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Research Article

On Extended *b*-Rectangular and Controlled Rectangular Fuzzy Metric-Like Spaces with Application

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In this article, we introduce the notions of extended *b*-rectangular and controlled rectangular fuzzy metric-like spaces that generalize many fuzzy metric spaces in the literature. We give examples to justify our newly defined fuzzy metric-like spaces and prove that these spaces are not Hausdorff. We use fuzzy contraction and prove Banach fixed point theorems in these spaces. As an application, we utilize our main results to solve the uniqueness of the solution of a differential equation occurring in the dynamic market equilibrium.

1. Introduction and Preliminaries

In 1965, Zadeh [1] introduced the concept of a fuzzy set that generalizes the concept of an ordinary set or crisp set. Many authors have used fuzzy sets in different branches of mathematics extensively. For example, Kaleva [2] gave the idea of fuzzy differential equations, Buckley and Feuring [3] introduced the concept of fuzzy partial differential equations, and Puri and Ralescu [4] introduced the differentials of fuzzy functions. Fuzzy metric space is one of the most studied topic in fuzzy set theory. The definition of fuzzy metric space was given by Kramosil and Michálek [5] in 1975 which was later modified by George and Veeramani [6]. In 1983, Grabiec [7] established and proved the fuzzy version of the Banach contraction principle. Many researchers have used and extended this version in many fuzzy metric spaces (see [8–15] and references therein). Branciari [16] generalized the definition of classical metric space by introducing rectangular or Branciari metric space and proved some fixed point results. As a generalization of a Branciari metric space, authors in [17] gave the notion of *b*-rectangular metric space.

Hitzler and Seda [18] introduced the idea of dislocated topology in which the self-distance between the points may not be zero. Amini-Harandi [19] introduced the definition of a metric-like space and proved related results. The notion of *b*-metric-like space was introduced by [20] as a generalization of metric-like space. Mlaiki et al. [21] generalized the definition of a rectangular metric space by defining rectangular metric-like space. The concept of a fuzzy metric-like space was introduced by Shukla and Abbas [14] as a generalization of [6] and proved related fixed point results. They generalized the definition of a fuzzy metric space in the sense that the self-distance may not be equal to one. The concept of fuzzy b-metric and fuzzy quasi-*b*-metric space was introduced by Nadaban [22]. The concepts of fuzzy double controlled metric space and fuzzy triple controlled metric space were given by Saleem et al. [13] and Furqan et al. [23], respectively. They also proved that these spaces are not Hausdorff.

Definition 1 (see [19]). Let F be a nonempty set, a mapping $\rho : \mathrm{F}\times\mathrm{F} \longrightarrow \mathbb{R}^{+}\cup\{0\}$ is called metric-like if, for all $\hbar_1, \hbar_2, \hbar_3 \in \mathrm{F}$, ρ satisfies the following:

$$\begin{aligned}
&\text{ML1 } \rho(\hbar_1,\hbar_2)=0 \Longrightarrow \hbar_1=\hbar_2,\\
&\text{ML2 } \rho(\hbar_1,\hbar_2)=\rho(\hbar_2,\hbar_1),\\
&\text{ML3 } \rho(\hbar_1,\hbar_3)\le\rho(\hbar_1,\hbar_2)+\rho(\hbar_2,\hbar_3).
\end{aligned}\tag{1}$$

The pair (F,ρ) is called a metric-like space.

Example 1 (see [19]). Let $F = \{0, 1\}$, and ρ is given by $\rho(\hbar_1, \hbar_2) = 2$ if $\hbar_1 = \hbar_2 = 0$, and $\rho(\hbar_1, \hbar_2) = 1$, otherwise. Then, (F, ρ) is a metric-like space.

Definition 2 (see [20]). Let F be a nonempty set and $b \geq 1$; a function $\rho : \mathrm{F} \times \mathrm{F} \longrightarrow \mathbb{R}^+ \cup \{0\}$ is called b-metric-like if $\hbar_1, \hbar_2, \hbar_3 \in \mathrm{F}, \rho$ satisfies the following:

$$\begin{gathered} \text{bML1}\ \rho(\hbar_1, \hbar_2) = 0 \Longrightarrow \hbar_1 = \hbar_2, \\ \text{bML2}\ \rho(\hbar_1, \hbar_2) = \rho(\hbar_2, \hbar_1), \\ \text{bML3}\ \rho(\hbar_1, \hbar_3) \leq b[\rho(\hbar_1, \hbar_2) + \rho(\hbar_2, \hbar_3)]. \end{gathered} \tag{2}$$

The pair (F, ρ) is called b-metric-like space.

Example 2 (see [20]). Let $F = [0,\infty)$ and $\rho : [0,\infty) \times [0,\infty) \times [0,\infty)$ be defined as $\rho(\hbar_1, \hbar_2) = (\hbar_1 + \hbar_2)^2$. Then, (F, ρ) is a b-metric-like space with $b = 2$.

Definition 3 (see [24]). Let F be a nonempty set and $\rho : \mathrm{F} \times \mathrm{F} \longrightarrow [0,\infty)$ be a function; then, ρ is said to be a rectangular metric-like space if it satisfies the following:

$$\text{RML1}\ \rho(\hbar_1, \hbar_2) = 0 \Longrightarrow \hbar_1 = \hbar_2,$$

$$\text{RML2}\ \rho(\hbar_1, \hbar_2) = \rho(\hbar_2, \hbar_1),$$

$$\begin{aligned} \text{RML3}\ \rho(\hbar_1, \hbar_4) &\leq \rho(\hbar_1, \hbar_2) + \rho(\hbar_2, \hbar_3) \\ &\quad + \rho(\hbar_3, \hbar_4), \text{for all distinct}\ \hbar_2, \hbar_3 \in \mathrm{F} \setminus \{\hbar_1, \hbar_4\}. \end{aligned} \tag{3}$$

The pair (F, ρ) is called rectangular metric-like space.

Definition 4 (see [25]). A binary operation $*$ on $[0, 1]$, where $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$, is called a continuous triangular norm, if for all $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \in [0, 1]$,the following conditions are satisfied:

$(*1) *(\hbar_1, \hbar_2) = *(\hbar_2, \hbar_1)$

$(*2) *(\hbar_1, *(\hbar_2, \hbar_3)) = *(*(\hbar_1, \hbar_2), \hbar_3)$

$(*3) *$ is continuous

$(*4) *(\hbar, 1) = \hbar$ for every $\hbar \in 0, 1]$,

$(*5) *(\hbar_1, \hbar_2) \leq *(\hbar_3, \hbar_4)$ whenever $\hbar_1 \leq \hbar_3, \hbar_2 \leq \hbar_4$.

2. Extended Fuzzy b-Rectangular Metric-Like Space

In this section, we introduce the definition of an extended fuzzy b-rectangular metric-like space.

Definition 5. Let $f : \mathrm{F} \times \mathrm{F} \longrightarrow [1,\infty)$ be a function; M_f is a fuzzy set on $F \times \mathrm{F} \times (0,\infty)$. Then, M_f is called an extended fuzzy b-rectangular metric-like with t-norm $*$, if for all $\hbar_1$, $\hbar_4 \in \mathrm{F}$ and all distinct $\hbar_3, \hbar_2 \in \mathrm{F} \setminus \{\hbar_1, \hbar_4\}$; M_f satisfies the following:

$$\begin{gathered} (M_f1)\ M_f(\hbar_1, \hbar_2, t) > 0, \\ (M_f2)\ \text{if}\ M_f(\hbar_1, \hbar_2, t) = 1\ \text{for all}\ t > 0, \text{then}\ \hbar_1 = \hbar_2, \\ (M_f3)\ M_f(\hbar_1, \hbar_2, t) = M_f(\hbar_2, \hbar_1, t), \\ \begin{aligned} &(M_f4)\ M_f(\hbar_1, \hbar_4, t + s + w) \\ &\quad \geq M_f\left(\hbar_1, \hbar_2, \frac{t}{f(\hbar_1, \hbar_4)}\right) * M_f\left(\hbar_2, \hbar_3, \frac{s}{f(\hbar_1, \hbar_4)}\right) \\ &\quad * M_f\left(\hbar_3, \hbar_4, \frac{w}{f(\hbar_1, \hbar_4)}\right), \text{for all}\ t, s, w > 0, \end{aligned} \\ (M_f5)\ M_f(\hbar_1, \hbar_2, \cdot) : (0,\infty) \longrightarrow [0, 1]\ \text{is continuous.} \end{gathered} \tag{4}$$

Then, $(\mathrm{F}, M_f, f, *)$ is called an extended fuzzy b-rectangular metric-like space.

Remark 6.

(i) By taking $f(x, y) = b \geq 1$, then extended fuzzy b-rectangular metric-like space reduces to fuzzy b-rectangular metric-like space.

(ii) By taking $f(x, y) = 1$, then extended fuzzy b-rectangular metric-like space reduces to fuzzy rectangular metric-like space.

The following example justifies Definition 5.

Example 3. Let $F = \{1, 2, 3, 4\}$. If we define $\rho : \mathrm{F} \times \mathrm{F} \longrightarrow [0,\infty)$ by $\rho(\hbar_1, \hbar_2) = (\hbar_1 + \hbar_2)^2$ for all $\hbar_1, \hbar_2$ in F and $f : \mathrm{F} \times \mathrm{F} \longrightarrow [1,\infty)$ by $\mathrm{f}(\hbar_1, \hbar_2) = \hbar_1^2 + \hbar_2^2 + 1$. Then, $M_f : \mathrm{F} \times \mathrm{F} \times (0,\infty) \longrightarrow [0, 1]$, given by

$$M_f(\hbar_1, \hbar_2, t) = \frac{t}{t + \rho(\hbar_1, \hbar_2)}, \quad \text{for all}\ t > 0, \tag{5}$$

is an extended fuzzy b-rectangular metric-like on F provided that $*$ is a minimum t-norm, that is, $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$.

Clearly,

$$\begin{gathered} f(1, 2) = f(2, 1) = 6, f(1, 3) = f(3, 1) = 11, f(1, 4) \\ = f(4, 1) = 18, \\ f(2, 3) = f(3, 2) = 14, f(2, 4) = f(4, 2) = 21, f(3, 4) \\ = f(4, 3) = 26, \\ f(1, 1) = 3, f(2, 2) = 9, f(3, 3) = 19, f(4, 4) = 33, \\ \rho(1, 2) = \rho(2, 1) = 9, \rho(2, 3) = \rho(3, 2) = 25, \rho(3, 4) \\ = \rho(4, 3) = 49, \\ \rho(1, 3) = \rho(3, 1) = 16, \rho(2, 4) = \rho(4, 2) = 36, \rho(1, 4) \\ = \rho(4, 1) = 25, \\ \rho(1, 1) = 4, \rho(2, 2) = 16, \rho(3, 3) = 36, \rho(4, 4) \\ = 64, \quad \text{for all}\ \hbar_1 \in F. \end{gathered} \tag{6}$$

Note that the first three axioms (M_f1–M_f3) clearly hold. To prove the axiom M_f4, we discuss the following cases:

Case 1. Let $\hbar_1 = 1, \hbar_3 = 4$. Then, we have

$$\begin{aligned} M_f(1,4,f(1,4)(t+s+w)) &= \frac{f(1,4)(t+s+w)}{f(1,4)(t+s+w)+\rho(1,4)} \\ &= \frac{18(t+s+w)}{18(t+s+w)+25} = 1 - \frac{25}{18(t+s+w)+225}. \end{aligned} \tag{7}$$

Also,

$$\begin{aligned} M_f(1,2,t) &= \frac{t}{t+\rho(1,2)} = \frac{t}{t+9} = 1 - \frac{9}{t+9}, \\ M_f(2,3,s) &= \frac{s}{s+\rho(2,3)} = \frac{s}{s+25} = 1 - \frac{25}{s+25}, \\ M_f(3,4,w) &= \frac{w}{w+\rho(3,4)} = \frac{w}{w+49} = 1 - \frac{49}{w+49}. \end{aligned} \tag{8}$$

Note that

$$\begin{aligned} M_f(1,4,f(1,4)(t+s+w)) &= 1 - \frac{25}{18(t+s+w)+25} \\ &= 1 - \frac{225}{162t+162s+162w+225}, \\ M_f(1,2,t) = 1 - \frac{9}{t+9} &= 1 - \frac{225}{25t+225}. \end{aligned} \tag{9}$$

Clearly,

$$1 - \frac{225}{162t+162s+162w+225} > 1 - \frac{225}{25t+225}. \tag{10}$$

That is,

$$M_f(1,4,f(1,4)(t+s+w)) > M_f(1,2,t). \tag{11}$$

Similarly, we have

$$\begin{aligned} M_f(1,4,f(1,4)(t+s+w)) &> M_f(2,3,s), \\ M_f(1,4,f(1,4)(t+s+w)) &> M_f(3,4,w). \end{aligned} \tag{12}$$

Hence,

$$\begin{aligned} &M_f(1,4,f(1,4)(t+s+w)) \\ &\quad > M_f(1,2,t) * M_f(2,3,s) * M_f(3,4,w). \end{aligned} \tag{13}$$

Case 2. Let $\hbar_1 = 2$, and $\hbar_3 = 4$. Then,

$$\begin{aligned} M_f(2,4,f(2,4)(t+s+w)) &= \frac{f(2,4)(t+s+w)}{f(2,4)(t+s+w)+\rho(2,4)} \\ &= \frac{21(t+s+w)}{21(t+s+w)+36}, \\ &= 1 - \frac{36}{21(t+s+w)+36}, \\ M_f(2,3,t) &= \frac{t}{t+\rho(2,3)} = \frac{t}{t+25} = 1 - \frac{25}{t+25}, \\ M_f(3,1,s) &= \frac{s}{s+\rho(3,1)} = \frac{s}{s+16} = 1 - \frac{16}{s+16}, \\ M_f(1,4,w) &= \frac{w}{w+\rho(1,4)} = \frac{w}{w+25} = 1 - \frac{25}{w+25}. \end{aligned} \tag{14}$$

Now,

$$\begin{aligned} M_f(2,4,f(2,4)(t+s+w)) &= 1 - \frac{36}{21(t+s+w)+36} \\ &= 1 - \frac{900}{525t+525s+525w+900}, \\ M_f(2,3,t) = 1 - \frac{25}{t+25} &= 1 - \frac{900}{36t+900}. \end{aligned} \tag{15}$$

Clearly,

$$1 - \frac{900}{525t+525s+525w+900} > 1 - \frac{900}{36t+900} \tag{16}$$

implies that

$$M_f(1,4,f(1,4)(t+s+w)) > M_f(1,2,t). \tag{17}$$

Similarly, we obtain that

$$\begin{aligned} M_f(2,4,f(2,4)(t+s+w)) &> M_f(3,1,s), \\ M_f(2,4,f(2,4)(t+s+w)) &> M_f(1,4,w). \end{aligned} \tag{18}$$

Hence,

$$\begin{aligned} &M_f(2,4,f(2,4)(t+s+w)) \\ &\quad > M_f(2,3,t) * M_f(3,1,s) * M_f(1,4,w). \end{aligned} \tag{19}$$

Case 3. If $\hbar_1 = 3$ and $\hbar_3 = 4$, then we have

$$\begin{aligned} M_f(3,4,f(3,4)(t+s+w)) &= \frac{f(3,4)(t+s+w)}{f(3,4)(t+s+w)+\rho(3,4)} \\ &= \frac{26(t+s+w)}{26(t+s+w)+49}, \\ &= 1-\frac{49}{26(t+s+w)+49}. \end{aligned} \tag{20}$$

Also,

$$\begin{aligned} M_f(3,1,t) &= \frac{t}{t+\rho(3,1)} = \frac{t}{t+16} = 1-\frac{16}{t+16}, \\ M_f(1,2,s) &= \frac{s}{s+\rho(1,2)} = \frac{s}{s+9} = 1-\frac{9}{s+9}, \\ M_f(2,4,w) &= \frac{w}{w+\rho(2,4)} = \frac{w}{w+4} = 1-\frac{36}{w+36}. \end{aligned} \tag{21}$$

Now,

$$\begin{aligned} 1-\frac{49}{26(t+s+w)+49} &= 1-\frac{784}{416(t+s+w)+784}, \\ M_f(3,1,t) = 1-\frac{16}{t+16} &= 1-\frac{784}{49t+784}. \end{aligned} \tag{22}$$

Clearly,

$$1-\frac{784}{416t+416s+416w+784} > 1-\frac{441}{49t+784} \tag{23}$$

implies that

$$M_f(3,4,f(1,4)(t+s+w)) > M_f(3,1,t). \tag{24}$$

Similarly,

$$\begin{aligned} M_f(3,4,f(3,4)(t+s+w)) &> M_f(1,2,s), \\ M_f(3,4,f(3,4)(t+s+w)) &> M_f(2,4,w). \end{aligned} \tag{25}$$

Hence,

$$\begin{aligned} &M_f(3,4,f(3,4)(t+s+w)) \\ &\quad > M_f(3,1,t) * M_f(1,2,s) * M_f(2,4,w). \end{aligned} \tag{26}$$

Thus, in each case, we have

$$\begin{aligned} &M_f(\hbar_1,\hbar_2,f(\hbar_1,\hbar_2)(t+s+w)) \\ &\quad \geq M_f(\hbar_1,\hbar_3,t) * M_f(\hbar_3,\hbar_4,s) * M_f(\hbar_4,\hbar_2,w), \end{aligned} \tag{27}$$

for all distinct $\hbar_1, \hbar_2, \hbar_3, \hbar_4 \in \mathrm{F}$ and $t, s, w > 0$. Hence, $(\mathrm{F}, M_f, f, *)$ is an extended fuzzy b-rectangular metric-like space.

Definition 7. Let $(F, M_f, f, *)$ be an extended fuzzy b-rectangular metric-like space and $\{\hbar_n\}$ be a sequence in F, then $\{\hbar_n\}$ is

(1) a convergent sequence, if there exists $\hbar$ in F such that

$$\lim_{n\longrightarrow\infty} M_f(\hbar_n,\hbar,t) = M_f(\hbar,\hbar,t), \quad \text{for all } t>0. \tag{28}$$

(2) a Cauchy sequence, if for all $t > 0$ and for $p \geq 1$,

$$\lim_{n\longrightarrow\infty} M_f\left(\hbar_{n+p},\hbar_n,t\right) \text{ exists and is finite.} \tag{29}$$

An extended fuzzy b-rectangular metric-like space is said to be complete, if every Cauchy sequence converges to some $h \in \mathrm{F}$.

Definition 8. Let $(\mathrm{F}, M_f, f, *)$ be an extended fuzzy b-rectangular metric metric-like space. A mapping $T : \mathrm{F} \longrightarrow \mathrm{F}$ is said to be a fuzzy contractive mapping if

$$\frac{1}{M_f(T\hbar_1,T\hbar_2,t)} - 1 \leq k\left[\frac{1}{M_f(\hbar_1,\hbar_2,t)} - 1\right] \quad \text{for all } \hbar_1,\hbar_2 \in F,\ t>0, \text{ and } k\in(0,1). \tag{30}$$

Theorem 9. *Let $(\mathrm{F}, M_f, f, *)$ be a complete extended fuzzy b-rectangular metric-like space with $f : \mathrm{F}\times\mathrm{F} \longrightarrow [1, 1/k)$ and $T : \mathrm{F} \longrightarrow \mathrm{F}$ be a fuzzy contractive mapping such that*

$$\lim_{t\longrightarrow\infty} M_f(\hbar_1,\hbar_2,t) = 1. \tag{31}$$

Then, T has a unique fixed point.

Proof. Let $\hbar_0$ be an arbitrary point in F. If $T\hbar_0 = \hbar_0$, then $\hbar_0$ is the required fixed point. If $T\hbar_0 \neq \hbar_0$, then there exists $\hbar_1 \in \mathrm{F}$ such that $T\hbar_0 = \hbar_1$. If $T\hbar_1 = \hbar_1$, then $\hbar_1$ is the required fixed point. Continuing in this way, we have the sequence $\{\hbar_n\}$ in F such that $T\hbar_n = T^{n+1}\hbar_0 = \hbar_{n+1}$. Let $t > 0$ and using inequality (30), we have

$$\begin{aligned} \frac{1}{M_f(\hbar_n,\hbar_{n+1},t)} - 1 &= \left[\frac{1}{M_f(T\hbar_{n-1},T\hbar_n,t)} - 1\right] \\ &\leq k\left[\frac{1}{M_f(\hbar_{n-1},\hbar_n,t)} - 1\right], \end{aligned} \tag{32}$$

so we have

$$\frac{1}{M_f(\hbar_n, \hbar_{n+1}, t)} \leq \frac{k}{M_f(\hbar_{n-1}, \hbar_n, t)} + 1 - k. \tag{33}$$

Now,

$$\begin{aligned} \frac{k}{M_f(\hbar_{n-1}, \hbar_n, t)} + 1 - k &= \frac{k}{M_f(T\hbar_{n-2}, T\hbar_{n-1}, t)} + 1 - k \\ &\leq k\left[\frac{k}{M_f(\hbar_{n-2}, \hbar_{n-1}, t)} + 1 - k\right], \end{aligned} \tag{34}$$

so we have

$$\frac{k}{M_f(\hbar_{n-1}, \hbar_n, t)} \leq \frac{k^2}{M_f(\hbar_{n-2}, \hbar_{n-1}, t)} + k(1-k) + (1-k). \tag{35}$$

Continuing in this way, we have

$$\begin{aligned} \frac{1}{M_f(\hbar_n, \hbar_{n+1}, t)} &\leq \frac{k^n}{M_f(\hbar_0, \hbar_1, t)} + k^{n-1}(1-k) \\ &\quad + k^{n-2}(1-k) + \cdots + k(1-k) + (1-k) \\ &= \frac{k^n}{M_f(\hbar_0, \hbar_1, t)} + \left(k^{n-1} + k^{n-2} + \cdots + k + 1\right) \\ &\quad \cdot (1-k) = \frac{k^n}{M_f(\hbar_0, \hbar_1, t)} + (1 - k^n), \end{aligned} \tag{36}$$

we have

$$\frac{1}{\left(k^n / M_f(\hbar_0, \hbar_1, t)\right) + (1 - k^n)} \leq M_f(\hbar_n, \hbar_{n+1}, t). \tag{37}$$

Working in the same way, we also can prove that

$$\frac{1}{k^{n-2} / M_f(\hbar_0, \hbar_2, t) + \left(1 - k^{n-2}\right)} \leq M_f(\hbar_{n-2}, \hbar_n, t). \tag{38}$$

□

Let $\{\hbar_n\}$ be a sequence in F; then, we have the following cases.

Case 1. If p is odd (say) $p = 2m + 1$ for $m \geq 1$, then

$$\begin{aligned} &M_f(\hbar_n, \hbar_{n+2m+1}, t) \\ &\quad \geq M_f\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m+1})}\right) * M_f\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+2}, \hbar_{n+2m+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m+1})}\right) \\ &\quad \geq M_f\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m+1})}\right) * M_f\left(\hbar_{n+2}, \hbar_{n+3}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+3}, \hbar_{n+4}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+4}, \hbar_{n+2m+1}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\quad \geq M_f\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m+1})}\right) * M_f\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+2}, \hbar_{n+3}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+3}, \hbar_{n+4}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+5}, \hbar_{n+6}, \frac{t/3^3}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1}) f(\hbar_{n+4}, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+6}, \hbar_{n+7}, \frac{t/3^4}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1}) f(\hbar_{n+4}, \hbar_{n+2m+1}) f(\hbar_{n+6}, \hbar_{n+2m+1})}\right) \\ &\qquad * M_f\left(\hbar_{n+7}, \hbar_{n+8}, \frac{t/3^4}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1}) f(\hbar_{n+4}, \hbar_{n+2m+1}) f(\hbar_{n+6}, \hbar_{n+2m+1})}\right) \cdots \\ &\qquad * M_f\left(\hbar_{n+2m}, \hbar_{n+2m+1}, \frac{t/3^m}{f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1}) \cdots f(\hbar_{n+2m-2}, \hbar_{n+2m+1})}\right). \\ &\qquad \vdots \end{aligned} \tag{39}$$

Using inequality (37), we have

$$\begin{aligned} &M_f(\hbar_n, \hbar_{n+2m+1}, t) \\ &\quad \geq \frac{1}{k^n / M_f(\hbar_0, \hbar_1, t/3 f(\hbar_n, \hbar_{n+2m+1})) + (1 - k^n)} * \frac{1}{k^{n+1} / M_f(\hbar_0, \hbar_1, t/3 f(\hbar_n, \hbar_{n+2m+1})) + \left(1 - k^{n+1}\right)} \\ &\qquad * \frac{1}{k^{n+2} / M_f\left(\hbar_0, \hbar_1, t/3^2 f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1})\right) + \left(1 - k^{n+2}\right)} * \cdots \\ &\qquad * \frac{1}{k^{n+p-1} / M_f(\hbar_0, \hbar_1, t/3^m f(\hbar_n, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1}) \cdots f(\hbar_{n+2m-2}, \hbar_{n+2m+1})) + \left(1 - k^{n+p-1}\right)}. \end{aligned} \tag{40}$$

Taking limit $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} M_f(\hbar_n, \hbar_{n+2m+1}, t) \geq 1 * 1 * \cdots * 1 = 1. \tag{41}$$

Case 2. If $p = 2m$,that is, p is even, then

$$\begin{aligned}
&M_f(\hbar_n, \hbar_{n+2m}, t) \\
&\quad \geq M_f\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m})}\right) \\
&\qquad * M_f\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m})}\right) * M_f\left(\hbar_{n+2}, \hbar_{n+2m}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m})}\right), \\
&\geq M_f\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m})}\right) \\
&\qquad * M_f\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m})}\right) * M_f\left(\hbar_{n+2}, \hbar_{n+3}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m})}\right) \\
&\qquad * M_f\left(\hbar_{n+3}, \hbar_{n+4}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m})}\right) * M_f\left(\hbar_{n+4}, \hbar_{n+2m}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m})}\right) \\
&\geq M_f\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m})}\right) \\
&\qquad * M_f\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_n, \hbar_{n+2m})}\right) * M_f\left(\hbar_{n+2}, \hbar_{n+3}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m})}\right) \\
&\qquad * M_f\left(\hbar_{n+3}, \hbar_{n+4}, \frac{t/3^2}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m})}\right) * M_f\left(\hbar_{n+4}, \hbar_{n+5}, \frac{t/3^3}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m}) f(\hbar_{n+4}, \hbar_{n+2m})}\right) \\
&\qquad * M_f\left(\hbar_{n+5}, \hbar_{n+6}, \frac{t/3^3}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m}) f(\hbar_{n+4}, \hbar_{n+2m})}\right) \\
&\qquad * M_f\left(\hbar_{n+6}, \hbar_{n+7}, \frac{t/3^4}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m}) f(\hbar_{n+4}, \hbar_{n+2m}) f(\hbar_{n+6}, \hbar_{n+2m})}\right) \cdots \\
&\qquad * M_f\left(\hbar_{n+2m-2}, \hbar_{n+2m}, \frac{t/3^{m-1}}{f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m}) \cdots f(\hbar_{n+2m-4}, \hbar_{n+2m})}\right).
\end{aligned} \tag{42}$$

Using inequality (37) and (38), we have

$$\begin{aligned}
M_f(\hbar_n, \hbar_{n+2m}, t) \geq{}& \frac{1}{k^n / M_f(\hbar_0, \hbar_1, t/3 f(\hbar_n, \hbar_{n+2m})) + (1 - k^n)} \\
&* \frac{1}{k^{n+1} / M_f(\hbar_0, \hbar_1, t/3 f(\hbar_n, \hbar_{n+2m})) + \left(1 - k^{n+1}\right)} \\
&* \frac{1}{k^{n+2} / M_f\left(\hbar_0, \hbar_1, t/3^2 f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m})\right) + \left(1 - k^{n+2}\right)} \\
&* \cdots * \frac{1}{k^{n+p-1} / M_f\left(\hbar_0, \hbar_2, t/3^{m-1} f(\hbar_n, \hbar_{n+2m}) f(\hbar_{n+2}, \hbar_{n+2m}) \cdots f(\hbar_{n+2m-4}, \hbar_{n+2m})\right) + \left(1 - k^{n+p-1}\right)}.
\end{aligned} \tag{43}$$

Taking limit $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} M_f(\hbar_n, \hbar_{n+2m}, t) \geq 1 * 1 * \cdots * 1 = 1. \tag{44}$$

Thus, $\{\hbar_n\}$ is a Cauchy sequence and converges to $\hbar \in \mathrm{F}$ (since F is complete). Now, we have to prove $\hbar$ is the fixed point of T. Consider

$$\begin{aligned}
\frac{1}{M_f(T\hbar_n, T\hbar, t)} - 1 &\leq k\left[\frac{1}{M_f(\hbar_n, \hbar, t)} - 1\right] \\
&= \frac{k}{M_f(\hbar_n, \hbar, t)} - k, \\
\frac{1}{M_f(T\hbar_n, T\hbar, t)} &\leq k\left[\frac{1}{M_f(\hbar_n, \hbar, t)} - 1\right] + 1 \\
&= \frac{k}{M_f(\hbar_n, \hbar, t)} + 1 - k,
\end{aligned} \tag{45}$$

so,

$$\frac{1}{k/M_f(\hbar_n, \hbar, t) + (1-k)} \leq M_f(T\hbar_n, T\hbar, t). \tag{46}$$

Now,

$$\begin{aligned}
M_f(\hbar, T\hbar, t) \geq\ & M_f\left(\hbar, \hbar_n, \frac{t/3}{f(\hbar, T\hbar)}\right) \\
& * M_f\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar, T\hbar)}\right) \\
& * M_f\left(\hbar_{n+1}, T\hbar, \frac{t/3}{f(\hbar, T\hbar)}\right) \\
\geq\ & M_f\left(\hbar, \hbar_n, \frac{t/3}{f(\hbar, T\hbar)}\right) \\
& * M_f\left(T\hbar_{n-1}, T\hbar_n, \frac{t/3}{f(\hbar, T\hbar)}\right) \\
& * M_f\left(T\hbar_n, T\hbar, \frac{t/3}{f(\hbar, T\hbar)}\right) \\
\geq\ & M_f\left(\hbar, \hbar_n, \frac{t/3}{f(\hbar, T\hbar)}\right) \\
& * \frac{1}{k/M_f(\hbar_n, \hbar, t) + (1-k)} \\
& * M_f\left(T\hbar_n, T\hbar, \frac{t/3}{f(\hbar, T\hbar)}\right).
\end{aligned} \tag{47}$$

Applying limit $n \longrightarrow \infty$ on the right-hand side, we have $M_f(\hbar, T\hbar, t) = 1$, which shows $\hbar$ is the fixed point of T. For uniqueness, let $\hbar'$ be an other fixed point of T such that $T\hbar' = \hbar'$ and consider,

$$\begin{aligned}
\frac{1}{M_f\left(\hbar, \hbar', t\right)} - 1 &= \frac{1}{M_f\left(T\hbar, T\hbar', t\right)} - 1 \\
&\leq k\left[\frac{1}{M_f\left(\hbar, \hbar', t\right)} - 1\right] \\
&< \frac{1}{M_f\left(\hbar, \hbar', t\right)} - 1,
\end{aligned} \tag{48}$$

which is a contradiction; thus, $\hbar$ is the only fixed point of T.

3. Fuzzy Controlled Rectangular Metric-Like Space

In this section, we will give the definition of fuzzy controlled rectangular metric like space.

Definition 10. Let $f : \mathrm{F} \times \mathrm{F} \longrightarrow [1, \infty)$ be a function and M_c is a fuzzy set on $F \times \mathrm{F} \times (0, \infty)$. Then, M_c is called fuzzy controlled rectangular metric-like with $*$ as continuous t-norm; if for any $\hbar_1, \hbar_2 \in \mathrm{F}$ and all distinct $\hbar_3, \hbar_4 \in \mathrm{F} \setminus \{\hbar_1, \hbar_2\}$, the following conditions are satisfied:

$$\begin{aligned}
& M_c1\ M_c(\hbar_1, \hbar_2, t) > 0, \\
& M_c2 \text{ if } M_c(\hbar_1, \hbar_2, t) = 1 \text{ for all } t > 0, \text{ then } \hbar_1 = \hbar_2, \\
& M_c3\ M_c(\hbar_1, \hbar_2, t) = M_c(\hbar_2, \hbar_1, t), \\
& M_c4\ M_c(\hbar_1, \hbar_2, t + s + w) \\
& \quad \geq M_c\left(\hbar_1, \hbar_3, \frac{t}{f(\hbar_1, \hbar_3)}\right) * M_c\left(\hbar_3, \hbar_4, \frac{s}{f(\hbar_3, \hbar_4)}\right) \\
& \quad * M_c\left(\hbar_4, \hbar_2, \frac{w}{f(\hbar_4, \hbar_2)}\right), \text{ for all } t, s, w > 0, \\
& M_c5\ M_c(\hbar_1, \hbar_2, \cdot) : (0, \infty) \longrightarrow [0, 1] \text{ is continuous.}
\end{aligned} \tag{49}$$

Then, $(\mathrm{F}, M_c, *)$ is called a fuzzy controlled rectangular metric-like space.

Remark 11.

(i) If we take $f(\hbar_1, \hbar_3) = f(\hbar_3, \hbar_4) = f(\hbar_4, \hbar_2) = f(\hbar_1, \hbar_2)$, then a fuzzy controlled rectangular metric-like space reduces to extended fuzzy b-rectangular metric-like space

(ii) If we take $f(\hbar_1, \hbar_3) = f(\hbar_3, \hbar_4) = f(\hbar_4, \hbar_2) = b \geq 1$, then a fuzzy controlled rectangular metric-like space reduces to fuzzy b-rectangular metric-$*$like space

(iii) If we take $f(\hbar_1, \hbar_3) = f(\hbar_3, \hbar_4) = f(\hbar_4, \hbar_2) = f(\hbar_1, \hbar_2) = 1$, then a fuzzy controlled rectangular metric-like space reduces to fuzzy rectangular metric-like space

Following example justifies the definition 10.

Example 4. Let $F = \mathbb{N} \cup \{0\}$, $f : F \times F \longrightarrow [1,\infty)$ and $t_1 * t_2 = t_1 \cdot t_2$. Define $M_c : F \times F \times (0,\infty) \longrightarrow [0,1]$ as

$$M_c(\hbar_1, \hbar_2, t) = \exp\left(-\frac{\hbar_1 + \hbar_2}{t}\right). \tag{50}$$

Then, $(F, M_c, *)$ is a fuzzy controlled rectangular metric-like space.

We will prove (M_c4) as (M_c1)–(M_c3) and (M_c5) are easy to prove.

Now,

$$\begin{aligned}(\hbar_1 + \hbar_4) &\le (\hbar_1 + \hbar_2) + (\hbar_2 + \hbar_3) + (\hbar_3 + \hbar_4)\\ &\le f(\hbar_1, \hbar_2)\frac{t+s+w}{t}(\hbar_1 + \hbar_2)\\ &\quad + f(\hbar_2, \hbar_3)\frac{t+s+w}{s}(\hbar_2 + \hbar_3)\\ &\quad + f(\hbar_3, \hbar_4)\frac{t+s+w}{w}(\hbar_3 + \hbar_4),\end{aligned} \tag{51}$$

so we have

$$\begin{aligned}\frac{(\hbar_1 + \hbar_4)}{t+s+w} &\le \frac{(\hbar_1 + \hbar_2)}{t/f(\hbar_1, \hbar_2)} + \frac{(\hbar_2 + \hbar_3)}{s/f(\hbar_2, \hbar_3)}\\ &\quad + \frac{(\hbar_3 + \hbar_4)}{w/f(\hbar_3, \hbar_4)} - \frac{(\hbar_1 + \hbar_4)}{t+s+w}\\ &\ge -\frac{(\hbar_1 + \hbar_2)}{t/f(\hbar_1, \hbar_2)} + \frac{(\hbar_2 + \hbar_3)}{s/f(\hbar_2, \hbar_3)}\\ &\quad + \frac{(\hbar_3 + \hbar_4)}{w/f(\hbar_3, \hbar_4)} \exp\left(-\frac{(\hbar_1 + \hbar_4)}{t+s+w}\right)\\ &\ge \exp\left(-\frac{(\hbar_1 + \hbar_2)}{t/f(\hbar_1, \hbar_2)}\right) \cdot \exp\left(-\frac{(\hbar_2 + \hbar_3)}{s/f(\hbar_2, \hbar_3)}\right)\\ &\quad \cdot \exp\left(-\frac{(\hbar_3 + \hbar_4)}{w/f(\hbar_3, \hbar_4)}\right).\end{aligned} \tag{52}$$

Thus

$$\begin{aligned}&M_c(\hbar_1, \hbar_4, t+s+w)\\ &\quad \ge M_c\left(\hbar_1, \hbar_2, \frac{t}{f(\hbar_1, \hbar_2)}\right) * M_c\left(\hbar_2, \hbar_3, \frac{s}{f(\hbar_2, \hbar_3)}\right)\\ &\quad * M_c\left(\hbar_3, \hbar_4, \frac{w}{f(\hbar_3, \hbar_4)}\right).\end{aligned} \tag{53}$$

Hence, $(F, M_c, *)$ is a fuzzy controlled rectangular metric-like space.

Example 5. Consider $F = \{0, 1, 2, 3\}$ and let $f : F \times F \longrightarrow [1,\infty)$ be a function defined as $f(\hbar_1, \hbar_2) = 1 + \hbar_1 + \hbar_2$, $f(\hbar_2, \hbar_3) = \hbar_2^2 + \hbar_3 + 2$, and $f(\hbar_3, \hbar_4) = \hbar_3^2 + \hbar_4^2 + 3$. Define a rectangular metric-like space by $\rho(\hbar, \hbar') = \max\{\hbar, \hbar'\}$. Now define $M_c : F \times F \times (0,\infty) \longrightarrow [0,1]$ as follows:

$$M_c(\hbar_1, \hbar_2, t) = \frac{t}{t + \rho(\hbar_1, \hbar_2)}. \tag{54}$$

Then, $(F, M_c, *)$ is a fuzzy controlled rectangular metric-like space with product t-norm.

Definition 12. Let $(F, M_c, *)$ be a fuzzy controlled rectangular metric-like space and $\{\hbar_n\}$ be a sequence in F; then, the sequence $\{\hbar_n\}$ is called a convergent sequence if

$$\lim_{n\longrightarrow\infty} M_c(\hbar_n, \hbar, t) = M_c(\hbar, \hbar, t). \tag{55}$$

Definition 13. A sequence $\{\hbar_n\}$ in F is called a Cauchy sequence if for $t > 0, p \ge 1$

$$\lim_{n\longrightarrow\infty} M_c(\hbar_n, \hbar_{n+p}, t) \text{ exists and is finite.} \tag{56}$$

If every Cauchy sequence converges in F, then $(F, M_c, *)$ is known as a complete fuzzy controlled rectangular metric-like space.

Definition 14. Let $(F, M_c, *)$ be a fuzzy controlled rectangular metric-like space. Then, an open ball $B(\hbar, r, t)$, with center $\hbar$ and radius r, is given by

$$B(\hbar, r, t) = \left\{\hbar' \in F : M_c\left(\hbar, \hbar', t\right) > 1 - r\right\}, \tag{57}$$

and the corresponding topology is defined as

$$\tau_{M_c} = \{C \subset F : B(\hbar, r, t) \subset C\}. \tag{58}$$

We show that a fuzzy controlled rectangular metric-like space need not be Hausdorff in the following example.

Example 6. Consider the fuzzy controlled rectangular metric-like space as in Example 5. Now, define an open ball $B_{\hbar_1}(\hbar_1, r_1, t_1)$ with center $\hbar_1 = 1$, radius $r_1 = 0.12$, and $t_1 = 10$ as

$$B_{\hbar_1}(1, 0.12, 10) = \left\{\hbar' \in F : M_c\left(1, \hbar', 10\right) > 0.88\right\}, \tag{59}$$

Let $0 \in F$, then $M_c(1, 0, 10) = 10/(10 + \max(1, 0)) = 10/(10 + 1) = 0.9$, so $0 \in B_{\hbar_1}(\hbar_1, r_1, t)$.

Let $1 \in F$, then $M_c(1, 1, 10) = 10/(10 + \max(1, 1)) = 10/(10 + 1) = 0.9$, so $1 \in B_{\hbar_1}(\hbar_1, r_1, t)$.

Let $2 \in F$, then $M_c(1, 2, 10) = 10/(10 + \max(1, 2)) = 10/(10 + 2) = 0.8333$, so $2 \notin B_{\hbar_1}(\hbar_1, r_1, t)$.

Let $3 \in F$, then $M_c(1, 3, 10) = 10/(10 + \max(1, 3)) = 10/(10 + 3) = 0.7692$, so $3 \notin B_{\hbar_1}(\hbar_1, r_1, t)$.

Hence,

$$B_{\hbar_1}(\hbar_1, r_1, t_1) = \{0, 1\}. \tag{60}$$

Now, define an open ball $B_{\hbar_2}(\hbar_2, r_2, t_2)$ with center $\hbar_2 = 2$, radius $r_2 = 0.4$, and $t_2 = 4$ as

$$B_{\hbar_2}(2, 0.4, 4) = \left\{\hbar' \in \mathrm{F} : M_c\left(2, \hbar', 4\right) > 0.6\right\}. \tag{61}$$

Let $0 \in F$, then $M_c(2, 0, 4) = 4/(4 + \max(2, 0)) = 4/(4 + 2) = 0.6666$, so $0 \in B_{\hbar_2}(\hbar_2, r_2, t)$.

Let $1 \in F$, then $M_c(2, 1, 4) = 4/(4 + \max(2, 1)) = 4/(4 + 2) = 0.6666$, so $1 \in B_{\hbar_2}(\hbar_2, r_2, t)$.

Let $2 \in F$, then $M_c(2, 2, 4) = 4/(4 + \max(2, 2)) = 4/(4 + 2) = 0.6666$, so $2 \in B_{\hbar_2}(\hbar_2, r_2, t)$.

Let $3 \in F$, then $M_c(2, 3, 4) = 4/(4 + \max(2, 3)) = 4/(4 + 3) = 0.5714$, so $3 \notin B_{\hbar_2}(\hbar_2, r_2, t)$.

Hence,

$$B_{\hbar_2}(\hbar_2, r_2, t_2) = \{0, 1, 2\}. \tag{62}$$

Now, $B_{\hbar_1}(\hbar_1, r_1, t) \cap B_{\hbar_2}(\hbar_2, r_2, t) = \{0, 1\} \cap \{0, 1, 2\} = \{0, 1\} \neq \varnothing$. Thus, a fuzzy controlled rectangular metric-like space is not Hausdorff.

Remark 15. In light of Remark 11, extended fuzzy rectangular b-metric-like, fuzzy rectangular b-metric-like, and a fuzzy rectangular metric-like spaces are also not Hausdorff.

The following is the Banach fixed point theorem in the settings of a fuzzy controlled rectangular metric-like space.

Theorem 16. *Let $f : \mathrm{F} \times \mathrm{F} \longrightarrow [1, 1/k)$, $(k \in (0, 1))$ be a function, and $(\mathrm{F}, M_c, *)$ be a complete fuzzy controlled rectangular metric-like space such that*

$$\lim_{t \longrightarrow \infty} M_c\left(\hbar, \hbar', t\right) = 1. \tag{63}$$

Assume further that $T : \mathrm{F} \longrightarrow \mathrm{F}$ be self-mapping such that for all $\hbar, \hbar' \in \mathrm{F}$,

$$M_c\left(T\hbar, T\hbar', kt\right) \geq M_c\left(\hbar, \hbar', t\right). \tag{64}$$

Then, T has a unique fixed point.

Proof. Let $\hbar_0$ be an arbitrary point in F; then, we have the iterative sequence $T\hbar_n = T^{n+1}\hbar_0 = \hbar_{n+1}$. Now,

$$\begin{aligned} M_c(\hbar_n, \hbar_{n+1}, t) &= M_c(T\hbar_{n-1}, T\hbar_n, t) \geq M_c\left(\hbar_{n-1}, \hbar_n, \frac{t}{k}\right) \\ &= M_c\left(T\hbar_{n-2}, T\hbar_{n-1}, \frac{t}{k}\right) \geq M_c\left(\hbar_{n-2}, \hbar_{n-1}, \frac{t}{k^2}\right) \\ &\geq M_c\left(\hbar_{n-3}, \hbar_{n-2}, \frac{t}{k^3}\right) \cdots \geq M_c\left(\hbar_0, \hbar_1, \frac{t}{k^n}\right), \end{aligned} \tag{65}$$

hence,

$$M_c(\hbar_n, \hbar_{n+1}, t) \geq M_c\left(\hbar_0, \hbar_1, \frac{t}{k^n}\right). \tag{66}$$

In the same way, we can prove that

$$M_c(\hbar_{n-2}, \hbar_n, t) \geq M_c\left(\hbar_0, \hbar_2, \frac{t}{k^{n-2}}\right). \tag{67}$$

□

Now writing $t = t/3 + t/3 + t/3$ and for any sequence $\{\hbar_n\}$, we have the cases below.

Case 1. When p is odd, $p = 2m + 1$(say), then

$$\begin{aligned} & M_c(\hbar_n, \hbar_{n+2m+1}, t) \\ &\geq M_c\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+1})}\right) \\ &\quad * M_c\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_{n+1}, \hbar_{n+2})}\right) \\ &\quad * M_c\left(\hbar_{n+2}, \hbar_{n+2m+1}, \frac{t/3}{f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\geq M_c\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+1})}\right) \\ &\quad * M_c\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_{n+1}, \hbar_{n+2})}\right) \\ &\quad * M_c\left(\hbar_{n+2}, \hbar_{n+3}, \frac{t/3^2}{f(\hbar_{n+2}, \hbar_{n+3}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+3}, \hbar_{n+4}, \frac{t/3^2}{f(\hbar_{n+3}, \hbar_{n+4}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+4}, \hbar_{n+2m+1}, \frac{t/3^2}{f(\hbar_{n+4}, \hbar_{n+2m+1}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\geq M_c\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+1})}\right) * M_c\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_{n+1}, \hbar_{n+2})}\right) \\ &\quad * M_c\left(\hbar_{n+2}, \hbar_{n+3}, \frac{t/3^2}{f(\hbar_{n+2}, \hbar_{n+3}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+3}, \hbar_{n+4}, \frac{t/3^2}{f(\hbar_{n+3}, \hbar_{n+4}) f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+4}, \hbar_{n+5}, \frac{t/3^3}{f(\hbar_{n+4}, \hbar_{n+5}) f(\hbar_{n+2}, \hbar_{n+2m+1}) f(\hbar_{n+4}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+5}, \hbar_{n+6}, \frac{t/3^3}{f(\hbar_{n+5}, \hbar_{n+6}) f(\hbar_{n+2}, \hbar_{n+2m+1}) f(\hbar_{n+4}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+6}, \hbar_{n+7}, \frac{t/3^4}{f(\hbar_{n+6}, \hbar_{n+7}) f(\hbar_{n+2}, \hbar_{n+2m+1}) f(\hbar_{n+4}, \hbar_{n+2m+1}) f(\hbar_{n+6}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+7}, \hbar_{n+8}, \frac{t/3^4}{f(\hbar_{n+7}, \hbar_{n+8}) f(\hbar_{n+2}, \hbar_{n+2m+1}) f(\hbar_{n+4}, \hbar_{n+2m+1}) f(\hbar_{n+6}, \hbar_{n+2m+1})}\right) \cdots \\ &\quad * M_c\left(\hbar_{n+2m-2}, \hbar_{n+2m-1}, \frac{t/3^m}{f(\hbar_{n+2m-2}, \hbar_{n+2m-1}) f(\hbar_{n+2m-2}, \hbar_{n+2m+1}) \cdots f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+2m-1}, \hbar_{n+2m}, \frac{t/3^m}{f(\hbar_{n+2m-1}, \hbar_{n+2m}) f(\hbar_{n+2m-2}, \hbar_{n+2m+1}) \cdots f(\hbar_{n+2}, \hbar_{n+2m+1})}\right) \\ &\quad * M_c\left(\hbar_{n+2m}, \hbar_{n+2m+1}, \frac{t/3^m}{f(\hbar_{n+2m}, \hbar_{n+2m+1}) f(\hbar_{n+2m-2}, \hbar_{n+2m+1}) \cdots f(\hbar_{n+2}, \hbar_{n+2m+1})}\right). \end{aligned} \tag{68}$$

Applying (66), we deduce

$$\begin{aligned}
M_c(\hbar_n, \hbar_{n+2m+1}, t) \geq{}& M_c\left(\hbar_0, \hbar_1, \frac{t/3}{f(\hbar_n, \hbar_{n+1})k^n}\right) * M_c\left(\hbar_0, \hbar_1, \frac{t/3}{f(\hbar_{n+1}, \hbar_{n+2})k^{n+1}}\right)\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^2}{f(\hbar_{n+2}, \hbar_{n+3})f(\hbar_{n+2}, \hbar_{n+2m+1})k^{n+2}}\right)\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^2}{f(\hbar_{n+3}, \hbar_{n+4})f(\hbar_{n+2}, \hbar_{n+2m+1})k^{n+3}}\right)\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^3}{f(\hbar_{n+4}, \hbar_{n+5})f(\hbar_{n+2}, \hbar_{n+2m+1})f(\hbar_{n+4}, \hbar_{n+2m+1})k^{n+4}}\right)\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^3}{f(\hbar_{n+5}, \hbar_{n+6})f(\hbar_{n+2}, \hbar_{n+2m+1})f(\hbar_{n+4}, \hbar_{n+2m+1})k^{n+5}}\right)\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^4}{f(\hbar_{n+6}, \hbar_{n+7})f(\hbar_{n+2}, \hbar_{n+2m+1})f(\hbar_{n+4}, \hbar_{n+2m+1})f(\hbar_{n+6}, \hbar_{n+2m+1})k^{n+6}}\right)\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^4}{f(\hbar_{n+7}, \hbar_{n+8})f(\hbar_{n+2}, \hbar_{n+2m+1})f(\hbar_{n+4}, \hbar_{n+2m+1})f(\hbar_{n+6}, \hbar_{n+2m+1})k^{n+7}}\right) \cdots\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^m}{f(\hbar_{n+2m-2}, \hbar_{n+2m-1})f(\hbar_{n+2m-2}, \hbar_{n+2m+1}) \cdots f(\hbar_{n+2}, \hbar_{n+2m+1})k^{n+2m-2}}\right)\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^m}{f(\hbar_{n+2m-1}, \hbar_{n+2m})f(\hbar_{n+2m-2}, \hbar_{n+2m+1}) \cdots f(\hbar_{n+2}, \hbar_{n+2m+1})k^{2m-1}}\right)\\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^m}{f(\hbar_{n+2m}, \hbar_{n+2m+1})f(\hbar_{n+2m-2}, \hbar_{n+2m+1}) \cdots f(\hbar_{n+2}, \hbar_{n+2m+1})k^{2m}}\right).
\end{aligned} \tag{69}$$

Case 2. When p is even, $p = 2m$ (say), then

$$\begin{aligned}
M_c(\hbar_n, \hbar_{n+2m}, t) \geq{}& M_c\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+1})}\right) * M_c\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_{n+1}, \hbar_{n+2})}\right)\\
&* M_c\left(\hbar_{n+2}, \hbar_{n+2m}, \frac{t/3}{f(\hbar_{n+2}, \hbar_{n+2m})}\right) \geq M_c\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+1})}\right) * M_c\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_{n+1}, \hbar_{n+2})}\right)\\
&* M_c\left(\hbar_{n+2}, \hbar_{n+3}, \frac{t/3^2}{f(\hbar_{n+2}, \hbar_{n+3})f(\hbar_{n+2}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+3}, \hbar_{n+4}, \frac{t/3^2}{f(\hbar_{n+3}, \hbar_{n+4})f(\hbar_{n+2}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+4}, \hbar_{n+2m+1}, \frac{t/3^2}{f(\hbar_{n+4}, \hbar_{n+2m+1})f(\hbar_{n+2}, \hbar_{n+2m})}\right) \geq M_c\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+1})}\right) * M_c\left(\hbar_{n+1}, \hbar_{n+2}, \frac{t/3}{f(\hbar_{n+1}, \hbar_{n+2})}\right)\\
&* M_c\left(\hbar_{n+2}, \hbar_{n+3}, \frac{t/3^2}{f(\hbar_{n+2}, \hbar_{n+3})f(\hbar_{n+2}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+3}, \hbar_{n+4}, \frac{t/3^2}{f(\hbar_{n+3}, \hbar_{n+4})f(\hbar_{n+2}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+4}, \hbar_{n+5}, \frac{t/3^3}{f(\hbar_{n+4}, \hbar_{n+5})f(\hbar_{n+2}, \hbar_{n+2m})f(\hbar_{n+4}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+5}, \hbar_{n+6}, \frac{t/3^3}{f(\hbar_{n+5}, \hbar_{n+6})f(\hbar_{n+2}, \hbar_{n+2m})f(\hbar_{n+4}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+6}, \hbar_{n+7}, \frac{t/3^4}{f(\hbar_{n+6}, \hbar_{n+7})f(\hbar_{n+2}, \hbar_{n+2m})f(\hbar_{n+4}, \hbar_{n+2m})f(\hbar_{n+6}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+7}, \hbar_{n+8}, \frac{t/3^4}{f(\hbar_{n+7}, \hbar_{n+8})f(\hbar_{n+2}, \hbar_{n+2m})f(\hbar_{n+4}, \hbar_{n+2m})f(\hbar_{n+6}, \hbar_{n+2m})}\right) \cdots\\
&* M_c\left(\hbar_{n+2m-4}, \hbar_{n+2m-3}, \frac{t/3^{m-1}}{f(\hbar_{n+2m-4}, \hbar_{n+2m-3})f(\hbar_{n+2m-4}, \hbar_{n+2m}) \cdots f(\hbar_{n+2}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+2m-3}, \hbar_{n+2m-2}, \frac{t/3^{m-1}}{f(\hbar_{n+2m-3}, \hbar_{n+2m-2})f(\hbar_{n+2m-4}, \hbar_{n+2m}) \cdots f(\hbar_{n+2}, \hbar_{n+2m})}\right)\\
&* M_c\left(\hbar_{n+2m-2}, \hbar_{n+2m}, \frac{t/3^{m-1}}{f(\hbar_{n+2m-2}, \hbar_{n+2m})f(\hbar_{n+2m-4}, \hbar_{n+2m}) \cdots f(\hbar_{n+2}, \hbar_{n+2m})}\right).
\end{aligned} \tag{70}$$

Now applying (66) and (67) on the right-hand side, we deduce

$$
\begin{aligned}
M_c(\hbar_n, \hbar_{n+2m}, t) \geq{}& M_c\left(\hbar_0, \hbar_1, \frac{t/3}{f(\hbar_n, \hbar_{n+1})k^n}\right) * M_c\left(\hbar_0, \hbar_1, \frac{t/3}{f(\hbar_{n+1}, \hbar_{n+2})k^{n+1}}\right) \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^2}{f(\hbar_{n+2}, \hbar_{n+3})f(\hbar_{n+2}, \hbar_{n+2m})k^{n+2}}\right) \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^2}{f(\hbar_{n+3}, \hbar_{n+4})f(\hbar_{n+2}, \hbar_{n+2m})k^{n+3}}\right) \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^3}{f(\hbar_{n+4}, \hbar_{n+5})f(\hbar_{n+2}, \hbar_{n+2m})f(\hbar_{n+4}, \hbar_{n+2m})k^{n+4}}\right) \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^3}{f(\hbar_{n+5}, \hbar_{n+6})f(\hbar_{n+2}, \hbar_{n+2m})f(\hbar_{n+4}, \hbar_{n+2m})k^{n+5}}\right) \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^4}{f(\hbar_{n+6}, \hbar_{n+7})f(\hbar_{n+2}, \hbar_{n+2m})f(\hbar_{n+4}, \hbar_{n+2m})f(\hbar_{n+6}, \hbar_{n+2m})k^{n+6}}\right) \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^4}{f(\hbar_{n+7}, \hbar_{n+8})f(\hbar_{n+2}, \hbar_{n+2m})f(\hbar_{n+4}, \hbar_{n+2m})f(\hbar_{n+6}, \hbar_{n+2m})k^{n+7}}\right) \cdots \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^m}{f(\hbar_{n+2m-4}, \hbar_{n+2m-3})f(\hbar_{n+2m-4}, \hbar_{n+2m}) \cdots f(\hbar_{n+2}, \hbar_{n+2m})k^{n+2m-2}}\right) \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^m}{f(\hbar_{n+2m-3}, \hbar_{n+2m-2})f(\hbar_{n+2m-4}, \hbar_{n+2m}) \cdots f(\hbar_{n+2}, \hbar_{n+2m})k^{2m-1}}\right) \\
&* M_c\left(\hbar_0, \hbar_1, \frac{t/3^m}{f(\hbar_{n+2m-2}, \hbar_{n+2m})f(\hbar_{n+2m-4}, \hbar_{n+2m}) \cdots f(\hbar_{n+2}, \hbar_{n+2m})k^{2m-2}}\right).
\end{aligned} \tag{71}
$$

Using (63) for each case, we obtain

$$\lim_{n\longrightarrow\infty} M_c\left(\hbar_n, \hbar_{n+p}, t\right) = 1 * 1 * \cdots * 1 = 1, \tag{72}$$

which shows $\{\hbar_n\}$ is a Cauchy sequence in F, as F is complete, so there exists $\hbar \in F$ such that

$$\lim_{n\longrightarrow\infty} M_c(\hbar_n, \hbar, t) = 1. \tag{73}$$

Now, to prove $\hbar$ is a fixed point of T. From (64),

$$
\begin{aligned}
&M_c(\hbar, T\hbar, t) \\
&\quad\geq M_c\left(\hbar, \hbar_n, \frac{t/3}{f(\hbar, \hbar_n)}\right) * M_c\left(\hbar_n, \hbar_{n+1}, \frac{t/3}{f(\hbar_n, \hbar_{n+1})}\right) \\
&\qquad * M_c\left(\hbar_{n+1}, T\hbar, \frac{t/3}{f(\hbar_{n+1}, T\hbar)}\right) \\
&\quad\geq M_c\left(\hbar, \hbar_n, \frac{t/3}{f(\hbar, \hbar_n)}\right) * M_c\left(T\hbar_{n-1}, T\hbar_n, \frac{t/3}{f(\hbar_n, \hbar_{n+1})}\right) \\
&\qquad * M_c\left(T\hbar_n, T\hbar, \frac{t/3}{f(\hbar_{n+1}, T\hbar)}\right) \\
&\quad\geq M_c\left(\hbar, \hbar_n, \frac{t/3}{f(\hbar, \hbar_n)}\right) * M_c\left(\hbar_{n-1}, \hbar_n, \frac{t/3}{f(\hbar_n, \hbar_{n+1})k}\right) \\
&\qquad * M_c\left(\hbar_n, \hbar, \frac{t/3}{f(\hbar_{n+1}, T\hbar)k}\right) \longrightarrow 1 * 1 * 1 = 1,
\end{aligned} \tag{74}
$$

as $n \longrightarrow \infty$ which shows $\hbar$ is a fixed point of T. For uniqueness, we assume T has $\hbar'$ as an other fixed point, then

$$\begin{aligned} M_c\left(\hbar, \hbar', t\right) &= M_c\left(T\hbar, T\hbar', t\right) \geq M_c\left(\hbar, \hbar', \frac{t}{k}\right), \\ &= M_c\left(T\hbar, T\hbar', \frac{t}{k}\right) \geq M_c\left(\hbar, \hbar', \frac{t}{k^2}\right) \\ &\geq \cdots \geq M_c\left(\hbar, \hbar', \frac{t}{k^n}\right), \longrightarrow 1, \end{aligned} \tag{75}$$

as $n \longrightarrow \infty$. Hence $\hbar = \hbar'$, so T has a unique fixed point.

Example 7. Let $I = [0, 1]$ and $T : I \longrightarrow I$ a mapping given by

$$T\hbar = \frac{\sqrt{k}\hbar}{2}. \tag{76}$$

Define $M_c : I \times I \times (0,\infty) \longrightarrow [0, 1]$ by

$$M_c\left(\hbar, \hbar', t\right) = e^{-\left(\hbar+\hbar'\right)^2/2}, \quad \text{for all } t > 0. \tag{77}$$

Note that $(\mathrm{F}, M_c,*)$ is a complete fuzzy controlled rectangular metric-like space and

$$\begin{aligned} M_c(T\hbar, Ty, kt) &= M_c\left(\frac{\sqrt{k}\hbar}{2}, \frac{\sqrt{k}\hbar'}{2}, kt\right) \\ &= e^{-\left(\frac{\sqrt{k}\hbar}{2}+\frac{\sqrt{k}\hbar'}{2}\right)^2/kt} = e^{-k\left(\hbar+\hbar'\right)^2/4kt} \\ &= e^{-\left(\hbar+\hbar'\right)^2/4t} \geq e^{-\left(\hbar+\hbar'\right)^2/t} \quad \text{for all } k \in (0,1) \\ &= M_c\left(\hbar, \hbar', t\right). \end{aligned} \tag{78}$$

Thus, all the conditions of Theorem 16 are satisfied. Moreover, $\hbar = 0$ is a unique fixed point of T.

4. Application to Dynamic Market Equilibrium

Due to its vast applications in many fields, the fixed point theory has been used to prove the uniqueness of the solutions of many problems. This section is devoted to prove the unique solution of a differential equation appearing in dynamic market equilibrium.

Let Q_s and Q_d denote the supply and demand of a certain item, respectively, affected by price and trends. The economist wants to know the current price $P(t)$ of the item which is falling or rising at a decreasing or increasing rate. Let $dP(t)/dt$ and $d^2P(t)/dt^2$ denote, respectively, the first and second derivatives of the price $P(t)$, and assume

$$\begin{aligned} Q_s &= g_1 + u_1 P(t) + e_1 \frac{dP(t)}{dt} + c_1 \frac{d^2P(t)}{dt^2}, \\ Q_d &= g_2 + u_2 P(t) + e_2 \frac{dP(t)}{dt} + c_2 \frac{d^2P(t)}{dt^2}, \end{aligned} \tag{79}$$

where g_1, g_2, c_1, c_2, e_1, and e_2 are constants. In equilibrium, $Q_s = Q_d$; hence, we have

$$\begin{aligned} g_1 &+ u_1 P(t) + e_1 \frac{dP(t)}{dt} + c_1 \frac{d^2P(t)}{dt^2} \\ &= g_2 + u_2 P(t) + e_2 \frac{dP(t)}{dt} + c_2 \frac{d^2P(t)}{dt^2}, \end{aligned} \tag{80}$$

which implies

$$\begin{aligned} (c_1 - c_2)\frac{d^2P(t)}{dt^2} &+ (e_1 - e_2)\frac{dP(t)}{dt} + (u_1 - u_2)P(t) \\ &= -(g_1 - g_2). \end{aligned} \tag{81}$$

Putting $c = c_1 - c_2, e = e_1 - e_2, u = u_1 - u_2$, and $g = g_1 - g_2$ in (81), we have

$$c\frac{d^2P(t)}{dt^2} + e\frac{dP(t)}{dt} + uP(t) = -g. \tag{82}$$

On dividing by c, we have the following initial value problem:

$$P'' + \frac{e}{c}P' + \frac{u}{c}P(t) = -\frac{g}{c}, \tag{83}$$

with $P(0) = 0, P'(0) = 0$. Note that equation (83) is equivalent to the following integral equation:

$$P(t) = \int_0^T \rho(t, r)F(t, r, P(r)), \tag{84}$$

where $\rho(t, r)$ is Green's function defined by

$$\rho(t, r) = \begin{cases} re^{(\mu/2)t-r}, & \text{if } 0 \leq r \leq t \leq T, \\ te^{(\mu/2)t-r}, & \text{if } 0 \leq t \leq r \leq T. \end{cases} \tag{85}$$

We will prove the uniqueness of the solution of the following integral equation:

$$P(t) = \int_0^T G(t, r, P(r))dr. \tag{86}$$

Denote $F = C([0, T])$, the space of all real-valued continuous functions defined over the interval $[0, T]$. Now, define a complete fuzzy controlled metric-like space with product t-norm as

$$M_c\left(\hbar(t), \hbar'(t), s\right) = e^{-\left(\sup_{t\in[0,T]} \left|\hbar(t)+\hbar'(t)\right|^2/s\right)}, \tag{87}$$

with controlled functions $f(\hbar_1, \hbar_2) = (1/3)(\hbar^2 + \hbar_2 + 3), f(\hbar_2, \hbar_3) = (1/3)(\hbar_2^2 + \hbar_3^2 + 4)$, and $f(\hbar_3, \hbar_4) = (1/3)(\hbar_3 + \hbar_4 + 5)$.

Theorem 17. *Consider the operator*

$$\Gamma P(t) = \int_0^T G(t, r, P(r))dr. \tag{88}$$

Assume the following conditions hold:

$$\begin{aligned} &G : [0, T] \times [0, T] \longrightarrow \mathbb{R}^+ \text{ is continuous}, \\ &|G(t, r, P(r)) + G(t, r, Q(r))|^2 \le f^2(t, r)|P(r) + Q(r)|^2, \end{aligned} \tag{89}$$

where

$$\sup_{t\in[0,T]} \int_0^T f^2(t, r)dr \le k < 1. \tag{90}$$

Then, integral equation (86) has a unique solution.

Proof. Let $P, Q \in \mathrm{F}$ and $s > 0$ and consider

$$\begin{aligned} &M_c(\Gamma P(t), \Gamma Q(t), ks) \\ &= e^{-\left(\sup_{t\in[0,T]} |\Gamma P(t)+\Gamma Q(t)|^2/ks\right)} \\ &= e^{-\left(\sup_{t\in[0,T]} \left|\int_0^T G(t,r,P(r))dr+\int_0^T G(t,r,Q(r))dr\right|^2/ks\right)} \\ &\ge e^{-\left(\sup_{t\in[0,T]} \int_0^T |G(t,r,P(r))dr+G(t,r,Q(r))|^2 dr/ks\right)} \\ &\ge e^{-\left(|P(r)+Q(r)|^2 \sup_{t\in[0,T]} \int_0^T f^2(t,r)dr/ks\right)} \\ &\ge e^{-\left(k|P(r)+Q(r)|^2/ks\right)} = e^{-\left(|P(r)+Q(r)|^2/s\right)} \\ &\ge e^{-\left(\sup_{r\in[0,T]} |P(r)+Q(r)|^2/s\right)} = M_c(P, Q, s). \end{aligned} \tag{91}$$

By the application of Theorem 16, problem (86) has a unique solution. □

5. Conclusion

In this article, the concepts of extended b-rectangular and fuzzy controlled rectangular metric-like spaces are given that extend numerous fuzzy metric-like spaces. We proved that these classes of fuzzy metric-like spaces are not Hausdorff, and we have given examples to support our main results and definitions; also, we proved the Banach fixed point theorem in these spaces by using different types of contractions. We apply our results to prove the uniqueness of the solution of an integral equation appearing in dynamic market equilibrium.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

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