\delta}_{a+;\rho,\beta}\big[\big(\tau^{a+1} - a^{s+1}\big)^{\frac{\mu}{k}}\big]\big)(x)$$
$$= \frac{(x^{a+1} - a^{s+1})^{\frac{\mu+\beta}{k}-1}\Gamma_k(\mu)}{(s+1)} E^{\delta}_{k,\rho,\beta+\mu}\big(\omega\big(x^{a+1} - a^{s+1}\big)^{\frac{\mu}{k}}\big) f(t)\,dt. \tag{29}$$

Proof From (14)

$$\big({}^{s}_{k}\varepsilon^{\omega;\delta}_{a+;\rho,\beta}\big[\big(\tau^{a+1} - a^{s+1}\big)^{\frac{\mu}{k}}\big]\big)(x)$$
$$= \frac{1}{k}\int_a^x \big(x^{a+1} - \tau^{s+1}\big)^{\frac{c}{k}-1} E^{\delta}_{k,\rho,\beta}\big(\omega\big(x^{a+1} - \tau^{s+1}\big)^{\frac{\rho}{k}}\big)\tau^s f(\tau)\,d\tau.$$

Therefore, we have

$$\big({}^{s}_{k}\varepsilon^{\omega;\delta}_{a+;\rho,\beta}\big[\big(\tau^{a+1} - a^{s+1}\big)^{\frac{\mu}{k}}\big]\big)(x)$$
$$= \frac{1}{k}\int_a^x \big(x^{a+1} - \tau^{s+1}\big)^{\frac{\beta}{k}-1}\big(\tau^{a+1} - a^{s+1}\big)^{\frac{\mu}{k}-1} E^{\delta}_{k,\rho,\beta}\big(\omega\big(x^{a+1} - \tau^{s+1}\big)^{\frac{\rho}{k}}\big)\tau^s\,d\tau$$
$$= \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\rho n + \beta)} \frac{\omega^n}{n!}\left(\frac{1}{k}\int_a^x \big(\tau^{a+1} - a^{s+1}\big)^{\frac{\mu}{k}-1}\big(x^{a+1} - \tau^{s+1}\big)^{\frac{\beta+\rho n}{k}-1}\tau^s\,d\tau\right)$$
$$= \sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{(s+1)\Gamma_k(\rho n + \beta)} \frac{\omega^n (x^{a+1} - a^{s+1})^{\frac{\beta+\rho n+\mu}{k}-1}}{n!} B_k(\beta + \rho n, \mu)$$
$$= \frac{(x^{a+1} - a^{s+1})^{\frac{\beta+\mu}{k}-1}\Gamma_k(\mu)}{(s+1)}\left\{\sum_{n=0}^{\infty} \frac{(\delta)_{n,k}}{\Gamma_k(\rho n + \beta)} \frac{\omega^n (x^{a+1} - a^{s+1})^{\frac{\rho n}{k}}}{\cdot} \frac{\Gamma_k(\mu)\Gamma_k(\rho n + \beta)}{\Gamma_k(\rho n + \beta + \mu)}\right\}$$
$$= \frac{(x^{a+1} - a^{s+1})^{\frac{\beta+\mu}{k}-1}\Gamma_k(\mu)}{(s+1)} E^{\delta}_{k,\rho,\beta+\mu}\big(\omega\big(x^{a+1} - a^{s+1}\big)^{\frac{\rho}{k}}\big),$$

which completes the desired proof. □

Theorem 4 *Suppose that $f \in L_1[a,b]$, $s \in \mathbb{R}\backslash\{-1\}$, $k > 0$, $\rho, \beta, \delta, \omega \in \mathbb{C}$, $\Re(\rho) > 0$, $\Re(\beta) > 0$, $\Re(\delta) > 0$ and $\Re(\mu) > 0$, then ${}^{s}_{k}\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f(x)$ exist for any $x \in [a,b]$.*

Proof Assume that $\Delta = [a,b] \times [a,b]$ and $P : \Delta \to \mathbb{R}$ such that $P(x,\tau) = [(x^{s+1} - \tau^{s+1})\tau^s]$ for all $x \in [a,b]$. It is obvious that $P = P_+ + P_-$ where

$$P_+(x,\tau) = \begin{cases} (x^{s+1} - \tau^{s+1})^{\frac{\beta}{k}-1}\tau^s; & a \le \tau \le x \le b, \\ 0; & a \le x \le t \le b \end{cases}$$

and

$$P_-(x,\tau) = \begin{cases} (\tau^{s+1} - x^{s+1})^{\frac{\beta}{k}-1}x^s; & a \le \tau \le x \le b, \\ 0; & a \le x \le \tau \le b. \end{cases}$$

As P is measurable on Δ, we can write

$$\begin{aligned} \int_a^b P(x,\tau)\,d\tau &= \int_a^x P(x,\tau)\,d\tau \\ &= \int_a^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\beta}{k}-1}\tau^s\,d\tau \\ &= \frac{k}{\beta}\left(x^{s+1} - \tau^{s+1}\right)^{\frac{\beta}{k}}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &\int_a^b P(x,\tau)E^{\delta}_{k,\rho,\beta}\left(\omega(x-\tau)^{\frac{\rho}{k}}\right)d\tau \\ &\quad= \int_a^x P(x,\tau)E^{\delta}_{k,\rho,\beta}\left(\omega(x-\tau)^{\frac{\rho}{k}}\right)d\tau \\ &\quad= \sum_{n=0}^{\infty}\frac{(\delta)_{n,k}\omega^n}{\Gamma_k(\rho n+\beta)n!}\int_a^x \left(x^{s+1} - \tau^{s+1}\right)^{\frac{\beta+\rho}{k}-1}\tau^s\,d\tau \\ &\quad= \sum_{n=0}^{\infty}\frac{(\delta)_{n,k}(\omega(x^{s+1}-a^{s+1})^{\frac{\rho}{k}})^n}{\Gamma_k(\rho n+\beta)n!}\frac{k}{\beta+\rho n}\left(x^{s+1} - a^{s+1}\right)^{\frac{\beta}{k}}. \end{aligned}$$

By using the repeated integral, we have

$$\begin{aligned} &\int_a^b \left(\int_a^b P(x,\tau)E^{\delta}_{k,\rho,\beta}\left(\omega(x-\tau)^{\frac{\rho}{k}}\right)\left|f(x)\right|d\tau\right)dx \\ &\quad= \int_a^b \left|f(x)\right|\left(\int_a^b P(x,\tau)E^{\delta}_{k,\rho,\beta}\left(\omega(x-\tau)^{\frac{\rho}{k}}\right)d\tau\right)dx \\ &\quad= \sum_{n=0}^{\infty}\frac{(\delta)_{n,k}(\omega)^n}{\Gamma_k(\rho n+\beta)n!}\frac{k}{\beta+\rho n} \\ &\qquad\times \int_a^b \left(x^{s+1} - a^{s+1}\right)^{\frac{\beta+\rho n}{k}}\left|f(x)\right|dx \\ &\quad\le \sum_{n=0}^{\infty}\frac{(\delta)_{n,k}(\omega(b^{s+1}-a^{s+1})^{\frac{\rho}{k}})^n}{\Gamma_k(\rho n+\beta)n!}\frac{k^2}{(\beta+\rho n)(\beta+\rho n+k)} \\ &\qquad\times \left(b^{s+1} - a^{s+1}\right)^{\frac{\beta}{k}+1}\int_a^b \left|f(x)\right|dx \end{aligned}$$

$$\leq \left(b^{s+1}-a^{s+1}\right)^{\frac{\beta}{k}+1}\sum_{n=0}^{\infty}\frac{(\delta)_{n,k}(\omega(b^{s+1}-a^{s+1})^{\frac{\rho}{k}})^n}{\Gamma_k(\rho n+\beta)n!}$$
$$\times \frac{k^2}{(\beta+\rho n)(\beta+\rho n+k)}\|f\|_1 \leq \infty.$$

Therefore the function $Q:\Delta\to\mathbb{R}$ such that $Q(x,t)=P(x,\tau)f(x)$ is integrable on Δ by Tonelli's theorem. Thus, by Fubini's theorem $\int_a^b P(x,\tau)E^{\delta}_{k,\rho,\beta}(\omega(x^{s+1}-\tau^{s+1})^{\frac{\rho}{k}})^n f(x)\,dx$ is an integrable function on $[a,b]$, as a function of $t\in[a,b]$. Thus, ${}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f(x)$ exists. □

Theorem 5 *For $\mu\in\mathbb{C}$, $\rho,\beta,\delta\in\mathbb{C}$, $\omega\in\mathbb{C}$, $\Re(\rho)>0$, $\Re(\beta)>0$, $\Re(\delta)>0$, $\Re(\mu)>0$, $k>0$, $s\in\mathbb{R}\backslash\{-1\}$, and $x>a$, the following result holds*:

$$\left({}^s_kI^{\mu}_{a+}\left[{}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f\right]\right)(x)=\frac{1}{(s+1)^{\frac{\mu}{k}}}\left({}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta+\mu}f\right)(x)=\left({}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}\left[{}^s_kI^{\mu}_{\rho+}f\right]\right)(x), \tag{30}$$

for any $f\in L(\rho,\beta)$.

Proof From equations (14) and (18), we observe

$$\begin{aligned}
&\left({}^s_kI^{\mu}_{a+}\left[{}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f\right]\right)(x)\\
&=\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^x\frac{[{}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f(\tau)]}{(x^{s+1}-\tau^{s+1})^{1-\frac{\mu}{k}}}\tau^s\,d\tau\\
&=\frac{(s+1)^{1-\frac{\mu}{k}}}{k^2\Gamma_k(\mu)}\int_a^x\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\mu}{k}-1}\\
&\quad\times\left[\int_\rho^\tau\left(\tau^{s+1}-u^{s+1}\right)^{\frac{\beta}{k}-1}E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-u^{s+1}\right)^{\frac{\rho}{k}}\right)f(u)u^s\,du\right]\tau^s\,d\tau.
\end{aligned}$$

By interchanging the order of integration, we obtain

$$\begin{aligned}
&\left({}^s_kI^{\mu}_{a+}\left[{}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f\right]\right)(x)\\
&=\frac{1}{k}\int_a^x\left[\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_u^x\left(x^{s+1}-t^{s+1}\right)^{\frac{\mu}{k}-1}\left(\tau^{s+1}-u^{s+1}\right)^{\frac{\beta}{k}-1}E^{\delta}_{k,\rho,\beta}\left(\omega\left(\tau^{s+1}-u^{s+1}\right)^{\frac{\rho}{k}}\right)\tau^s\,d\tau\right]\\
&\quad\times u^sf(u)\,du.
\end{aligned}$$

By applying (26), we have

$$\begin{aligned}
&\left({}^s_kI^{\mu}_{a+}\left[{}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f\right]\right)(x)\\
&=\left[\frac{1}{k(s+1)^{\frac{\mu}{k}}}\int_a^x\left(x^{s+1}-u^{s+1}\right)^{\frac{\mu+\beta}{k}-1}E^{\delta}_{k,\rho,\beta+\mu}\left(\omega\left(x^{s+1}-u^{s+1}\right)^{\frac{\rho}{k}}\right)u^sf(u)\,du\right];
\end{aligned}$$

thus, we get

$$\left({}^s_kI^{\mu}_{a+}\left[{}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta}f\right]\right)(x)=\frac{1}{(s+1)^{\frac{\mu}{k}}}\left({}^s_k\varepsilon^{\omega;\delta}_{a+;\rho,\beta+\mu}f\right)(x). \tag{31}$$

To prove the second part, consider the rhs of (30) then by applying (14), we get

$$\begin{aligned}
&\left({}_{k}^{s}\varepsilon_{a+;\rho,\beta}^{\omega;\delta}\left[{}_{k}^{s}I_{a+}^{\mu}f\right]\right)(x) \\
&= \frac{1}{k}\int_a^x \left(x^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1} E_{k,\rho,\beta}^{\delta}\left(\omega\left(x^{s+1}-t^{s+1}\right)^{\frac{\rho}{k}}\right)\left[{}_{k}^{s}I_{a+}^{\mu}f\right](\tau)\tau^s\,d\tau \\
&= \frac{1}{k}\int_a^x \left(x^{s+1}-t^{s+1}\right)^{\frac{\beta}{k}-1} E_{k,\rho,\beta}^{\delta}\left(\omega\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\rho}{k}}\right) \\
&\quad\times\left(\frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)}\int_a^\tau \frac{f(u)}{(\tau^{s+1}-u^{s+1})^{1-\frac{\mu}{k}}}u^s\,du\right)d\tau.
\end{aligned}$$

By interchanging the order of integration, we have

$$\begin{aligned}
&\left({}_{k}^{s}\varepsilon_{a+;\rho,\beta}^{\omega;\delta,q}\left[{}_{k}^{s}I_{a+}^{\mu}f\right]\right)(x) \\
&= \frac{1}{k}\int_a^x \frac{(s+1)^{1-\frac{\mu}{k}}}{k\Gamma_k(\mu)} \\
&\quad\times\left[\int_u^x \left(x^{s+1}-\tau^{s+1}\right)^{\frac{\beta}{k}-1}\left(\tau^{s+1}-u^{s+1}\right)^{\frac{\mu}{k}-1}E_{k,\rho,\beta}^{\delta}\left(\omega\left(x^{s+1}-\tau^{s+1}\right)^{\frac{\rho}{k}}\right)\tau^s\,d\tau\right] \\
&\quad\times u^s f(u)\,du.
\end{aligned}$$

Again by making the use of (18) and applying (26), we obtain

$$\left({}_{k}^{s}\varepsilon_{a+;\rho,\beta}^{\omega;\delta}\left[{}_{k}^{s}I_{\rho+}^{\mu}f\right]\right)(x) = \frac{1}{(s+1)^{\frac{\mu}{k}}}\left({}_{k}^{s}\varepsilon_{a+;\rho,\beta+\mu}^{\omega;\delta}\right)f(x). \tag{32}$$

Thus (31) and (32) complete the desired proof of (30). □

4 Conclusion

We conclude that, if $s = 0$, then the obtained results reduce to the well-known results introduced by [34]. Similarly if $k = 1$ and $s = 0$, then the obtained results reduce to the well-known results of the Mittag-Leffler function defined by [4, 31].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors have contributed equally to this manuscript. They read and approved the final manuscript.

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