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Numerical Construction of Lyapunov Functions Using Homotopy Continuation Method

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Abstract

Lyapunov functions are commonly involved in the analysis of the stability of linear and nonlinear dynamical systems. Despite the fact that there is no generic procedure for creating these functions, many authors use polynomials in p-forms as candidates for constructing Lyapunov functions, while others restrict the construction to quadratic forms. We proposed a method for constructing polynomial Lyapunov functions that are not necessary in a form by focusing on the positive and negative definiteness of the Lyapunov candidate and the Hessian of its derivative, as well as employing the sum of square decomposition. The idea of Newton polytopes was used to transform the problem into a system of algebraic equations that were solved using the polynomial homotopy continuation method. Our method can produce several possibilities of Lyapunov functions for a given candidate. The sample test conducted demonstrates that the method developed is promising.

Keywords: $p-form$ Sum of Square Homotopy PHClab Polytope.
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1. Introduction

One of the basic requirements of designing both linear and nonlinear control systems is that they remain stable at all times. Stability analysis of a nonlinear system is complicated and typically require establishing

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the existence of a Lyapunov function, see [23, 12, 21]. Thus, the stability analysis of a particular nonlinear system equilibrium state is reduced to the investigation of the properties of its corresponding Lyapunov function. This approach possesses great power in applications [11, 22, 14]. To date, there exists no general approach for obtaining Lyapunov functions which often makes the construction of such functions to be intractable problems.

It is well known that there are globally stable systems for which no Lyapunov function can be found, see, for instance, [15] and the references therein. To this end, many researchers have proposed different types of approaches on how to construct Lyapunov functions. For example, Zhenyi *et al* reported in [9] proposed a method for constructing Lyapunov function in quadratic form by using positive polynomials. These polynomials are constructed using the technique of Zhikun *et al* reported in [20] and by constructing and solving a semi-algebraic system using cylindrical algebraic decomposition (CAD) reported by Collins in [6]. Recently an algorithm for construction of Lyapunov functions in $p-form$ was developed in [13]. In that report, the authors utilize using Sum Of Square technique (SOS) and the requirement for the candidate to be in quadratic form was removed. SOS decomposition has been employed by many researchers to construct polynomial and non-polynomial Lyapunov functions for decades, see for instance [16] and the references therein. The reason why SOS decomposition is extensively used in the construction of Lyapunov functions may not be unconnected to the fact that SOS decomposition is strongly linked to positive semi definiteness of polynomial, [1, 2]. SOS decomposition generally requires decomposing polynomials in terms of its monomials. Given that, most of the monomials are not required in the final calculations, and the idea of Newton polytope is frequently used in pruning unwanted monomials, see [3, 18, 7, 5].

In this current study, we focus on developing an approach that extends the search of polynomial Lyapunov function to polynomials of even degree that are not necessarily quadratic nor a $p-form$. This extends the work of [9, 13] by removing the requirements of the Lyapunov candidates to be quadratic or in $p-form$. We use SOS decomposition and Newton polytope to have decomposition with fewer monomials. The problem is eventually transformed into a system of algebraic equations where, we use PHClab, which is a software package for Polynomial Homotopy Continuation, to numerically solve systems of polynomial equations, see [8] for a detailed discussion on PHClab.

The paper is organized as follows. In Section 2, we give preliminaries, in Section 3, we present our proposed modified algorithm for computing Lyapunov functions, in Section 4, we give the implementation of our algorithm, and in Section 5, we present the conclusion of the paper.

Consider the following system, where $\dot{\mathbf{x}}$ denotes the time derivative of $\mathbf{x}$ with respect to time,

$$\dot{\mathbf{x}} = f(\mathbf{x}). \tag{1}$$

Here, $\mathbf{x} \in \mathbf{R}^n$ and $f(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))^T$ is a vector. A point $\mathbf{x}^*$ is considered to be an equilibrium point for 1 if $f(\mathbf{x}^*) = \mathbf{0}$. Without loss of generality, we assume the origin $\mathbf{0}$ is the equilibrium of the given system.

Theorem 1.1. *[17] Let $F(x) \in \mathbf{R}[\boldsymbol{x}]$ be a polynomial of degree $2d$, then, $F(x)$ is SOS iff there exist a symmetric positive definite (PSD) matrix Q such that*

$$F(x) = zQz^T. \tag{2}$$

Definition 1.2. *[19] A Newton polytope (or cage) of a polynomial $p = \sum_{\alpha \in A} c_\alpha x^\alpha$ over a set A is the convex hull of A which is denoted as $C(p) := convhull(A)$.*

The reduced Newton polytope is given as $\frac{1}{2}C(p) := \left\{\frac{1}{2}\alpha : \alpha \in C(p)\right\}$. Theorem 1.3 is useful for reduction in the number of monomials.

Theorem 1.3. *[19] If $p = \sum_{i=1}^{m} f_i^2$ then the vertices of $C(p)$ are vectors whose entries are even numbers and $C(f_1) \subseteq \frac{1}{2}C(p)$.*

Definition 1.2 and theorems 1.1 and 1.3 are used for pruning unnecessary monomials from any SOS decomposition. Without pruning, SOS decomposition may be time consuming for polynomials with many terms, involves more detailed discussion can found in [19].

2. Lyapunov function by SOS decomposition and Newton polytope

In this section, we propose an approach that utilizes the sum of square decomposition efficiently to search for polynomial Lyapunov functions which are neither quadratic nor p-form. The following is the main theorem of our work.

Theorem 2.1 (Main: Theorem)**.** *Let $V(\boldsymbol{x})$ be a polynomial of degree $2d$ whose degree of all monomials is greater than or equal to two (≥ 2) for a given autonomous polynomial system of differential equations. If V is a SOS and the Hessian of $\dot{V}$ evaluated at the equilibrium point $\left(Hess(\frac{d}{dt}V)|_{\boldsymbol{x}=0}\right)$ is negative definite then V is a Lyapunov function.*

Proof. $V(\mathbf{x})$ is a SOS, so from Theorem 1 there exist a symmetric Matrix Q with real entries such that

$$Q \geq 0, \text{and}, V(\mathbf{x}) = z(\mathbf{x})^T Q z(\mathbf{x}).$$

This implies that $V(\mathbf{x})$ is positive definite. Now it is sufficient to prove that there is neighborhood $U \subseteq \mathbf{R}^n$ such that

$$\frac{d}{dt}V(\mathbf{x}) < \mathbf{0}, \text{for all} \mathbf{x} \in U - \{\mathbf{0}\}.$$

Firstly let

$$\frac{d}{dt}V(\mathbf{x}) = \sum_{i=1}^{n} \frac{\delta}{\delta x_i} V(\mathbf{x}) f_i.$$

Thus for any arbitrary but fixed j, $1 \leq j \leq n$

$$\frac{\delta\left(\frac{d}{dt}V(\mathbf{x})\right)}{\delta x_j} = \frac{\delta\left(\sum_{i=1}^{n} \frac{\delta}{\delta x_i} V(\mathbf{x}) f_i(\mathbf{x})\right)}{\delta x_j}$$

$$= \sum_{i=1}^{n} \frac{\delta\left(\frac{\delta}{\delta x_i} V(\mathbf{x}) f_i(\mathbf{x})\right)}{\delta x_j}.$$

Let $r = \frac{\delta V(\mathbf{x})}{\delta x_i}$, $s = f_i$, $\frac{\delta r}{\delta x_j} = \frac{\delta^2 V(\mathbf{x})}{\delta x_j \delta x_i}$ and $\frac{\delta s}{\delta x_j} = \frac{\delta f_i}{\delta x_j}$ then using product rule of differentiation, we have

$$\frac{\delta \frac{\delta V(\mathbf{x})}{\delta x_i} f_i}{\delta x_j} = sr' + rs'$$

$$= \frac{\delta^2 V(\mathbf{x})}{\delta x_j \delta x_i} f_i + \frac{\delta V(\mathbf{x})}{\delta x_i} \frac{\delta f_i}{\delta x_j}.$$

So that

$$\frac{\delta\left(\frac{dV(\mathbf{x})}{dt}\right)}{\delta x_j} = \sum_{i=1}^{n} \left(\frac{\delta^2 V(\mathbf{x})}{\delta x_j \delta x_i} f_i + \frac{\delta V(\mathbf{x})}{\delta x_i} \frac{\delta f_i}{\delta x_j}\right). \tag{3}$$

Evaluating 3 at $\mathbf{x} = \mathbf{0}$, we have

$$\frac{\delta\left(\frac{dV(\mathbf{x})}{dt}\right)|_{\mathbf{x}=0}}{\delta x_j}$$

$$= \sum_{i=1}^{n} \left(\frac{\delta^2 V(\mathbf{0})}{\delta x_j \delta x_i} f_i(\mathbf{0}) + \frac{\delta V(\mathbf{0})}{\delta x_i} \frac{\delta f_i(\mathbf{0})}{\delta x_j} \right). \tag{4}$$

Then from 1 and the assumption $f_i(\mathbf{0}) = \mathbf{0}$ and $V(\mathbf{x})$ is a polynomial, the first derivative of any polynomial evaluated at $\mathbf{0}$ is also $\mathbf{0}$. Hence

$$\frac{\delta \left(\frac{dV(\mathbf{x})}{dt}\right)|_{\mathbf{x}=0}}{\delta x_j} = \mathbf{0}.$$

For a fixed arbitrary j, $\frac{\delta\left(\frac{dV(x)}{dt}\right)|_{\mathbf{x}=0}}{\delta x_j} = 0$. Since,

$$\left(\frac{\delta}{\delta x_1} \left(\frac{dV(\mathbf{x})}{dt} \right), \ldots, \frac{\delta}{\delta x_n} \left(\frac{dV(\mathbf{x})}{dt} \right) \right) |_{\mathbf{x}=\mathbf{0}} = \mathbf{0}$$

and $Hess\left(\frac{dV(\mathbf{x})}{dt}\right)|_{\mathbf{x}=\mathbf{0}}$ is negative definite. Then by the extremum theory [16], there is a neighborhood U of the origin such that $\frac{dV(\mathbf{x})}{dt} < \frac{dV(\mathbf{0})}{dt} = 0$ for all $x \in \mathbf{U} - 0$. Hence, $V(x)$ is a Lyapunov function. □

The achievement of the theorem 2.1 are as follows.

1. It does not require the Lyapunov candidate to be either quadratic or a form.
2. Finding a Lyapunov function using our method does not require the computation of two Hessian as in the work of [9].

From theorem 2.1 we state the following Corollary.

Corollary 2.2. *Let $V(x)$ be a polynomial of degree $2d$ with the degree of all monomials (≥ 2) for a given differential system. If V is a SOS and the Hessian of V evaluated at the equilibrium point $\left(Hess(\frac{d}{dt}V)|_{\boldsymbol{x}=0}\right)$ is negative semi-definite then V is a Lyapunov like function.*

2.1. Construction of positive polynomial systems

Now our approach of constructing Lyapunov functions begins with a symmetric $n \times n$ matrix say A whose definiteness is to be determined. There are many techniques for doing this one of them is using the principal minors. The following theorem from [4] can be used to do just that;

Theorem 2.3. *[4] Let A be a symmetric $n \times n$ matrix, and denote by D_k the leading principal minor of order $k, 1 \leq k \leq n$. Then we have*

- *A is positive definite $\Leftrightarrow D_k > 0$ for all leading principal minors*
- *A is negative definite $\Leftrightarrow (-1)^k D_k > 0$ for all leading principal minors*
- *A is positive semi-definite $\Leftrightarrow D_k \geq 0$ for all leading principal minors*
- *A is negative semi-definite*
  $\Leftrightarrow (-1)^k D_k \geq 0$ for all leading principal minors.

Using theorem 2.3 an $n \times n$ matrix A is positive definite if and only if the following inequality is true

$$Ineq_1 = \{D_1 > 0, D_2 > 0, ..., D_n > 0\}. \tag{5}$$

Similarly, the Hessian matrix of A is is negative definite, if and only if its principal minor H_k satisfies

$$Ineq_2 = \{-H_1 > 0, H_2 > 0, \ldots, (-1)^n H_n > 0\}. \tag{6}$$

Combining inequalities 5 and 6, we have

$$\begin{aligned} Ineq &= \{A_1 > 0, A_2 > 0, \ldots, A_n > 0, \\ &- H_1 > 0, H_2 > 0, \ldots, (-1)^n H_n > 0\}. \end{aligned} \tag{7}$$

Now finding a solution for the parameters such that the symmetric matrix A positive definite and H negative definite is equivalent to finding a solution of the semi-algebraic system given in 7. The inequality in 7 can be converted to a system of equations by adding slack variables

$$\mathbf{x} = (x_1, x_2, \ldots, x_m), 0 \leq m \leq 2n,$$

and using these variables, we re-write equation 7 as

$$ES_1 = \left\{s_1 - x_1^2 = 0, \ldots, s_n - x_m^2 = 0\right\}, \tag{8}$$

where $s_1 = A_1, s_2 = A_2$, and so on. If we can find one real solution $(\bar{\mathbf{a}}, \bar{\mathbf{x}})$ of the system (8) with at least one nonzero element in $\bar{\mathbf{x}}$, then the point $\bar{\mathbf{a}}$ satisfies

$$Ineq = \{s_1(\bar{\mathbf{a}}) > 0, \ldots, s_m(\bar{\mathbf{a}}) > 0\}, \tag{9}$$

this means that there exists a Lyapunov function at the equilibrium for the given system. Below we present our propose algorithm for finding such Lyapunov functions.

2.2. Propose algorithm for constructing Lyapunov functions

In this section, we present our proposed algorithm for construction of a Lyapunov function where the candidates are not necessarily in a $p - form$ nor quadratic, unlike the work in [9]. The algorithm slightly differs from the one presented in [9], hence we only highlighted the main differences.

Input:- A differential system as defined and a tolerance.

Output:- A Lyapunov function or unknown.

1. Given a polynomial p of degree $2d (d \in \mathbf{N})$ with the degree of all monomials (≥ 2).
2. Compute the SOS decomposition of p, the Hessian of of p and its derivative.
3. Follow the procedure outlined in [9].

3. Experiment

In this section, we present some examples to demonstrate the efficiency of our algorithm. We use a personal computer with 2.20 GHz CPU processor, 4 GB RAM, and codes were written utilizing PHClab in MATLAB R2013a to carry out our calculations. We terminate our calculations whenever a Lyapunov function is found or a failure when the number of iterations is over 100.

Example 3.1. *This is an example from [**?**]*

$$\begin{aligned} \dot{x} &= -x + y + xy, \\ \dot{y} &= -x - x^2. \end{aligned} \tag{10}$$

We assumed the Lyapunov candidate to be $V(x, y) = x^4 + ax^2 + bxy + cy^2 + y^3$ *which is a polynomial of degree 4 in two variables* ($n = 2$) *that is neither quadratic nor a p-form. We compute the SOS decomposition as follows*

1. $dim(V) = n = 2$, $degV = 4 = 2d$, *where* $d = 2$.
2. *Number of monomials in z is* $|\wedge_Z| = \binom{n+d}{d} = 6$.

3. *The list of all monomials in two variables with degree* ≤ 2 *is* $X = \left[1, x, y, x^2, xy, y^2\right]$. *These are all of the power-products that could occur in an arbitrary 2-dimensional polynomial of degree* 4.
4. *The set of monomial degree vectors for* V *is* $X :=$

$$\left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \end{pmatrix} \right\}$$

5. *The Newton polytope* $C(V)$ *is the quadrilateral with vertices* $\left\{ \binom{4}{0}, \binom{2}{0}, \binom{0}{2}, \binom{0}{3} \right\}$.
6. *The reduced Newton polytope* $\frac{1}{2}C(V) \cap \mathbf{N}^n$ *is a triangle with vertices* $\left\{ \binom{0}{1}, \binom{1}{0}, \binom{2}{0} \right\}$.
7. *Set the monomials in the reduced Newton polytope* $Z = [x, y, x^2]$.
8. V *is an SOS if we can find a* 3×3 *real, symmetric matrix* Q *known as the Gram matrix such that*

$$V = zQz^T. \tag{11}$$

That is, we are looking for some real, symmetric,

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

such that

$$\begin{pmatrix} x & y & x^2 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ x^2 \end{pmatrix} = V(x,y) = x^4 + ax^2 + bxy + cy^2 + y^3.$$

By multiplying through, and simplifying we have

$$V(x,y) = q_{11}x^2 + (q_{12} + q_{21})xy + (q_{13} + q_{31})x^2 + (q_{23} + q_{32})x^2y + q_{22}y^2 + q_{33}x^4.$$

Comparing coefficients,

$$q_{12} + q_{21} = b, q_{13} + q_{31} = a, q_{23} + q_{32} = 0, q_{22} = c, q_{33} = 1, q_{11} = a.$$

Moreover, we know that Q *must be symmetric, so we can strengthen our linear constraint system even further:*

$$2q_{12} = b, 2q_{13} = a, 2q_{23} = 0, q_{22} = c, q_{33} = 1, q_{11} = a.$$

For $V(x,y) = x^4 + ax^2 + bxy + cy^2 + y^3$ *to be SOS we must have*

$$Q = \begin{pmatrix} a & \frac{b}{2} & \frac{a}{2} \\ \frac{b}{2} & c & 0 \\ \frac{a}{2} & 0 & 1 \end{pmatrix}$$

to be positive semi-definite. Thus we have the semi-algebraic inequality

$$\begin{aligned} Ineq_1 &= \left[a > 0, -\frac{b^2}{4} + ac > 0, \right. \\ &- \left. \frac{b^2}{4} + ac - \frac{a^2c}{4} > 0 \right]. \end{aligned} \tag{12}$$

For the Hessian of the derivative

$$H = \begin{pmatrix} -4a-2b & 2a-b-2c \\ 2a-b-2c & 2b \end{pmatrix}$$, *we have the following semi-algebraic system*

$$Ineq_2 = [4a+2b > 0, -4a^2 - 4ab + 8ac - 5b^2 - 4bc - 4c^2 > 0]. \tag{13}$$

Combining 12 and 13 we have the following $[a > 0, -\frac{b^2}{4} + ac > 0, -\frac{b^2}{4} + ac - \frac{a^2c}{4} > 0, 4a+2b > 0, -4a^2 - 4ab + 8ac - 5b^2 - 4bc - 4c^2 > 0]$ *which we converted into the following system*

$$ES_1 = \begin{cases} a - x_1^2 = 0, \\ -\frac{b^2}{4} + ac - x_2^2 = 0, \\ -\frac{b^2}{4} + ac - \frac{a^2c}{4} - x_3^2 = 0, \\ 4a + 2b - x_4^2 = 0, \\ -4a^2 - 4ab + 8ac - 5b^2 - 4bc- \\ 4c^2 - x_5^2 = 0. \end{cases}$$

We then constructed three hyperplane $\{h_1, h_2, h_3\}$ *denote by* ES_2 *as*

$$h_1 = 0.540708809411a + 0.06328640275578b + 0.0549791162158c + 0.941086405860x_1 + 0.169279620130x_2 + 0.961699048039x_3 + 0.505103290781x_4 + 0.659778101676x_5.$$

$$h_2 = 0.692959778799a + 0.65457115166656b + 0.0340062393965c + 0.839223301894x_1 + 0.082575528455x_2 + 0.275900397578x_3 + 0.271008171267x_4 + 0.890554884863x_5.$$

$$h_3 = 0.575002278003a + 0.37734604504476b + 0.5428655373597c + 0.826034942560x_1 + 0.846906747132x_2 + 0.057982096941x_3 + 0.834334490277x_4 + 0.555949898229x_5.$$

step4 We compute the solution of $\{ES_1, ES_2\}$ *using the Homotopy approximation method through PHC solver in MATLAB, see [8] where we obtained the following results in 1.715900 seconds.* $a = 1.2908$, $b = -1.6672$ *and* $c = 0.9920$. *Hence*

$$V(x,y) = x^4 + 1.2908x^2 - 1.6672xy + 0.9920y^2 + y^3 \tag{14}$$

is a Lyapunov function for Example 3.1. The trajectories for this Example 3.1 is shown in Figure 1 demonstrating global asymptotic stability.

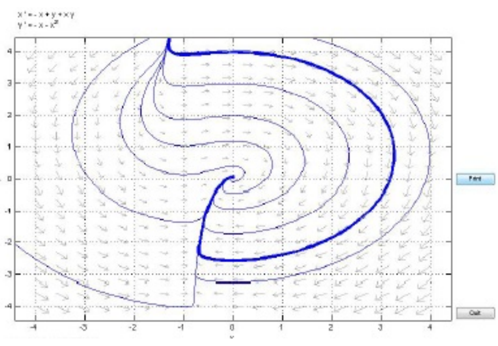

Figure 1: Phase Portrait of system presented in Example 3.1

Example 3.2. *Consider the following non-linear dynamic system which has an equilibrium point at the origin which also fits the Moore Greitzer jet engine model see [10]. The equation takes the form*

$$\begin{aligned} \dot{x} &= -y - \frac{3}{2}x^2 - \frac{1}{2}x^3, \\ \dot{y} &= 3x - y. \end{aligned} \tag{15}$$

We choose a polynomial of degree 4 as Lyapunov candidate as $V(x,y) = x^4 + ax^2 + bx^2y^2 + cxy + dx^2y + exy^2 + fy^2 + y^4$. *A Lyapunov function of degree 4 is found using our algorithm as*

$$V(x,y) = x^4 + 1.6983x^2 + 1.3319x^2y^2 - 0.0213xy - 1.0221x^2y + 0.1501xy^2 + 0.6225y^2 + y^4.$$

The trajectories for Example 3.2 is shown in Figure 2, also demonstrating global asymptotic stability. Our

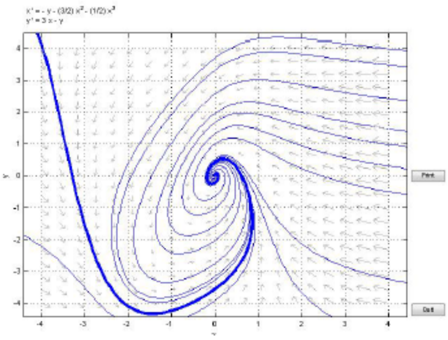

Figure 2: Phase Portrait of system presented in Example 3.2

method can produce several Lyapunov functions at the same time. We illustrate this in ***Examples 3, 4 5, and 6****. In these examples, we provide the differential systems and propose the respective Lyapunov candidates. The results of the calculation of the coefficients, the number of solutions (N), eigenvalues of the matrices Q and H and the C. P. U. time are presented in Table 1.*

Example 3.3. *This is an example from [20].*

$$\begin{aligned} \dot{x} &= -x - 3y + 2x + yz, \\ \dot{y} &= 3x - y + z + xz, \\ \dot{z} &= -2x + y - z + xy, \end{aligned} \tag{16}$$

with Lyapunov candidate as $V(x,y,z) = x^2 + axy + xz + cy^2 + dyz^2 + ez^2$.

Example 3.4. *A noisy time series of the chaotic Lorenz model introduced in* 1963 *by Edward Lorenz as a simple model of atmospheric convection see [**?**]. The model equations takes the form*

$$\begin{aligned} \dot{x_1} &= -\lambda_1 x_1 + \lambda_1 x_2, \\ \dot{x_2} &= \lambda_2 x_1 - x_2 + x_3^2 - x_1 x_3, \\ \dot{x_3} &= -\lambda_3 x_3 + x_1 x_2, \end{aligned} \tag{17}$$

we choose a Lyapunov candidate as $V(x_1,x_2,x_3) = ax_1^2 + x_1^3x_3 + x_1x_2x_3 + cx_1x_2 + dx_2^2 + bx_3^2$ *where the* $\lambda_i, i = 1 \dots 3$ *are usually regarded as positive numbers, here we assigned arbitrary positive values as* $\lambda_1 = 2.3399, \lambda_2 = 48.9372, \lambda_3 = 21.0961$.

Example 3.5. *Consider the system*

$$\begin{aligned} \dot{x} &= -x + x^2 z, \\ \dot{y} &= z, \\ \dot{z} &= -y - z - x^3, \end{aligned} \tag{18}$$

with the Lyapunov candidate as $V(x,y,z) = x^4 + ax^2 + bxy + cxz + dy^2 + eyz + z^2 + y^4$.

Example 3.6. *Consider the system*

$$\begin{aligned} \dot{x} &= -x + y^3, \\ \dot{y} &= -y - \frac{x}{\sqrt{(1+x^2)}} + z^3, \\ \dot{z} &= -z + x, \end{aligned} \tag{19}$$

with the Lyapunov candidate as $V(x,y,z) = x^6+x^3y+ax^2+bxyz+cy^2+dyz+ez^2+fxz+z^6+xz^3+z^5x+gx^2y$.

Table 1: Lyapunov functions for Examples 3.3, 3.4, 3.5 and 3.6, the number of solutions and the C. P. U. times.

Ex.	Lyapunov Function	$Eig(Q)$	$Eig(H)$	N	T(s)
3.3	$V(x,y) = x^2 + 0.2300xy$ $+xz + 0.7832y^2$ $+1.0305yz^2 + 2.7704z^2$	0.3645 0.8132 2.8759	-14.6595 -3.5527 -0.0028	72	2.8252
3.4	$V(x_1,x_2,x_3) = 18.0095x_1^2$ $+x_1x_2x_3 - 139.8403x_1x_2$ $+27.6294x_2^2 + 194.4470x_3^2$ $+x_1^3x_3$	2.1995 18.0095 219.8769 6.6708	-16408.2200 -14620.3998 -0.0003	48	1.9106
3.5	$V(x,y,z) = x^4 + 6.5279x^2$ $+2.2828xy - 0.9553xz$ $+0.7487y^2 + 0.8014yz$ $+z^2 + y^4$	0.0597 0.7522 0.9935 1.7772 2.0701	-2.5022 -0.0735	769	78.0545
3.6	$V(x,y,z) = x^6 + x^3y$ $+2.7831x^2 + 4.1090y^2$ $+1.1040yz + 7.3379z^2$ $-2.3671xz + z^6$ $+xz^3 + z^5x.$	0.8535 0.9205 2.6272 4.1188 7.7099	-42.3248 -19.3191 -0.0108	543	95.8067

From our results, we can say that our algorithm for constructing Lyapunov function where the candidates are not required to be quadratic or $p-form$ is successful. This has removed two important restrictions in search of Lyapunov functions reported in the literature, see for instance [13, 9, 20]. We included the eigenvalues of Q and H to show that the definiteness of the two matrices is satisfied, unlike the work reported in [9]. We choose homotopy continuation method in solving the system of algebraic equations resulting from our SOS decomposition because the method is known to be an efficient numerical technique for approximating all isolated solutions of a polynomial system, [8].

4. Conclusion

In this work, we presented a modified algorithm for constructing Lyapunov functions. Our main contribution in this paper is that we were able to address the fact that a polynomial Lyapunov candidate must not necessarily be quadratic or $p-forms$. We provided some examples to illustrate the usefulness of the proposed method.

Availability of data and material

Not applicable.

Conflicts of interest

The authors declare no conflict of interest.

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Author contributions

All the authors read and approved the final manuscript.

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