TWMS J. Pure Appl. Math. V.11, N.2, 2020, pp.204-212

ON INTERPOLATIVE BOYD-WONG AND MATKOWSKI TYPE CONTRACTIONS

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Abstract. By using an interpolation approach, we recognize Boyd-Wong and Matkowski type contractions and we prove the related fixed point theorems in the class of metric spaces. The obtained results are supported by some examples. We also give the partial metric case according to our results.

Keywords: Boyd-Wong contraction, Matkowski contraction, interpolative contraction, fixed point.

AMS Subject Classification: 46T99, 47H10, 54H25.

1. Introduction and preliminaries

Fixed point theory is one of the common cornerstones of three fields: nonlinear functional analysis, topology and applied mathematics, see e.g. [9, 13, 16, 19, 20, 21, 26, 27]. The interpolative Kannan contraction has been initiated in [17], where a unique fixed point result has been proved. This idea was extended and generalized in [1]-[15], [20]. Very recently, the authors in [18] gave a counter-example showing that the fixed point may be not unique. One of generalizations of the Banach Contraction Principle [9] is due to Hardy-Rogers [16] (see [13]).

Theorem 1.1. *Let (X,d) be a complete metric space. Let $T : X \to X$ be a given mapping such that*

$$d(Tx,Ty) \le \alpha d(x,y) + \beta d(x,Tx) + \gamma d(y,Ty) + \delta[\frac{1}{2}(d(x,Ty) + d(y,Tx))],$$

for all $x,y \in X$, where $\alpha,\beta,\gamma,\delta$ are non-negative reals such that $\alpha+\beta+\gamma+\delta < 1$. Then T has a unique fixed point in X.

The following known lemma is useful in the sequel.

Lemma 1.1. [11] *Let (X,d) be a complete metric space and $\{s_n\}$ be a sequence in X such that $\lim_{n\to+\infty} d(s_n,s_{n+1}) = 0$. If $\{s_n\}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$ and subsequences $\{m(k)\}$ and $\{n(k)\}$ in $\mathbb{N}$ with $n(k) > m(k) > k$ such that $d(s_{n(k)},s_{m(k)}) \ge \varepsilon$ and $d(s_{n(k)-1},s_{m(k)}) < \varepsilon$, so that the following holds:*

(i) $\lim_{k\to+\infty} d(s_{n(k)-1},s_{m(k)-1}) = \varepsilon$;
(ii) $\lim_{k\to+\infty} d(s_{n(k)},s_{m(k)}) = \varepsilon$;

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Manuscript received June 2020.

(iii) $\lim_{k\to+\infty} d(s_{n(k)}, s_{m(k)-1}) = \varepsilon$;
(iv) $\lim_{k\to+\infty} d(s_{m(k)}, s_{n(k)-1}) = \varepsilon$.

In this paper, we initiate the concept of *interpolative Boyd-Wong and Matkovski type contractions* (see [14]). We also provide some examples illustrating the obtained results. We also extend our results to partial metric spaces.

2. Main results

2.1. **Interpolative Boyd–Wong type contractions.** We start this section by introducing the notion of *interpolative Boyd-Wong type contractions.* First, let Ψ be the set of functions $\psi\colon [0,+\infty) \to [0,+\infty)$ such that [10]
(ψ_1) $\psi(0) = 0$;
(ψ_2) $\psi(t) < t$ for each $t > 0$;
(ψ_3) ψ is upper semi-continuous from the right.

Definition 2.1. *Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an* interpolative Boyd-Wong type contraction, *if there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and a nondecreasing function $\psi \in \Psi$ such that*

$$d(Tx,Ty) \le \psi\Big([d(x,y)]^{\beta} \cdot [d(x,Tx)]^{\alpha} \cdot [d(y,Ty)]^{\gamma} \cdot \left[\frac{1}{2}(d(x,Ty) + d(y,Tx))\right]^{1-\alpha-\beta-\gamma} \Big), \quad (1)$$

for all $x, y \in X\backslash Fix(T)$.

Theorem 2.1. *Let (X, d) be a complete metric space and T be an interpolative Boyd-Wong type contraction. Then T has a fixed point in X.*

Proof. Starting from $x_0 \in X$, consider $\{x_n\}$ given as $x_n = T^n(x_0)$ for each positive integer n. If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. The proof is completed. From now, assume that $x_n \ne x_{n+1}$ for all $n \ge 0$. By substituting the values $x = x_n$ and $y = x_{n-1}$ in (12), we find that

$$\begin{aligned} d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) &\le \psi([d(x_n,x_{n-1})]^{\beta}\, [d(x_n,Tx_n)]^{\alpha} \cdot [d(x_{n-1},Tx_{n-1})]^{\gamma}) \\ &\quad \cdot \left[\tfrac{1}{2}(d(x_n,x_n) + d(x_{n-1},x_{n+1}))\right]^{1-\alpha-\beta-\gamma}) \\ &\le \psi([d(x_n,x_{n-1})]^{\beta} \cdot [d(x_n,x_{n+1})]^{\alpha} \cdot [d(x_{n-1},x_n)]^{\gamma} \\ &\quad \cdot \left[\tfrac{1}{2}(d(x_{n-1},x_n) + d(x_n,x_{n+1}))\right]^{1-\alpha-\beta-\gamma}). \end{aligned} \quad (2)$$

Suppose that $d(x_{n-1},x_n) < d(x_n,x_{n+1})$ for some $n \ge 1$. Thus,

$$\frac{1}{2}(d(x_{n-1},x_n) + d(x_n,x_{n+1})) \le d(x_n,x_{n+1}).$$

Consequently, the inequality (2) yields that

$$0 < d(x_n,x_{n+1}) \le \psi([d(x_{n-1},x_n)]^{\beta+\gamma} \cdot [d(x_n,x_{n+1})]^{1-\beta-\gamma}) \le [d(x_{n-1},x_n)]^{\beta+\gamma} \cdot [d(x_n,x_{n+1})]^{1-\beta-\gamma}, \quad (3)$$

so, $[d(x_n,x_{n+1})]^{\beta+\gamma} \le [d(x_{n-1},x_n)]^{\beta+\gamma}$, which is a contradiction. Thus, we have $d(x_n,x_{n+1}) \le d(x_{n-1},x_n)$ for all $n \ge 1$. Hence, $\{d(x_{n-1},x_n)\}$ is a non-increasing sequence with positive terms. Set $\ell =: \lim\limits_{n\to+\infty} d(x_{n-1},x_n)$. We have

$$\frac{1}{2}(d(x_{n-1},x_n) + d(x_n,x_{n+1})) \le d(x_{n-1},x_n), \quad \text{for all } n \ge 1.$$

By a simple elimination, the inequality (2) implies that

$$d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)). \tag{4}$$

Since ψ is upper semi-continuous from the right, we have

$$\ell = \lim_{n\to+\infty} d(x_n, x_{n+1}) \leq \limsup_{n\to+\infty} \psi(d(x_{n-1}, x_n)) \leq \psi(\ell) < \ell,$$

which is a contradiction, so we get that

$$\ell = \lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \tag{5}$$

In what follows, we shall prove that $\{x_n\}$ is a Cauchy sequence.

We argue by contradiction, that is, $\{x_n\}$ is not a Cauchy sequence. This means that there exists $\varepsilon > 0$ for which we can find subsequences of integers (m_k) and (m_k) with $n_k > m_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{6}$$

Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (6). Then

$$d(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{7}$$

By substituting the values $x = x_{n_k-1}$ and $y = x_{m_k-1}$ in (12), we find that

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &= d(Tx_{n_k-1}, Tx_{m_k-1}) \\ &\leq \psi([d(x_{n_k-1}, x_{m_k-1})]^\beta [d(x_{n_k-1}, Tx_{n_k-1})]^\alpha \cdot [d(x_{m_k-1}, Tx_{m_k-1})]^\gamma) \\ &\cdot \left[\tfrac{1}{2}(d(x_{n_k-1}, Tx_{m_k-1}) + d(x_{m_k-1}, Tx_{n_k-1}))\right]^{1-\alpha-\beta-\gamma}) \\ &= \psi([d(x_{n_k-1}, x_{m_k-1})]^\beta [d(x_{n_k-1}, x_{n_k})]^\alpha \cdot [d(x_{m_k-1}, x_{m_k})]^\gamma) \\ &\cdot \left[\tfrac{1}{2}(d(x_{n_k-1}, x_{m_k}) + d(x_{m_k-1}, x_{n_k}))\right]^{1-\alpha-\beta-\gamma}). \end{aligned} \tag{8}$$

Using the upper semi-continuity of ψ, (5) and Lemma 1.1, we obtain that

$$\varepsilon \leq \psi(0) = 0,$$

which is a contradiction. Thus, $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), so there exists $u \in X$ such that $\lim_{n\to+\infty} d(x_n, u) = 0$. Suppose that $x \neq Tx$. Since $x_n \neq Tx_n$ for each $n \geq 0$, by letting $x = x_n$ and $y = u$ in (12), we have

$$\begin{aligned} d(x_{n+1}, Tu) = d(Tx_n, Tu) &\leq \psi([d(x_n, u)]^\beta \cdot [d(x_n, Tx_n)]^\alpha \cdot [d(u, Tu)]^\gamma \\ &\cdot \left[\frac{1}{2}(d(x_n, Tu) + d(u, x_{n+1}))\right]^{1-\alpha-\beta-\gamma}). \end{aligned} \tag{9}$$

Letting $n \to +\infty$ in the inequality (9) and using the upper semi-continuity of ψ, we find out

$$d(u, Tu) \leq \psi(0) = 0,$$

which is a contradiction. Thus, $Tu = u$.

□

Proceeding as the proof of Theorem 2.1, we state the following result.

Corollary 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose there exist $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$ and a nondecreasing function $\psi \in \Psi$ such that*

$$d(Tx, Ty) \leq \phi\Big([d(x, y)]^\beta \cdot [d(x, Tx)]^\alpha \cdot [d(y, Ty)]^{1-\alpha-\beta}\Big), \tag{10}$$

for all $x, y \in X \backslash Fix(T)$. Then T has a fixed point in X.

Taking $\psi(t) = \lambda t$ (where $\lambda \in [0, 1)$) in Theorem 2.1, we have

Corollary 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and $\lambda \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda [d(x, y)]^{\beta} \cdot [d(x, Tx)]^{\alpha} \cdot [d(y, Ty)]^{\gamma} \cdot \left[\frac{1}{2}(d(x, Ty) + d(y, Tx))\right]^{1-\alpha-\beta-\gamma}, \quad (11)$$

for all $x, y \in X \backslash Fix(T)$. Then T has a fixed point in X.

2.2. **Interpolative Matkowski type contractions.** Let Φ be the set of functions $\phi \colon [0, +\infty) \to [0, +\infty)$ such that
(ϕ_1) ϕ is nondecreasing,
(ϕ_2) $\lim_{n\to+\infty} \phi^n(t) = 0$ for each $t > 0$.

In order to state our next theorem we shall need the following well-known and easy, but useful, observation (by Matkowski [22, 23]).

Lemma 2.1. [6, 19] *Let $\phi \in \Phi$. Then $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.*

Definition 2.1. *Let (X, d) be a metric space. We say that the self-mapping $T : X \to X$ is an* interpolative Matkowski type contraction*, if there exist $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$ and $\phi \in \Phi$ such that*

$$d(Tx, Ty) \leq \phi\Big([d(x, y)]^{\beta} \cdot [d(x, Tx)]^{\alpha} \cdot [d(y, Ty)]^{\gamma} \cdot \left[\frac{1}{2}(d(x, Ty) + d(y, Tx))\right]^{1-\alpha-\beta-\gamma} \Big), \quad (12)$$

for all $x, y \in X \backslash Fix(T)$.

On what follows we state and prove the main result of this section.

Theorem 2.2. *Let (X, d) be a complete metric space and T be an interpolative Matkowski type contraction. Then T has a fixed point in X.*

Proof. Following the related lines in the proof of Theorem 2.1, without loss of generality, we construct a sequence $\{x_n = T^n x_0\}$ such that $x_n \neq x_{n+1}$. From (4), we have

$$0 < d(x_n, x_{n+1}) \leq \phi(d(x_{n-1}, x_n)) \leq \phi^n(d(x_0, x_1)). \quad (13)$$

Since $\phi \in \Phi$, we have $\lim\limits_{n\to+\infty} \phi^n(d(x_0, x_1)) = 0$. Hence, we find that

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = 0. \quad (14)$$

Now, we shall show that $\{x_n\}$ is Cauchy. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$,

$$\phi^n(d(x_0, x_1)) < \varepsilon - \phi(\varepsilon).$$

By (13), this implies that

$$0 < d(x_n, x_{n+1}) < \varepsilon - \phi(\varepsilon). \quad (15)$$

We claim that

$$0 < d(x_n, x_m) < \varepsilon, \quad \text{for all } m \geq n \geq n_0. \quad (16)$$

We prove (16) by induction. Since $\varepsilon - \phi(\varepsilon) < \varepsilon$, by using (15), we conclude that (16) holds when $m = n+1$. Now suppose that (16) holds for $m = k$. For $m = k+1$, we have

$$\begin{aligned} d(x_n, x_{k+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{k+1}) \\ &\leq \varepsilon - \phi(\varepsilon) + d(Tx_n, Tx_k) \\ &\leq \varepsilon - \phi(\varepsilon) + \phi([d(x_n, x_k)]^{\beta} \cdot [d(x_n, Tx_n)]^{\alpha} \cdot [d(x_k, Tx_k)]^{\gamma} \\ &\cdot \left[\frac{1}{2}(d(x_n, Tx_k) + d(x_k, Tx_n))\right]^{1-\alpha-\beta-\gamma}) \\ &= \varepsilon - \phi(\varepsilon) + \phi([d(x_n, x_k)]^{\beta} \cdot [d(x_n, x_{n+1})]^{\alpha} \cdot [d(x_k, x_{k+1})]^{\gamma} \\ &\cdot \left[\frac{1}{2}(d(x_n, x_{k+1}) + d(x_k, x_{n+1}))\right]^{1-\alpha-\beta-\gamma}). \end{aligned}$$

Since $k \geq n \geq n_0$, we have

$$[d(x_n, x_k)]^{\beta} \cdot [d(x_n, x_{n+1})]^{\alpha} \cdot [d(x_k, x_{k+1})]^{\gamma} \cdot \left[\frac{1}{2}(d(x_n, x_{k+1}) + d(x_k, x_{n+1}))\right]^{1-\alpha-\beta-\gamma} < \varepsilon.$$

Thus, we find

$$d(x_n, x_{k+1}) \leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon,$$

so (16) holds for $m = k+1$. By (16), the sequence $\{x_n\}$ is Cauchy, so there exists $u \in X$ such that $\lim_{n\to+\infty} d(x_n, u) = 0$. Suppose that $u \neq Tu$. Since $x_n \neq Tx_n$ for each $n \geq 0$, by (12), we have

$$\begin{aligned} d(x_{n+1}, Tu) = d(Tx_n, Tu) &\leq \phi([d(x_n, u)]^{\beta} \cdot [d(x_n, Tx_n)]^{\alpha} \cdot [d(u, Tu)]^{\gamma} \\ &\cdot \left[\frac{1}{2}(d(x_n, Tu) + d(u, x_{n+1}))\right]^{1-\alpha-\beta-\gamma}). \end{aligned} \tag{17}$$

It is obvious that there exists $N \in \mathbb{N}$ such that for each $n \geq N$,

$$[d(x_n, u)]^{\beta} \cdot [d(x_n, Tx_n)]^{\alpha} \cdot [d(u, Tu)]^{\gamma} \cdot \left[\frac{1}{2}(d(x_n, Tu) + d(u, x_{n+1}))\right]^{1-\alpha-\beta-\gamma} < d(u, Tu).$$

Since ϕ is nondecreasing, by insertion of this last inequality in (17), we get that

$$d(x_{n+1}, Tu) \leq \phi(d(u, Tu)), \quad \text{for all } n \geq N.$$

Letting $n \to +\infty$, we obtain that

$$0 < d(u, Tu) \leq \phi(d(u, Tu)),$$

which is a contradiction. Thus, $Tu = u$.

□

Proceeding as the proof of Theorem 2.2, we state the following result.

Corollary 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. Suppose there exist $\alpha, \beta \in (0,1)$ with $\alpha + \beta < 1$ and $\phi \in \Phi$ such that*

$$d(Tx, Ty) \leq \phi\Big([d(x, y)]^{\beta} \cdot [d(x, Tx)]^{\alpha} \cdot [d(y, Ty)]^{1-\alpha-\beta}\Big), \tag{18}$$

for all $x, y \in X \backslash Fix(T)$. Then T has a fixed point in X.

Following [24] (see also [2, 3, 4, 5, 12, 25, 26] for more details on partial metric spaces), the partial metric case of Theorem 2.1 is

Theorem 2.3. *Let (X,p) be a complete partial metric space and $T: X \to X$ be a self-mapping. Suppose there exist $\alpha, \beta, \gamma \in (0,1)$ with $\alpha+\beta+\gamma<1$ and a nondecreasing function $\psi \in \Psi$ such that*

$$p\left(Tx,Ty\right) \leq \psi\Big(\left[p\left(x,y\right)\right]^{\beta} \cdot \left[p\left(x,Tx\right)\right]^{\alpha} \cdot \left[p\left(y,Ty\right)\right]^{\gamma} \cdot \left[\frac{1}{2}(p\left(x,Ty\right)+p\left(y,Tx\right))\right]^{1-\alpha-\beta-\gamma} \Big), \quad (19)$$

for all $x,y \in X \backslash Fix(T)$. Then T has a fixed point in X.

Again, the partial metric case of Theorem 2.2 is

Theorem 2.4. *Let (X,p) be a complete partial metric space and $T: X \to X$ be a self-mapping. Suppose there exist $\alpha, \beta, \gamma \in (0,1)$ with $\alpha+\beta+\gamma<1$ and $\phi \in \Phi$ such that*

$$p\left(Tx,Ty\right) \leq \phi\Big(\left[p\left(x,y\right)\right]^{\beta} \cdot \left[p\left(x,Tx\right)\right]^{\alpha} \cdot \left[p\left(y,Ty\right)\right]^{\gamma} \cdot \left[\frac{1}{2}(p\left(x,Ty\right)+p\left(y,Tx\right))\right]^{1-\alpha-\beta-\gamma} \Big), \quad (20)$$

for all $x,y \in X \backslash Fix(T)$. Then T has a fixed point in X.

Another consequence of Theorem 2.3 (or Theorem 2.4) is

Corollary 2.4. *Let (X,p) be a complete partial metric space and $T: X \to X$ be a self-mapping. Suppose there exist $\alpha, \beta, \gamma \in (0,1)$ with $\alpha+\beta+\gamma<1$ and $\lambda \in [0,1)$ such that*

$$p\left(Tx,Ty\right) \leq \lambda \left[p\left(x,y\right)\right]^{\beta} \cdot \left[p\left(x,Tx\right)\right]^{\alpha} \cdot \left[p\left(y,Ty\right)\right]^{\gamma} \cdot \left[\frac{1}{2}(p\left(x,Ty\right)+p\left(y,Tx\right))\right]^{1-\alpha-\beta-\gamma}, \quad (21)$$

for all $x,y \in X \backslash Fix(T)$. Then T has a fixed point in X.

In the following examples, the fixed point exists, but the Hardy-Rogers result [16] is not applicable.

Example 2.1. *Let $X=\{x,y,z,w\}$ be a set endowed with a metric d such that*

$$\begin{aligned} d(x,x) &= d(y,y) = d(z,z) = d(w,w) = 0, \\ d(y,x) &= d(x,y) = 3, \\ d(z,x) &= d(x,z) = 4, \\ d(y,z) &= d(z,y) = \tfrac{3}{2} \\ d(w,x) &= d(x,w) = \tfrac{5}{2} \\ d(w,y) &= d(y,w) = 2 \\ d(w,z) &= d(z,w) = \tfrac{3}{2}. \end{aligned}$$

We define a self-mapping T on X by $T: \begin{pmatrix} x & y & z & w \\ x & w & x & y \end{pmatrix}$. It is clear that T is not a Hardy-Rogers contraction. Indeed, there is no $\lambda \in [0, \frac{1}{4})$ where $\lambda = \min\{\alpha,\beta,\gamma\}$ such that the following inequality is fulfilled:

$$d(Tw,Tz) = d(y,x) = 3 \leq \lambda (d(w,z) + d(Tw,w) + d(z,Tz) + \frac{d(Tz,w)+d(Tw,z)}{2}) = \frac{41}{4}\lambda.$$

On the other hand, for $\alpha = \frac{1}{32} = \gamma$, $\beta = \frac{3}{4}$ and $\psi(t) = \frac{9t}{10}$, the self-mapping T forms an interpolative Boyd-Wong (resp. Matkowski) type contraction and x is the desired fixed point of T.

Example 2.2. *Let $X = \{1, 2, 11, 21\}$ be a set endowed with the classical metric $d(x, y) = |x - y|$, that is,*

d(x, y)	*1*	*2*	*11*	*21*
1	*0*	*1*	*10*	*20*
2	*1*	*0*	*9*	*19*
11	*10*	*9*	*0*	*10*
21	*20*	*19*	*10*	*0*

We define a self-mapping T on X by $T : \begin{pmatrix} 1 & 2 & 11 & 21 \\ 1 & 2 & 1 & 2 \end{pmatrix}$. It is clear that T is not a Hardy-Rogers contraction. Indeed, there is no $\lambda \in [0, \frac{1}{4})$ such that the following inequality is fulfilled:

$$d(T1, T2) = d(1, 2) = 1 \leq \lambda \left[d(1, 2) + d(T1, 1) + d(2, T2) + \frac{d(T1, 2) + d(T2, 1)}{2} \right] = 2\lambda.$$

Choose $\alpha = \beta = \gamma = \frac{1}{4}$. Consider $\phi : [0, \infty) \to [0, \infty)$ defined by

$$\phi(t) = \begin{cases} \frac{7}{10}t, & 0 \leq t \leq 1 \\ \frac{t}{4}, & t > 1. \end{cases}$$

Mention that $\phi \in \Phi$. Let $x, y \in X \backslash Fix(T)$, then

$$(x, y) \in \{(11, 11), (21, 21), (11, 21), (21, 11)\}.$$

Without loss of generality, let $(x, y) = (11, 21)$. In this case, we have

$$d(Tx, Ty) = 1 \leq \phi(26600^{\frac{1}{4}}) =$$

$$\phi\Big([d(x, y)]^{\beta} \cdot [d(x, Tx)]^{\alpha} \cdot [d(y, Ty)]^{\gamma} \cdot \left[\frac{1}{2}(d(x, Ty) + d(y, Tx)) \right]^{1-\alpha-\beta-\gamma} \Big).$$

That is, T is an interpolative Matkowski type contraction and $\{1, 2\}$ is the set of fixed points of T.

3. Conclusion

In this paper, we revisit two famous contractions, Boyd-Wong contraction and Matkowski contraction, by using the idea of the interpolative contraction in the setting of complete metric space. These proposed contraction can bring new framework to the metric fixed point theory. Further, well-known applications of fixed point theory can be revisited and extended.

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