OPEN ACCESS
Fractals, Vol. 30, No. 5 (2022) 2240129 (7 pages)

DOI: 10.1142/S0218348X22501298

ON AN EXTENSION OF THE OPERATOR WITH MITTAG-LEFFLER KERNEL

MOHAMMED AL-REFAI
Department of Mathematics
Yarmouk University, Irbid, Jordan
m_alrefai@yu.edu.jo

DUMITRU BALEANU*
Department of Mathematics and Computer Science
Cankaya University, Angara, Turkey

Institute of Space Sciences
Magurele-Bucharest, Romania

Department of Medical Research
China Medical University Hospital
China Medical University
Taichung, Taiwan
dumitru@cankaya.edu.tr

Received June 13, 2021
Accepted October 30, 2021
Published May 31, 2022

*Corresponding author.

Abstract

Dealing with nonsingular kernels is not an easy task due to their restrictions at origin. In this short paper, we suggest an extension of the fractional operator involving the Mittag-Leffler kernel which admits integrable singular kernel at the origin. New solutions of the related differential equations were reported together with some perspectives from the modelling viewpoint.

Keywords: Fractional Calculus; Mittag-Leffler Kernel.

1. INTRODUCTION

Fractional calculus has a respectable history of 325 years but still has tremendous open problems at both theoretical and applied viewpoints (see for example Refs. 1–5 and the references therein). Fractional calculus is an extension of meaning[6] and in our opinion, several types of fractional operators can be suggested. During the last years, several classifications of fractional operators were proposed and the reader can see for example, Ref. 7 and the references therein. Since we have more experimental results to verify the validity of the fractional models, some researchers concluded that it is not possible to use a single fractional operator, e.g. Caputo ones, to describe all type of complex phenomena in science and engineering.[8] Among several point of views regarding the meaning of fractional calculus,[1–5] perhaps the one expressed by Liouville in 1832 is valuable, namely, he created a fractional operator to be used successfully for some applied complex problems from geometry and physics.[9] Generalizing the work from Ref. 8 and taking into account the fundamental work of Boltzmann[10] the operator with Mittag-Leffler kernel was suggested in Ref. 11. It is well known that several kernels, singular or nonsingular, describe complex processes with memory effect. However, for nonsingular kernels, we can have some problems with the initialization. Like mentioned earlier in Ref. 12 for all types of equations of the following form:

$$\int_a^x K(x,t)y(t)dt = f(x), \quad a \neq x \neq b, \tag{1}$$

where $K(x,t)$ and $f(x)$ are continuous, if $K(a,a) \neq 0$ then $f(a) = 0$.[12] This condition leads to some unnatural restrictions within the related differential equations involving nonsingular kernels. For more details regarding this important unsolved yet issue, the readers can see p. 3 of Ref. 12. Very recently, Refs. 13 and 14 solved the above issue for the operator introduced in Ref. 8. Thus, taking into account the above-mentioned problems of nonsingular operators, in this paper, we introduce a modification of the operator with Mittag-Leffler kernel.

The main aim of this paper is to introduce a modification of ABC fractional operator and to prove that the related fractional differential equations based of this new operator can be easily initialized and new type of solutions can be reported.

2. MAIN RESULTS

2.1. The Role of the Space

The Caputo fractional derivative of order $0 < \alpha < 1$, is defined by Refs. 1–3

$$({}^C D_0^\alpha f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (s-t)^{-\alpha} f'(s)ds, \quad t > 0.$$

It is known that if $f \in C^1[0,1]$, then

$$\lim_{t\to 0^+} ({}^C D_0^\alpha f)(t) = 0,$$

and thus the homogeneous fractional differential equation

$$\left({}^C D_0^\alpha f\right)(t) = \lambda f(t),$$

possesses only the trivial solution on the space $C^1[0,1]$. Also, the nonhomogeneous equation

$$\left({}^C D_0^\alpha f\right)(t) = -\lambda f(t) + h(t),$$

possesses a solution in $C^1[0,1]$ provided that $-\lambda f(0) + h(0) = 0$. However, the space $C^1[0,1]$ is too restrictive for the Caputo derivative and a more wider space is recommended. For instance, in the space $\chi(f) = \{f : f' \in L^1[0,1]\}$, the fractional initial value problem

$$\left({}^C D_0^\alpha f\right)(t) = -\lambda f(t) + h(t), \quad t \in (0,T],$$
$$f(0) = f_0,$$

$0 < \alpha < 1$, possesses the unique solution

$$f(t) = f_0 E_{\alpha,1}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} \times E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) h(s) ds,$$

and the corresponding homogeneous equation possesses the nontrivial solution $f(t) = f_0 E_{\alpha,1}(-\lambda t^\alpha)$.

The idea is: even if we consider a fractional derivative with singular kernel, the homogeneous equation might have only the zero solution in a certain space, which indicates the role of the space to be considered. The Atangana–Baleanu fractional derivative of order $0 < \alpha < 1$ of Caputo sense is defined by Ref. 11

$$\left({}^{\mathrm{ABC}}D_0^\alpha f\right)(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t E_\alpha(-\mu_\alpha (t-s)^\alpha) \times f'(s) ds, \quad t \geq 0, \tag{2}$$

where $\mu_\alpha = \frac{\alpha}{1-\alpha}$, and $B(\alpha)$ is a normalization function with $B(0) = B(1) = 1$. The derivative is defined for $t \geq 0$, as the kernel $k(t) = E_\alpha(-\mu_\alpha t^\alpha)$ is nonsingular. Because $k(t)$ is nonsingular one can easily show that

$$\left({}^{\mathrm{ABC}}D_0^\alpha f\right)(0) = 0.$$

Thus, the homogeneous fractional differential equation

$$\left({}^{\mathrm{ABC}}D_0^\alpha f\right)(t) = \lambda f(t),$$

possesses only the trivial solution, and the nonhomogeneous equation

$$\left({}^{\mathrm{ABC}}D_0^\alpha f\right)(t) = -\lambda f(t) + h(t),$$

possesses a solution provided that $-\lambda f(0) + h(0) = 0$, no matter is the space, see Ref. 15.

Using the standard integration by parts in (2) and taking into account the derivative of the Mittag-Leffler function, the ABC-derivative can be written as, see Ref. 20,

$$\left({}^{\mathrm{ABC}}D_0^\alpha f\right)(t) = \frac{B(\alpha)}{1-\alpha}\Big[f(t) - E_\alpha(-\mu_\alpha t^\alpha) f(0) - \mu_\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha (t-s)^\alpha) f(s) ds\Big]. \tag{3}$$

The definitions in Eqs. (2) and (3) are equivalent in a space like $H^1(0,T)$. However, the space of the function f in Eq. (3) can be extended to a more wider space in which

$$\lim_{t\to 0^+} \left({}^{\mathrm{ABC}}D_0^\alpha f\right)(t) \neq 0.$$

In that case, we avoid the extra conditions needed to guarantee the existence of solutions to the associated fractional differential equations.

Remark 1. We remark here that under certain conditions, we can do integration by parts for the fractional derivatives with singular kernels as well. For instance if $f \in C^1[0,T]$ and $f(t_0) = 0$, $t_0 \in (0,T]$ then integration by parts of the Caputo derivative $(D_0^\alpha f)(t_0)$, $0 < \alpha < 1$, was performed in Refs. 16–18.

2.2. The Modified ABC Fractional Operator in $L^1(0,T)$

The expression of the ABC fractional derivative is presented below.

Definition 2. Let $f \in L^1(0,T)$, the modified Atangana–Baleanu derivative of order $0 < \alpha < 1$, in Caputo sense is defined by

$$\left({}^{\mathrm{MABC}}D_0^\alpha f\right)(t) = \frac{B(\alpha)}{1-\alpha}\Big[f(t) - E_\alpha(-\mu_\alpha t^\alpha) f(0) - \mu_\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha (t-s)^\alpha) f(s) ds\Big]. \tag{4}$$

We remark here that the kernel in the MABC-derivative $k(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha t^\alpha)$ has integrable singularity at the origin. By direct calculations, we observed that the integral operator corresponding to (2) is the same as the original ABC integral.[11] On the same line of taught, the higher order MABC-derivatives can be defined in the following manner.

Definition 3. Let $f^{(n-1)} \in L^1(0,T)$, the modified Atangana–Baleanu derivative of order $n-1 < \delta < n$, in Caputo sense is defined by

$$\left({}^{\mathrm{MABC}}D_0^\delta f\right)(t) = \frac{B(\alpha)}{1-\alpha}\Big[f^{(n-1)}(t) - E_\alpha(-\mu_\alpha t^\alpha) f^{(n-1)}(0) - \mu_\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha (t-s)^\alpha) \times f^{(n-1)}(s) ds\Big], \tag{5}$$

where $\delta = \alpha + n - 1$.

Below we present first some illustrative examples.

Example 1. Let us consider the constant function $f(t) = C$ and $0 < \alpha < 1$. We have

$$\begin{aligned}&\left({}^{\mathrm{MABC}}D_0^{\alpha}C\right)(t)\\&\quad= \frac{B(\alpha)}{1-\alpha}\bigg[C - E_\alpha(-\mu_\alpha t^\alpha)C\\&\qquad - \mu_\alpha\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_\alpha(t-s)^\alpha)C ds\bigg].\end{aligned} \tag{6}$$

Since

$$\begin{aligned}&\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_\alpha(t-s)^\alpha)ds\\&\quad= -\frac{1}{\mu_\alpha}(E_\alpha(-\mu_\alpha t^\alpha)-1),\end{aligned}$$

then

$$\left({}^{\mathrm{MABC}}D_0^{\alpha}C\right)(t) = 0.$$

In the above example, we indicate that $\left({}^{\mathrm{MABC}}D_0^{\alpha}C\right)(t) = 0$, a result which is expected as $f(t) = C \in H^1(0,T)$. In the following example, we show that $\left({}^{\mathrm{MABC}}D_0^{\alpha}f\right)(t)$ doesn't vanish at $t = 0$.

Example 2. Consider

$$f(t) = \begin{cases} t^{-\frac{1}{2}}, & t \neq 0,\\ A, & t = 0.\end{cases} \tag{7}$$

$A \in \mathbb{R}$. For $\alpha = \frac{1}{2}$, and $B(\alpha) = 1$, we have $\mu_\alpha = \frac{\alpha}{1-\alpha} = 1$, and thus

$$\begin{aligned}&\left({}^{\mathrm{MABC}}D_0^{\frac{1}{2}}f\right)(t)\\&\quad= 2\bigg[f(t) - AE_\alpha(-t^{\frac{1}{2}})\\&\qquad - \int_0^t (t-s)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}(-(t-s)^{\frac{1}{2}})s^{-\frac{1}{2}}ds\bigg]\\&\quad= 2\big[f(t) - AE_\alpha(-t^{\frac{1}{2}}) - \sqrt{\pi}E_{\frac{1}{2}}(-t^{\frac{1}{2}})\big]\\&\quad= 2\big[f(t) - (A+\sqrt{\pi})E_{\frac{1}{2}}(-t^{\frac{1}{2}})\big].\end{aligned}$$

Since $E_{\frac{1}{2}}(0) = 1$, we have

$$\begin{aligned}\left({}^{\mathrm{MABC}}D_0^{\frac{1}{2}}f\right)(0) &= 2\big[f(0) - (A+\sqrt{\pi})E_{\frac{1}{2}}(0)\big]\\&= -2\sqrt{\pi} \neq 0.\end{aligned}$$

Figure 1 depicts the graph of $f(t)$ and the MABC-derivative of $f(t)$ for $A = 1$.

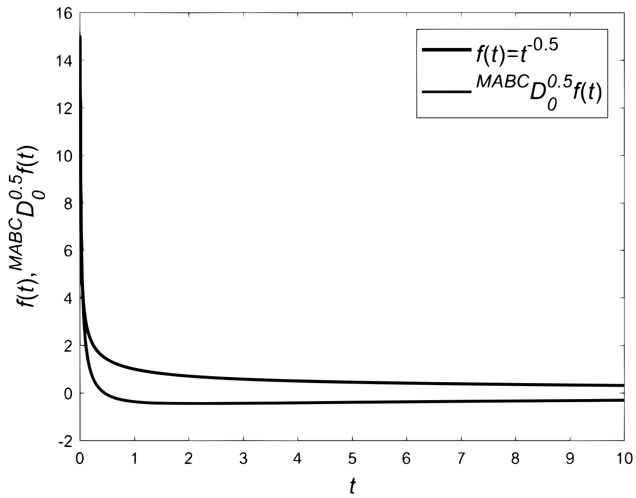

Fig. 1 The graph of $f(t)$ and the MABC-derivative of $f(t)$ in Example 2.

Remark 4. The above example indicates that the fractional initial value problem

$$\begin{aligned}\left({}^{\mathrm{MABC}}D_0^{\frac{1}{2}}u\right)(t) - 2u(t) &= -2(A+\sqrt{\pi})E_{\frac{1}{2}}(-t^{\frac{1}{2}}),\\ u(0) &= A,\end{aligned}$$

possesses the solution $f(t)$ given in Eq. (7), while the corresponding initial value problem with the original ABC-derivative

$$\begin{aligned}({}^{\mathrm{ABC}}D_0^{\frac{1}{2}}u)(t) - 2u(t) &= -2(A+\sqrt{\pi})E_{\frac{1}{2}}(-t^{\frac{1}{2}}),\\ u(0) &= A,\end{aligned}$$

possesses no solution.

Example 3. Consider

$$f(t) = \begin{cases} t^{\frac{1}{2}}\sin\left(\frac{1}{t}\right), & t \neq 0,\\ 0, & t = 0.\end{cases}$$

Because $f \in C[0,1]$, then $\left({}^{\mathrm{MABC}}D_0^{\alpha}f\right)(t)$ exists for all $0 < \alpha < 1$, and $t \in [0,1]$, whereas $({}^{\mathrm{ABC}}D_0^{\alpha}f)(t)$ doesn't exist.

It is known that for $0 < \alpha < 1$, the equation $({}^{\mathrm{ABC}}D_0^{\alpha}u)(t) = C$, $C \in \mathbb{R} - \{0\}$, possesses no solution, which is not the case with the MABC-derivative as indicated in the following example.

Example 4. For $0 < \alpha < 1$, the solution of

$$\left({}^{\mathrm{MABC}}D_0^{\alpha}u\right)(t) = C, \quad C \in \mathbb{R},\ t > 0$$

is given by

$$u(t) = C\frac{1-\alpha}{B(\alpha)}\begin{cases}1+\mu_\alpha\dfrac{t^\alpha}{\Gamma(1+\alpha)}, & t>0,\\ 0, & t=0.\end{cases}$$

Proof. Since $u(0)=0$, we have for $t>0$,

$$\begin{aligned}\mathcal{L}\big({}^{\mathrm{MABC}}D_0^\alpha u;s\big) &= \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha}{s^\alpha+\mu_\alpha}\mathcal{L}(u;s)\\ &= C\frac{s^\alpha}{s^\alpha+\mu_\alpha}\left(\frac{1}{s}+\frac{\mu_\alpha}{s^{1+\alpha}}\right)\\ &= \frac{C}{s} = \mathcal{L}(C),\end{aligned}$$

which completes the proof. □

In the following, we show that the homogenous fractional initial value problem possesses a nonzero solution. We will use the following known formulas:

$$\mathcal{L}(E_\alpha(\gamma t^\alpha)) = \frac{s^{\alpha-1}}{s^\alpha-\gamma},\quad \left|\frac{\gamma}{s^\alpha}\right|<1, \tag{8}$$

$$\mathcal{L}(t^{\alpha-1}E_{\alpha,\alpha}(\gamma t^\alpha)) = \frac{1}{s^\alpha-\gamma},\quad \left|\frac{\gamma}{s^\alpha}\right|<1, \tag{9}$$

$$\begin{aligned}&\mathcal{L}\big({}^{\mathrm{MABC}}D_0^\alpha f;s\big)\\ &\quad= \mathcal{L}\big({}^{\mathrm{ABC}}D_0^\alpha f;s\big)\\ &\quad= \frac{B(\alpha)}{1-\alpha}\frac{s^\alpha\mathcal{L}(f;s)-f(0)s^{\alpha-1}}{s^\alpha+\mu_\alpha},\quad \left|\frac{\mu_\alpha}{s^\alpha}\right|<1.\end{aligned} \tag{10}$$

Lemma 5. *Consider the fractional initial value problem*

$$\big({}^{\mathrm{MABC}}D_0^\alpha u\big)(t) = \lambda u(t),\quad t>0,\ \ u(0)=u_0,$$

where $0<\alpha<1$.

(1) *For* $\lambda = \frac{B(\alpha)}{1-\alpha}$, *the solution is given by*

$$u(t) = u_0\begin{cases}-\dfrac{t^{-\alpha}}{\mu_\alpha\Gamma(1-\alpha)}, & t\neq 0,\\ 1, & t=0.\end{cases}$$

(2) *For* $\lambda \neq \frac{B(\alpha)}{1-\alpha}$, *the solution is given by*

$$u(t) = u_0\begin{cases}\dfrac{E_\alpha(\mu_\alpha\frac{\delta_\alpha}{1-\delta_\alpha}t^\alpha)}{1-\delta_\alpha}, & t\neq 0,\\ 1, & t=0,\end{cases}$$

where $\delta_\alpha = \frac{\lambda(1-\alpha)}{B(\alpha)}$.

Proof.

(1) Given that

$$\begin{aligned}&\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_\alpha(t-s)^\alpha)s^{-\alpha}ds\\ &\quad= \Gamma(1-\alpha)E_\alpha(-\mu_\alpha t^\alpha),\end{aligned} \tag{11}$$

we have for $t>0$,

$$\begin{aligned}&\big({}^{\mathrm{MABC}}D_0^\alpha u\big)(t)\\ &= \frac{B(\alpha)}{1-\alpha}\bigg(u(t)-E_\alpha(-\mu_\alpha t^\alpha)u_0\\ &\quad -\mu_\alpha\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\mu_\alpha(t-s)^\alpha)\\ &\quad \times\left(-\frac{u_0}{\mu_\alpha\Gamma(1-\alpha)}s^{-\alpha}\right)ds\bigg)\\ &= \frac{B(\alpha)}{1-\alpha}(u(t)-E_\alpha(-\mu_\alpha t^\alpha)u_0\\ &\quad +E_\alpha(-\mu_\alpha t^\alpha)u_0)\\ &= \lambda u(t),\end{aligned}$$

which completes the proof.

(2) Using Eqs. (8) and (10) we have for $t>0$,

$$\begin{aligned}&\mathcal{L}\big({}^{\mathrm{MABC}}D_0^\alpha u;s\big)\\ &\quad= \frac{B(\alpha)}{1-\alpha}\frac{1}{s^\alpha+\mu_\alpha}\bigg(\frac{u_0}{1-\delta_\alpha}s^\alpha\\ &\qquad \times\frac{s^{\alpha-1}}{s^\alpha-\mu_\alpha\frac{\delta_\alpha}{1-\delta_\alpha}}-u_0s^{\alpha-1}\bigg)\\ &\quad= \frac{B(\alpha)}{1-\alpha}u_0\frac{\delta_\alpha}{1-\delta_\alpha}\frac{s^{\alpha-1}}{s^\alpha-\mu_\alpha\frac{\delta_\alpha}{1-\delta_\alpha}}\\ &\quad= \lambda\frac{u_0}{1-\delta_\alpha}\frac{s^{\alpha-1}}{s^\alpha-\mu_\alpha\frac{\delta_\alpha}{1-\delta_\alpha}}\\ &\quad= \lambda\frac{u_0}{1-\delta_\alpha}\mathcal{L}\left(E_\alpha\left(\mu_\alpha\frac{\delta_\alpha}{1-\delta_\alpha}t^\alpha\right)\right),\end{aligned}$$

which completes the proof. □

Lemma 6. *Consider the linear fractional initial value problem*

$$\big({}^{\mathrm{MABC}}D_0^\alpha u\big)(t)+\lambda u(t) = g(t),\quad t>0,\ \ u(0)=u_0.$$

For $0<\alpha<1$, *and* $\lambda\neq -\frac{B(\alpha)}{1-\alpha}$, *the solution of the above fractional initial value problem is given by*

$$u(t) = \begin{cases}\hat{u}, & t\neq 0,\\ u_0, & t=0,\end{cases} \tag{12}$$

where

$$\hat{u} = u_0 \frac{B(\alpha)}{r_\alpha} E_\alpha\left(-\frac{\lambda\alpha}{r_\alpha} t^\alpha\right) + \frac{1-\alpha}{r_\alpha} g(t) + \frac{1-\alpha}{r_\alpha} \times \left(\mu_\alpha - \frac{\lambda\alpha}{r_\alpha}\right)\left(t^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{\lambda\alpha}{r_\alpha} t^\alpha\right)\right) * g,$$

and $r_\alpha = B(\alpha) + \lambda(1-\alpha)$.

Proof. Using Eqs. (8) and (9) one can easily verify that

$$\mathcal{L}(\hat{u}; s) = \frac{u_0 B(\alpha) s^{\alpha-1} + (1-\alpha)(s^\alpha + \mu_\alpha)\mathcal{L}(g; s)}{r_\alpha s^\alpha + \lambda\alpha}. \tag{13}$$

Using Eq. (10) we have

$$\mathcal{L}\left({}^{\mathrm{MABC}}D_0^\alpha u + \lambda u; s\right) = \frac{B(\alpha)}{1-\alpha} \frac{s^\alpha \mathcal{L}(\hat{u}; s) - s^{\alpha-1} u_0}{s^\alpha + \mu_\alpha} + \lambda \mathcal{L}(\hat{u}; s). \tag{14}$$

Direct calculations will lead to

$$\mathcal{L}\left({}^{\mathrm{MABC}}D_0^\alpha f + \lambda u; s\right) = \frac{1}{(1-\alpha)(s^\alpha + \mu_\alpha)}((r_\alpha s^\alpha + \lambda\alpha)\mathcal{L}(\hat{u}; s) - B(\alpha) u_0 s^{\alpha-1}). \tag{15}$$

By substituting Eq.(13) in Eq. (15) we have

$$\begin{aligned}\mathcal{L}\left({}^{\mathrm{MABC}}D_0^\alpha f + \lambda u; s\right) &= \frac{1}{(1-\alpha)(s^\alpha + \mu_\alpha)}(u_0 B(\alpha) s^{\alpha-1} + (1-\alpha) \times (s^\alpha + \mu_\alpha)\mathcal{L}(g; s) - u_0 B(\alpha) s^{\alpha-1}) \\ &= \mathcal{L}(g; s),\end{aligned}$$

which completes the proof. □

Remark 7. If $g \in C[0, T]$, then

$$\hat{u}(0) = \frac{1}{r_\alpha}(u_0 B(\alpha) + (1-\alpha) g(0)).$$

If we add the extra condition

$$\lambda u_0 = g(0), \tag{16}$$

then $\hat{u}(0) = u_0$, and the solution given in Eq. (12) is continuous. This solution is the same solution obtained for the associated initial value problem with the ABC-derivative, and the condition in (16) is the necessary condition to guarantee the existence of a solution.

For the associated fractional integral operator, we use the one obtained in Ref. 19.

Definition 8. For $f \in L^1(0, \infty)$, $n-1 < \delta < n$, $n \in \mathbb{N}$, $\alpha = \delta - n + 1$, the modified Atangana–Baleanu fractional integral operator is defined by

$$\begin{aligned}\left({}^{\mathrm{MAB}}I_0^\delta f\right)(t) &= \frac{1-\alpha}{B(\alpha)}\Bigg(\left({}^{RL}I_0^{n-1} f\right)(t) + \mu_\alpha\left({}^{RL}I_0^{n+\alpha-1} f\right)(t) - f(0)\left(\frac{t^{n-1}}{\Gamma(n)} + \mu_\alpha \frac{t^{n+\alpha-1}}{\Gamma(n+\alpha)}\right)\Bigg), \\ &= \frac{1-\alpha}{B(\alpha)}\left({}^{RL}I_0^{n-1}(f - f(0))\right)(t) + \mu_\alpha\left({}^{RL}I_0^{n+\alpha-1}(f - f(0))\right)(t).\end{aligned} \tag{17}$$

Lemma 9 (Ref. 19). *For* $f^{(n)} \in L^1(0, \infty)$, *and* $n-1 < \delta < n$, $n \in \mathbb{N}$, $\alpha = \delta - n + 1$, *the following holds true*:

$$\left({}^{\mathrm{MAB}}I_0^\delta \ {}^{\mathrm{MABC}}D_0^\delta f\right)(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0) \frac{t^k}{k!}, \tag{18}$$

$$\left({}^{\mathrm{MABC}}D_0^\delta \ {}^{\mathrm{MAB}}I_0^\delta f\right)(t) = f(t) - f(0). \tag{19}$$

2.3. Infinite Series Representation

We present an infinite series representation of the MABC-derivative using the Riemann–Liouville integrals. We have

$$\begin{aligned}&\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha (t-s)^\alpha) f(s) ds \\ &\quad = \int_0^t (t-s)^{\alpha-1} \sum_{k=0}^\infty \frac{(-\mu_\alpha)^k (t-s)^{\alpha k}}{\Gamma(\alpha k + \alpha)} f(s) ds \\ &\quad = \sum_{k=0}^\infty \frac{(-\mu_\alpha)^k}{\Gamma(\alpha k + \alpha)} \int_0^t (t-s)^{\alpha(k+1)-1} f(s) ds \\ &\quad = \sum_{k=0}^\infty (-\mu_\alpha)^k \left({}^{RL}I_0^{\alpha(k+1)} f\right)(t),\end{aligned} \tag{20}$$

and thus

$$\begin{aligned}&\left({}^{\mathrm{MABC}}D_0^\alpha f\right)(t) \\ &\quad = \frac{B(\alpha)}{1-\alpha}\Bigg[f(t) - E_\alpha(-\mu_\alpha) f(0) - \mu_\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\mu_\alpha (t-s)^\alpha) f(s) ds\Bigg]\end{aligned}$$

$$= \frac{B(\alpha)}{1-\alpha}\Bigg[f(t) - E_\alpha(-\mu_\alpha)f(0) - \sum_{k=0}^{\infty}(-\mu_\alpha)^{k+1}\left({}^{RL}I_0^{\alpha(k+1)}f\right)(t)\Bigg]. \tag{21}$$

The above-mentioned formulas are useful to derive the Leibniz and chain rules for the modified version of ABC operator.

3. CONCLUSION

We have introduced the MABC-fractional derivative which is an extension to the ABC-derivative in a more wider space. The kernel of the MABC-derivative has integrable singularity at the origin. The modification of ABC leads us to some new solutions of the corresponding fractional differential equations and the fundamental role of the space can be clearly stated. Besides, the results obtained in this present manuscript show once more the importance of the Caputo like type derivatives. We show that the solutions of several fractional equations with the MABC-derivative which are not solvable with the ABC-derivative. For instance, the homogenous fractional differential equations with the MABC-derivative admit a nonzero solution, and certain linear fractional equations admit solutions without imposing extra conditions. The integral operator associated to the MABC-derivative is the same as the one corresponding to the ABC-derivative. We also report an infinite series representation of the MABC-derivative. From the modelling viewpoint the new suggested modification will bring some light for both problems with and without singularity at origin. As a result, we will be able to characterize better the dynamics of complex phenomena by using MABC-derivative. In this way, the range of applicability of these operators will be considerably enlarged.

REFERENCES

1. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach, Yverdon, 1993).
2. I. Podlubny, *Fractional Differential Equations* (Academic Press, San Diego, 1999).
3. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Vol. 204 (Elsevier Science B.V., Amsterdam, 2006).
4. R. L. Magin, *Fractional Calculus in Bioengineering* (Begell House Publishers, 2006).
5. D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, 2nd edn. (World Scientific, New York, 2017).
6. B. Ross, *A Brief History and Exposition of the Fundamental Theory of Fractional Calculus: Fractional Calculus and its Applications*, Lecture Notes in Mathematics (Springer-Verlag, Berlin, Heidelberg, 1975), pp. 1–36.
7. D. Baleanu and A. Fernandez, On fractional operators and their classification, *Mathematics* **7** (2019) 830, doi:10.339/math7090830.
8. M. Caputo and M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.* **1**(2) (2015) 73–85.
9. J. Liouville, Mémoire sur quelques questions de géométrie et de la mecanique, et sur un nouveau genre de calcul pour resoudre ces questions, *J. Ec. Polytech.* **13** (1832) 1–69.
10. S. Boltzmann, Zur theorie der elastischen nachwirkung, *Sitzber. Akad. Wiss. Wien Math. Naturw. Kl* **70** (1874) 275; *Pogg. Ann. Phys.* **7** (1876) 624.
11. A. Atangana and D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, *Thermal Sci.* **20**(2) (2016) 763–769.
12. A. D. Polyanin and A. V. Manzhirov, *Handbook of Integral Equations*, 2nd edn. (Taylor and Francisc Group, 2008).
13. M. Caputo and M. Fabrizio, On the singular kernels for fractional derivatives. Some applications to partial differential equations, *Progr. Fract. Differ. Appl.* **7**(2) (2021) 79–82.
14. J. J. Nieto and J. Losada, Fractional integral associated to fractional derivatives with nonsingular kernels, *Progr. Fract. Differ. Appl.* **7**(3) (2021) 137–143.
15. M. Al-Refai, Comparison principles for differential equations involving Caputo fractional derivative with Mittag-Leffler non-singular kernel, *Electron. J. Differ. Equ.* **36** (2018) 1–10.
16. M. Al-Refai, On the fractional derivatives at extreme points, *Electron. J. Qual. Theory Differ. Equ.* **55** (2012) 1–5.
17. M. Al-Refai and Y. Luchko, Maximum principle for the fractional diffusion equations with the Riemann–Liouville fractional derivative and its applications, *Fract. Calc. Appl. Anal.* **17**(2) (2014) 483–498.
18. Y. Luchko, Maximum principle for the generalized time-fractional diffusion equation, *J. Math. Anal. Appl.* **351** (2009) 218–223.
19. M. Al-Refai, Proper inverse operators of fractional derivatives with nonsingular kernels, *Rend. Circ. Mat. Palermo (2)* (2021), doi:10.1007/s12215-021-00638-2.
20. K. Diethelm, R. Garrappa, A. Giusti, M. Stynes, Why fractional derivatives with nonsingular kernels should not be used, *Frac. Calc. Appl. Anal.* **23**(3) 610–634.