Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 7(5): 7421–7440.
DOI: 10.3934/math.2022415
Received: 29 November 2021
Revised: 04 January 2022
Accepted: 09 January 2022
Published: 14 February 2022

Research article

On soliton solutions of fractional-order nonlinear model appears in physical sciences

Naeem Ullah[1], **Muhammad Imran Asjad**[1,*], **Jan Awrejcewicz**[2], **Taseer Muhammad**[3] **and Dumitru Baleanu**[4,5,6]

[1] Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan

[2] Department of Automation, Biomechanics, and Mechatronics, Faculty of Mechanical Engineering, Lodz University of Technology, Lodz 90924, Poland

[3] Department of Mathematics, College of Sciences, King Khalid University, Abha 61413, Saudi Arabia

[4] Department of Mathematics, Cankaya University, Balgat, Ankara, Turkey

[5] Institute of Space Sciences, Magurele, Bucharest, Romania

[6] Department of Medical Research, China Medical University Hospita, China Medical University, Taichung, Taiwan

* **Correspondence:** Email: imran.asjad@umt.edu.pk; Tel: +924235212801.

Abstract: In wave theory, the higher dimensional non-linear models are very important to define the physical phenomena of waves. Herein study we have built the various solitons solutions of (4+1)-dimensional fractional-order Fokas equation by using two analytical techniques that is, the Sardar-subequation method and new extended hyperbolic function method. Different types of novel solitons are attained such as, singular soliton, bright soliton, dark soliton, and periodic soliton. To understand the physical behavior, we have plotted 2D and 3D graphs of some selected solutions. From results we concluded that the proposed methods are straightforward, simple, and efficient. Moreover, this paper offers a hint, how we can convert the fractional-order PDE into an ODE to acquire the exact solutions. Also, the proposed methods and results can be help to examine the advance fractional-order models which seem in optics, hydrodynamics, plasma and wave theory etc.

Keywords: Fokas equation; Sardar-subequation method; new extended hyperbolic function method; solitons solutions
Mathematics Subject Classification: 35Q51, 35Q53

1. Introduction

Currently, the study of non-linear fractional differential equations (NLFDEs) play a key role in the arena of solitary wave theory. The analysis of such models [1–3], is significant to understand the everyday physical phenomena in non-linear evolution models. Nowadays technological developments have exposed that, fractional calculus have developed as a dominant instrument in numerous fields of applied sciences and engineering [4, 5]. A big advantage of fractional models is that their narrative is more suitable than the ordinary models [6]. These properties motivate our interest to yield consideration in a noteworthy and appropriate model for (4+1)-dimensional fractional-order non-linear Fokas equation. Non-linear Fokas equation is one of the well-known physical models [7, 8] that defines a huge diversity of phenomena, such as non-linear optics, fluid mechanics, condensed matter, and plasmas. The properties of fractional derivatives are complex, therefore, it is not easy to solve fractional-order problems using novel derivatives. As for as we know, there is no regular scheme to answer the fractional-order non-linear models. A number of novel arithmetical and analytical schemes [9–20], are suggested for fractional-order models by using ordinary calculus, however, it's static task for the scholars. Fokas [21] improved the Lax couples of integrable non-linear KP and DS equations, and consequent the simple wave equation. In wave systems, DS and KP models are proposed to define the surface and internal waves in canals with altering depth and breadth [22, 23]. The Fokas equation has many applications in quantum mechanics and plasma physics to explore the transmission of solitons. The important applications of the Fokas equation lie in real world problems considered as a higher dimensional integrable model in mathematical physics. The importance of the Fokas equation suggests that the idea of complexifying time can be investigated in the context of modern field theories via the existence of integrable nonlinear equations in four spatial dimensions involving complex time [24, 25]. In this paper, we deal the non-linear fractional-order (4+1)-dimensional Fokas equation (FE),

$$4\frac{\partial^{2\alpha}u}{\partial t^{\alpha}\partial x^{\alpha}} - \frac{\partial^{3\alpha}}{\partial x^{3\alpha}}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{\partial^{3}u}{\partial y^{3}}\right) + 12\frac{\partial^{\alpha}u}{\partial x^{\alpha}}\frac{\partial u}{\partial y} + 12u\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(\frac{\partial u}{\partial y}\right) - 6\frac{\partial^{2}u}{\partial z\partial w} = 0, \tag{1.1}$$

where u is a function and wave velocity physically also $0 < \alpha < 1$. While $\alpha = 1$ then Eq (1.1) converts to simple (4+1)-dimensional FE. Assume this general Eq (1.1) to designate the gesture of waves in multifaceted broadcasting. It shows the distortion or dispersion of the external wave. So, various types of solution of FE will offer us proposals to instruct the multifaceted system of DS and KD models. Also, according above-stated evidences and the significant uses of the FE, it is crucial to discover the innovative solitons solutions of Eq (1.1). In recent times, physicists and mathematicians have great concern to discover the Eq (1.1), such as Zhang et al. [26] applied the sub-equation method (SM) to determine the solution of Eq (1.1) using some derivative in the form of rational, trigonometric, and functions of hyperbolic. Zhang et al. [27] utilized the new SM, in fractional form of ($G0 = G$)-expansion scheme. Choi et al. [28] applied the SM using different schemes and attained the novel analytical solutions of Eq (1.1). Zhang [29] used Exp-function method to explore the solitary wave solution of Eq (1.1). Zhao et al. [30] suggested the improved fractional ($G0 = G$)-expansion scheme and acquire the exact solutions. Solitons are used to denote the particle-like properties of non-linear pulses. The significance of solitons is due to their existence in a variety of non-linear differential equations representing many complex nonlinear phenomena, such as acoustics, optics, convictive fluids, condensed matter, and solid-state physics. There are numerous type of solitary wave and solitons solutions [31–41].

The study of non-linear PDEs has become much significant in pure and applied mathematics from many years. Through the aid of computer technology, the new horizons are opened in the field of applied sciences for the mathematicians. There has been a growing interest in PDEs which are frequently used in engineering sciences and mathematical physics. The PDEs arise commonly in biological and physical sciences as well. Many problems of physics, chemistry, and engineering lead naturally to the resolution of PDEs. In the light of the literature review, there are various models of PDEs, particularly, PDEs whose exact solutions need to be discussed. The application of non-linear PDEs in mathematical modeling has induced a motivation for research in the field of differential problems.

The prominent concern of this existing study is to utilize the novel meanings of fractional-order derivative, named conformable derivative, for space-time fractional-order (4 + 1)-dimensional FE. As our best knowledge the considered methods have not been utilized for such model in the literature. Several type of solutions in the form of bright soliton, dark soliton, combined dark-bright soliton, periodic soliton and general solitary waves solutions of FE are missing in the literature. As compared our obtained results with previous existing results in literature, some of results are not available in literature with other methods lies in [42–46]. In this article our goal is to discover novel various soliton solutions of (4+1)-dimensional fractional-order non-linear Fokas equation using Sardar-subequation [47, 48] and new extended hyperbolic function method [49–52]. As a result, novel soliton solutions are more generalized and in different form which have never been obtained before. To the best of our knowledge, these novel properties and interesting structures are investigated for the first time in (4+1)-dimensional Fokas equation. Our results enrich the variety of the dynamics of higher-dimensional non-linear wave field. It is hoped that these results will provide some valuable information in the higher-dimensional non-linear field.

The layouts of this paper are: Section 2 contains the governing model. Section 3 consists on the analysis of the methods. In section 4 the applications of the proposed methods are presented. Section 5 describes results and discussion and conclusion of this work is presented in section 6.

2. Conformable derivative

Progress in fractional-calculus is more beneficial for scholars to express the physical phenomena with novel techniques. Newly, Caputo et al. [53] modified the Caputo derivative [54], Atangana-Baleanu (AB) presented a novel fractional derivative (FD), known as AB derivative [55]. In 2014, Khalil et al. [56] proposed a conformable derivative. The Caputo, Riemann-Liouville (RL) [57, 58], and further derivatives do not follow the simple instructions, which are supposed by Newtonian kind simple derivative. Such as:

(1) For RL derivative, $D_t^{\alpha} P \neq 0$, P is any constant.

(2) For Caputo and RL derivative,

$$D_t^{\alpha}\,(f(t)g(t)) \;\neq\; f(t)D_t^{\alpha}\,g(t) + g(t)\,D_t^{\alpha} f(t).$$

(3) For Caputo and RL derivative,

$$\frac{D_t^{\alpha} f(t)}{g(t)} \neq \frac{g(t)D_t^{\alpha} f(t) - f(t)D_t^{\alpha} g(t)}{g(t)^2}.$$

Definition 2.1: [56] The FD of a function $k = k(t) : [0, \infty) \longrightarrow \Re$ of order $\varsigma > 0$ is described as

$$D_t^{\varsigma} k(t) = \lim_{\epsilon \to 0} \frac{k(\epsilon t^{1-\varsigma} + t) - k(t)}{\epsilon}, \quad 0 < \varsigma \leq 1. \tag{2.1}$$

At $\varsigma = 1$, the fractional-order derivative converted to integer-order derivative.

Theorem 2.2: We have the following properties when $\varsigma \in (0, 1]$ and $k,\ k_1$ are conformable functions as follows:

•$D_t^{\varsigma} k(t) = t^{1-\varsigma} \frac{dk(t)}{dt}$.
•$D_t^{\varsigma} t^h = h t^{n-\varsigma}, \forall h \in R$.
•$D_t^{\varsigma} c = 0$, $\forall$ constant functions $k(t) = 0$.
•$D_t^{\varsigma}(p_1 * k(t) + p_2 * k_1(t)) = p_1 * D_t^{\varsigma} k(t) + p_2 * D_t^{\varsigma} k_1(t), \quad \forall p_1, p_2 \in R$.
•$D_t^{\varsigma}(k(t)k_1(t)) = k(t)D_t^{\varsigma} k_1(t) + k_1(t)D_t^{\varsigma} h(t)$.
•$\frac{D_t^{\varsigma} k(t)}{k_1(t)} = \frac{k_1(t)D_t^{\varsigma} - k(t)D_t^{\varsigma} k_1(t)}{k_1(t)^2}$.
•$D_t^{\varsigma}(k \ o \ k_1)(t) = t^{1-\varsigma} k_1'(t) k'(k_1(t))$.

3. Analysis of the methods

In this section, we have analyzed two methods that are applied to construct novel solitons solutions of the given model.

3.1. Sardar-subequation method

In this section, we describe the offered methods to solve the fractional model, we assume the proposed model as

$$H(u, \frac{\partial^{\alpha} u}{\partial t^{\alpha}}, \frac{\partial^{\alpha} u}{\partial x^{\alpha}}, \frac{\partial^{\alpha} u}{\partial y^{\alpha}}, \frac{\partial^{\alpha} u}{\partial z^{\alpha}}, \frac{\partial^{\alpha} u}{\partial w^{\alpha}}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \frac{\partial^{2\alpha} u}{\partial y^{2\alpha}}, ...) = 0,$$

where u is a function. Using following wave transformations in (3.1)

$$u(x, y, z, w, t) = U(\eta), \quad \eta = \beta_1 \frac{x^{\alpha}}{\alpha} + \beta_2 \frac{y^{\alpha}}{\alpha} + \beta_3 \frac{z^{\alpha}}{\alpha} + \beta_4 \frac{w^{\alpha}}{\alpha} - v \frac{t^{\alpha}}{\alpha},$$

here v is frequency while β_1, β_2, β_3, β_4 are wave lengths, respectively. Using (3.2) into (3.1), we yield an ODE as follows

$$P(U, U', U'', ...) = 0. \tag{3.1}$$

Where U', U'' and U''' are the first, second and the third derivatives of U, respectively w. r. t η and so on.

$$U(\eta) = \sum_{j=0}^{M} F_j \Phi^i(\eta), \tag{3.2}$$

where F_j are constants and $\Phi(\eta)$ accept the (3.7) as follows

$$(\Phi'(\eta))^2 = \epsilon + \delta\Phi^2(\eta) + \Phi^4(\eta), \tag{3.3}$$

here ϵ and δ are constants.

Case 1: When $\delta > 0$ *and* $\epsilon = 0$, then

$$\Phi_1^{\pm}(\eta) = \pm\sqrt{-\delta pq}\; sech_{pq}(\sqrt{\delta}\eta),$$

$$\Phi_2^{\pm}(\eta) = \pm\sqrt{\delta pq}\; csch_{pq}(\sqrt{\delta}\eta),$$

where, $sech_{pq}(\eta) = \frac{2}{pe^{\eta}+qe^{-\eta}},\ csch_{pq}(\eta) = \frac{2}{pe^{\eta}-qe^{-\eta}}$.

Case 2: When $\delta < 0$ *and* $\epsilon = 0$, then

$$\Phi_3^{\pm}(\eta) = \pm\sqrt{-\delta pq}\; sec_{pq}(\sqrt{-\delta}\eta),$$

$$\Phi_4^{\pm}(\eta) = \pm\sqrt{-\delta pq}\; csc_{pq}(\sqrt{-\delta}\eta),$$

where, $sec_{pq}(\eta) = \frac{2}{pe^{\iota\eta}+qe^{-\iota\eta}},\ csc_{pq}(\eta) = \frac{2\iota}{pe^{\iota\eta}-qe^{-\iota\eta}}$.

Case 3: When $\delta < 0$ *and* $\epsilon = \frac{\delta^2}{4b}$, then

$$\Phi_5^{\pm}(\eta) = \pm\sqrt{-\frac{\delta}{2}} tanh_{pq}(\sqrt{-\frac{\delta}{2}}\ \eta),$$

$$\Phi_6^{\pm}(\eta) = \pm\sqrt{-\frac{\delta}{2}} coth_{pq}(\sqrt{-\frac{\delta}{2}}\ \eta),$$

$$\Phi_7^{\pm}(\eta) = \pm\sqrt{-\frac{\delta}{2}}\Big(tanh_{pq}(\sqrt{-2\delta}\ \eta) \pm \iota\sqrt{pq}\ sech_{pq}(\sqrt{-2\delta}\ \eta)\Big),$$

$$\Phi_8^{\pm}(\eta) = \pm\sqrt{-\frac{\delta}{2}}\Big(coth_{pq}(\sqrt{-2\delta}\ \eta) \pm \sqrt{pq}\ csch_{pq}(\sqrt{-2\delta}\ \eta)\Big),$$

$$\Phi_9^{\pm}(\eta) = \pm\sqrt{-\frac{\delta}{8}}\Big(tanh_{pq}(\sqrt{-\frac{\delta}{8}}\ \eta) + coth_{pq}(\sqrt{-\frac{\delta}{8}}\ \eta)\Big),$$

where, $tanh_{pq}(\eta) = \frac{pe^{\eta}-qe^{-\eta}}{pe^{\eta}+qe^{-\eta}},\ coth_{pq}(\eta) = \frac{pe^{\eta}+qe^{-\eta}}{pe^{\eta}-qe^{-\eta}}$.

Case 4: When $\delta > 0$ *and* $\epsilon = \frac{\delta^2}{4}$, then

$$\Phi_{10}^{\pm}(\eta) = \pm\sqrt{\frac{\delta}{2}} tan_{pq}(\sqrt{\frac{\delta}{2}}\ \eta),$$

$$\Phi_{11}^{\pm}(\eta) = \pm\sqrt{\frac{\delta}{2}} cot_{pq}(\sqrt{\frac{\delta}{2}}\,\eta),$$

$$\Phi_{12}^{\pm}(\eta) = \pm\sqrt{\frac{\delta}{2}}\Big(tan_{pq}(\sqrt{2\delta}\,\eta) \pm \sqrt{pq}\; sec_{pq}(\sqrt{2\delta}\,\eta)\Big),$$

$$\Phi_{13}^{\pm}(\eta) = \pm\sqrt{\frac{\delta}{2}}\Big(cot_{pq}(\sqrt{2\delta}\,\eta) \pm \sqrt{pq}\; csc_{pq}(\sqrt{2\delta}\,\eta)\Big),$$

$$\Phi_{14}^{\pm}(\eta) = \pm\sqrt{\frac{\delta}{8}}\Big(tan_{pq}(\sqrt{\frac{\delta}{8}}\,\eta) + cot_{pq}(\sqrt{\frac{\delta}{8}}\,\eta)\Big),$$

where, $tan_{pq}(\eta) = -\iota\frac{pe^{\iota\eta}-qe^{-\iota\eta}}{pe^{\iota\eta}+qe^{-\iota\eta}}$, $cot_{pq}(\eta) = \iota\frac{pe^{\iota\eta}+qe^{-\iota\eta}}{pe^{\iota\eta}-qe^{-\iota\eta}}$.

3.2. New EHFM

The new EHFM has two phases as follows.

Form 1: Consider FPDE is taken in (3.1) and using the wave transformations in (3.2) to obtain (3.3). Assume (3.3) admits the solution as follows:

$$U(\eta) = \sum_{j=0}^{M} F_j \Phi^i(\eta), \tag{3.4}$$

where F_j are constants and $\Phi(\eta)$ accept the (3.7)

$$\frac{d\Phi}{d\eta} = \Phi\sqrt{\Lambda + \Delta\Phi^2},\ \Lambda, \Theta \in R. \tag{3.5}$$

On balancing in (3.3) the value of N is obtained. Inserting (3.6) into (3.3) along with (3.7), yields a set of equations. By resolving the equations, we get the solutions that accepts (3.5), as

Set 1: When $\Lambda > 0$ and $\Delta > 0$,

$$\Phi(\eta) = -\sqrt{\frac{\Lambda}{\Delta}}\; csch(\sqrt{\Lambda}(\eta + \eta_0)). \tag{3.6}$$

Set 2: When $\Lambda < 0$ and $\Delta > 0$,

$$\Phi(\eta) = \sqrt{\frac{-\Lambda}{\Delta}} sec(\sqrt{-\Lambda}(\eta + \eta_0)). \tag{3.7}$$

Set 3: When $\Lambda > 0$ and $\Delta < 0$,

$$\Phi(\eta) = \sqrt{\frac{\Lambda}{-\Delta}} sech(\sqrt{\Lambda}(\eta + \eta_0)). \tag{3.8}$$

Set 4: When $\Lambda < 0$ and $\Delta > 0$,

$$\Phi(\eta) = \sqrt{\frac{-\Lambda}{\Delta}} csc(\sqrt{-\Lambda}(\eta + \eta_0)). \tag{3.9}$$

Set 5: When $\Lambda > 0$ and $\Delta = 0$,

$$\Phi(\eta) = exp\,(\sqrt{\Lambda}(\eta + \eta_0)). \tag{3.10}$$

Set 6: When $\Lambda < 0$ and $\Delta = 0$,

$$\Phi(\eta) = cos(\sqrt{-\Lambda}(\eta + \eta_0)) + \iota sin(\sqrt{-\Lambda}(\eta + \eta_0)). \tag{3.11}$$

Set 7: When $\Lambda = 0$ and $\Delta > 0$,

$$\Phi(\eta) = \pm \frac{1}{(\sqrt{\Delta}(\eta + \eta_0))}. \tag{3.12}$$

Set 8: When $\Lambda = 0$ and $\Delta < 0$,

$$\Phi(\eta) = \pm \frac{\iota}{(\sqrt{-\Delta}(\eta + \eta_0))}. \tag{3.13}$$

Form 2: Adopting the procedure as earlier, consider that $\Phi(\eta)$ accept the following ODE as

$$\frac{d\Phi}{d\eta} = \Lambda + \Delta\Phi^2,\ \Lambda, \Delta \in R. \tag{3.14}$$

Substituting (3.6) into (3.3) along with (3.16), gets a set of equations with the values of $F_j\ (j = 1,\ 2,\ 3, ...M)$. We assume that (3.5) has solutions as

Set 1: When $\Lambda\Delta > 0$,

$$\Phi(\eta) = sn(\Lambda)\sqrt{\frac{\Lambda}{\Delta}} tan(\sqrt{\Lambda\Delta}(\eta + \eta_0)). \tag{3.15}$$

Set 2: When $\Lambda\Delta > 0$,

$$\Phi(\eta) = -sn(\Lambda)\sqrt{\frac{\Lambda}{\Delta}} cot(\sqrt{\Lambda\Delta}(\eta + \eta_0)). \tag{3.16}$$

Set 3: When $\Lambda\Delta < 0$,

$$\Phi(\eta) = sn(\Lambda)\sqrt{\frac{\Lambda}{-\Delta}} tanh(\sqrt{-\Lambda\Delta}(\eta + \eta_0)). \tag{3.17}$$

Set 4: When $\Lambda\Delta < 0$,

$$\Phi(\eta) = sn(\Lambda)\sqrt{\frac{\Lambda}{-\Delta}} coth(\sqrt{-\Lambda\Delta}(\eta + \eta_0)). \tag{3.18}$$

Set 5: When $\Lambda = 0$ and $\Delta > 0$,

$$\Phi(\eta) = -\frac{1}{\Delta(\eta + \eta_0)}. \tag{3.19}$$

Set 6: When $\Lambda \in R$ and $\Delta = 0$,

$$\Phi(\eta) = \Lambda(\eta + \eta_0). \tag{3.20}$$

Note: *sn* is well-known sign function.

4. Application

Here, we construct the solitons solutions of fractional-order Fokas equation. The Eq (1.1) with Eq (3.2) becomes

$$\beta_1\beta_2(\beta_1^2-\beta_2^2)U''+(4\beta_1 v+6\beta_3\beta_4)U-6\beta_1\beta_2U^2=0. \tag{4.1}$$

4.1. Application of the SSM

Now, we apply the SSM to solve the (4+1)-dimensional FE. Using balance method on (4.1), gets $M=2$, so (3.6) decreases to

$$U(\eta)=F_0+F_1\Phi(\eta)+F_2\Phi(\eta)^2, \tag{4.2}$$

where F_0, F_1 and F_2 are constants. Inserting (4.2) into (4.1) and comparing the coefficients of $\Phi(\eta)$ to zero, we get a set of equations in $F_0,\ F_1,\ F_2,\ \epsilon\ and\ v$.

Working on the set of equations, yields

$$F_0=F_0,\ \ F_1=0,\ \ F_2=(\beta_1^2-\beta_2^2),$$

$$\epsilon=\frac{2\delta\beta_1^2F_0-2\delta\beta_2^2F_0-3F_0^2}{(\beta_1^2-\beta_2^2)^2},$$

$$v=\frac{-2\delta\beta_1^3\beta_2+2\delta\beta_1\beta_2^3-3\beta_3\beta_4+6\beta_1\beta_2F_0}{2\beta_1}. \tag{4.3}$$

Case 1: When $\delta>0$ *and* $\epsilon=0$, then

$$u_{1,1}^{\pm}(x,y,z,w,t)=F_0+(\beta_1^2-\beta_2^2)\Big(\pm\sqrt{-pq\delta}\ sech_{pq}(\sqrt{\delta}\ (\eta))\Big)^2, \tag{4.4}$$

$$u_{1,2}^{\pm}(x,y,z,w,t)=F_0+(\beta_1^2-\beta_2^2)\Big(\pm\sqrt{pq\delta}\ csch_{pq}(\sqrt{\delta}\ (\eta))\Big)^2. \tag{4.5}$$

Case 2: When $\delta<0$ *and* $\epsilon=0$, then

$$u_{1,3}^{\pm}(x,y,z,w,t)=F_0+(\beta_1^2-\beta_2^2)\Big(\pm\sqrt{-pq\delta}\ sec_{pq}(\sqrt{-\delta}\ (\eta))\Big)^2, \tag{4.6}$$

$$u_{1,4}^{\pm}(x,y,z,w,t)=F_0+(\beta_1^2-\beta_2^2)\Big(\pm\sqrt{-pq\delta}\ csc_{pq}(\sqrt{-\delta}\ (\eta))\Big)^2. \tag{4.7}$$

Case 3: When $\delta<0$ *and* $\epsilon=\frac{\delta^2}{4b}$, then

$$u_{1,5}^{\pm}(x,y,z,w,t)=F_0+(\beta_1^2-\beta_2^2)\Big(\pm\sqrt{-\frac{\delta}{2}}tanh_{pq}(\sqrt{-\frac{\delta}{2}}\ (\eta)\Big)^2, \tag{4.8}$$

$$u^{\pm}_{1,6}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{-\frac{\delta}{2}} coth_{pq}(\sqrt{-\frac{\delta}{2}}\ (\eta)\Big)^2, \tag{4.9}$$

$$u^{\pm}_{1,7}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{-\frac{\delta}{2}}\Big(tanh_{pq}(\sqrt{-2\delta}\ (\eta)) \pm\iota \sqrt{pq}\ sech_{pq}(\sqrt{-2\delta}\ (\eta))\Big)\Big)^2, \tag{4.10}$$

$$u^{\pm}_{1,8}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{-\frac{\delta}{2}}\Big(coth_{pq}(\sqrt{-2\delta}\ (\eta)) \pm \sqrt{pq}\ csch_{pq}(\sqrt{-2\delta}\ (\eta))\Big)\Big)^2, \tag{4.11}$$

$$u^{\pm}_{1,9}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{-\frac{\delta}{8}}\Big(tanh_{pq}(\sqrt{-\frac{\delta}{8}}\ (\eta)) + coth_{pq}(\sqrt{-\frac{\delta}{8}}\ (\eta))\Big)\Big)^2. \tag{4.12}$$

Case 4: When $\delta > 0$ *and* $\epsilon = \frac{\delta^2}{4}$, then

$$u^{\pm}_{1,10}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{\frac{\delta}{2}} tan_{pq}(\sqrt{\frac{\delta}{2}}\ (\eta))\Big)^2, \tag{4.13}$$

$$u^{\pm}_{1,11}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{\frac{\delta}{2}} cot_{pq}(\sqrt{\frac{\delta}{2}}\ (\eta))\Big)^2, \tag{4.14}$$

$$u^{\pm}_{1,12}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{\frac{\delta}{2}}\Big(tan_{pq}(\sqrt{2\delta}\ (\eta)) \pm \sqrt{pq}\ sec_{pq}(\sqrt{2\delta}\ (\eta))\Big)\Big)^2, \tag{4.15}$$

$$u^{\pm}_{1,13}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{\frac{\delta}{2}}\Big(cot_{pq}(\sqrt{2\delta}\ (\eta)) \pm \sqrt{pq}\ csc_{pq}(\sqrt{2\delta}\ (\eta))\Big)\Big)^2, \tag{4.16}$$

$$u^{\pm}_{1,14}(x,y,z,w,t) = F_0 + (\beta_1^2 - \beta_2^2)\Big(\pm \sqrt{\frac{\delta}{8}}\Big(tan_{pq}(\sqrt{\frac{\delta}{8}}\ (\eta)) + cot_{pq}(\sqrt{\frac{\delta}{8}}\ (\eta))\Big)\Big)^2. \tag{4.17}$$

4.2. Application of the new EHFM

Form 1: Now, we apply the new EHFM to solve the (4+1)-dimensional FE. Using balance method in (4.1), yields $M = 2$, so (3.6) changes to

$$U(\eta) = F_0 + F_1\Phi(\eta) + F_2\Phi(\eta)^2, \tag{4.18}$$

where $F_0,\ F_1$ and F_2 are constants. Inserting (4.18) into (4.1) and associating the constants of $\Phi(\eta)$ with zero, we yield the equations in $F_0,\ F_1,\ F_2,\ \Lambda$ *and* Δ.

On working set of equations, we achieve

$$F_0 = \frac{2v\beta_1 + 3\beta_3\beta_4}{3\beta_1\beta_2},\quad F_1 = 0,$$

$$F_2 = F_2,\quad \Lambda = \frac{2v\beta_1 + 3\beta_3\beta_4}{2\beta_1\beta_2(\beta_1^2 - \beta_2^2)}.$$

$$\Delta = \frac{F_2}{(\beta_1^2 - \beta_2^2)}. \tag{4.19}$$

Set 1: When $\Lambda > 0$ and $\Delta > 0$,

$$u_1(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4}{3\beta_1\beta_2} + F_2\Big(\sqrt{\frac{2v\beta_1 + 3\beta_3\beta_4}{2F_2\beta_1\beta_2}}\, csch(\sqrt{\frac{2v\beta_1 + 3\beta_3\beta_4}{2\beta_1\beta_2(\beta_1^2 - \beta_2^2)}}(\eta + \eta_0))\Big)^2. \tag{4.20}$$

Set 2: When $\Lambda < 0$ and $\Delta > 0$,

$$u_2(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4}{3\beta_1\beta_2} + F_2\Big(\sqrt{\frac{2v\beta_1 + 3\beta_3\beta_4}{2F_2\beta_1\beta_2}}\, sec(\sqrt{-\frac{2v\beta_1 + 3\beta_3\beta_4}{2\beta_1\beta_2(\beta_1^2 - \beta_2^2)}}(\eta + \eta_0))\Big)^2. \tag{4.21}$$

Set 3: When $\Lambda > 0$ and $\Delta < 0$,

$$u_3(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4}{3\beta_1\beta_2} + F_2\Big(\sqrt{\frac{2v\beta_1 + 3\beta_3\beta_4}{2F_2\beta_1\beta_2}}\, sech(\sqrt{-\frac{2v\beta_1 + 3\beta_3\beta_4}{2\beta_1\beta_2(\beta_1^2 - \beta_2^2)}}(\eta + \eta_0))\Big)^2. \tag{4.22}$$

Set 4: When $\Lambda < 0$ and $\Delta < 0$,

$$u_4(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4}{3\beta_1\beta_2} + F_2\Big(\sqrt{\frac{2v\beta_1 + 3\beta_3\beta_4}{2F_2\beta_1\beta_2}}$$

$$csc(\sqrt{-\frac{2v\beta_1+3\beta_3\beta_4}{2\beta_1\beta_2(\beta_1^2-\beta_2^2)}}(\eta+\eta_0))\Big)^2. \tag{4.23}$$

Set 5: When $\Lambda > 0$ and $\Delta = 0$,

$$u_5(x,y,z,w,t) = \frac{2v\beta_1+3\beta_3\beta_4}{3\beta_1\beta_2} + F_2\Big(\exp(\sqrt{\frac{2v\beta_1+3\beta_3\beta_4}{2\beta_1\beta_2(\beta_1^2-\beta_2^2)}}(\eta+\eta_0))\Big)^2. \tag{4.24}$$

Set 6: When $\Lambda < 0$ and $\Delta = 0$,

$$u_6(x,y,z,w,t) = \frac{2v\beta_1+3\beta_3\beta_4}{3\beta_1\beta_2} + F_2\Big(\cos(\sqrt{-\frac{2v\beta_1+3\beta_3\beta_4}{2\beta_1\beta_2(\beta_1^2-\beta_2^2)}}(\eta+\eta_0)) + \iota sin(\sqrt{-\frac{2v\beta_1+3\beta_3\beta_4}{2\beta_1\beta_2(\beta_1^2-\beta_2^2)}}(\eta+\eta_0))\Big)^2. \tag{4.25}$$

Set 7: When $\Lambda = 0$ and $\Delta > 0$,

$$u_7(x,y,z,w,t) = \frac{2v\beta_1+3\beta_3\beta_4}{3\beta_1\beta_2} + F_2\Big(\pm\frac{1}{(\sqrt{\frac{F_2}{(\beta_1^2-\beta_2^2)}}(\eta+\eta_0)}\Big)^2. \tag{4.26}$$

Set 8: When $\Lambda = 0$ and $\Delta < 0$,

$$u_8(x,y,z,w,t) = \frac{2v\beta_1+3\beta_3\beta_4}{3\beta_1\beta_2} + F_2\Big(\pm\frac{\iota}{(\sqrt{-\frac{F_2}{(\beta_1^2-\beta_2^2)}}(\eta+\eta_0)}\Big)^2, \tag{4.27}$$

where $\eta = \beta_1\frac{x^\alpha}{\alpha} + \beta_2\frac{y^\alpha}{\alpha} + \beta_3\frac{z^\alpha}{\alpha} + \beta_4\frac{w^\alpha}{\alpha} - v\frac{t^\alpha}{\alpha}$.

Form 2: Utilizing balance method in (4.1), gives $M = 2$, so (3.6) converts to

$$u(\eta) = F_0 + F_1\Phi(\eta) + F_2\Phi(\eta)^2, \tag{4.28}$$

where F_0, F_1 and F_2 are numbers. Inserting (4.28) into (4.1) and associating the constants of $\Phi(\eta)$ with zero, we attain set of equations in $F_0,\ F_1,\ F_2,\ \Lambda,\ and\ \Delta$.

On working the set of equations, we achieve

$$F_0 = \frac{2v\beta_1+3\beta_3\beta_4-2\sqrt{(2v\beta_1+3\beta_3\beta_4)^2}}{6\beta_1\beta_2},\quad F_1 = 0,$$

$$F_2 = \frac{(2v\beta_1+3\beta_3\beta_4)^2}{4\Lambda^2\beta_1^2\beta_2^2(\beta_1^2-\beta_2^2)},\quad \Lambda = \Lambda,$$

$$\Delta = \frac{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{2\Lambda\beta_1^3\beta_2 - 2\Lambda\beta_1\beta_2^3}. \tag{4.29}$$

Set 1: When $\Lambda\Delta > 0$,

$$u_9(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4 - 2\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{6\beta_1\beta_2} + \frac{(2v\beta_1 + 3\beta_3\beta_4)^2}{4\Lambda^2\beta_1^2\beta_2^2(\beta_1^2 - \beta_2^2)} \Big(\chi\sqrt{\frac{2\Lambda^2\beta_1\beta_2(\beta_1^2 - \beta_2^2)}{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}} tan(\frac{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{2\beta_1^3\beta_2 - 2\beta_1 - \beta_2^3}(\eta + \eta_0))\Big)^2. \tag{4.30}$$

Set 2: When $\Lambda\Delta > 0$,

$$u_{10}(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4 - 2\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{6\beta_1\beta_2} + \frac{(2v\beta_1 + 3\beta_3\beta_4)^2}{4\Lambda^2\beta_1^2\beta_2^2(\beta_1^2 - \beta_2^2)} \Big(-\chi\sqrt{\frac{2\Lambda^2\beta_1\beta_2(\beta_1^2 - \beta_2^2)}{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}} cot(\frac{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{2\beta_1^3\beta_2 - 2\beta_1 - \beta_2^3}(\eta + \eta_0))\Big)^2. \tag{4.31}$$

Set 3: When $\Lambda\Delta < 0$,

$$u_{11}(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4 - 2\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{6\beta_1\beta_2} + \frac{(2v\beta_1 + 3\beta_3\beta_4)^2}{4\Lambda^2\beta_1^2\beta_2^2(\beta_1^2 - \beta_2^2)} \Big(\chi\sqrt{\frac{2\Lambda^2\beta_1\beta_2(\beta_1^2 - \beta_2^2)}{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}} tanh(\frac{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{2\beta_1^3\beta_2 - 2\beta_1 - \beta_2^3}(\eta + \eta_0))\Big)^2. \tag{4.32}$$

Set 4: When $\Lambda\Delta < 0$,

$$u_{12}(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4 - 2\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{6\beta_1\beta_2} + \frac{(2v\beta_1 + 3\beta_3\beta_4)^2}{4\Lambda^2\beta_1^2\beta_2^2(\beta_1^2 - \beta_2^2)} \Big(\chi\sqrt{\frac{2\Lambda^2\beta_1\beta_2(\beta_1^2 - \beta_2^2)}{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}} coth(\frac{\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{2\beta_1^3\beta_2 - 2\beta_1 - \beta_2^3}(\eta + \eta_0))\Big)^2. \tag{4.33}$$

Set 5: When $\Lambda = 0$ and $\Delta > 0$,

$$u_{13}(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4 - 2\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{6\beta_1\beta_2} + \frac{(2v\beta_1 + 3\beta_3\beta_4)^2}{4\Lambda^2\beta_1^2\beta_2^2(\beta_1^2 - \beta_2^2)} \Big(-\frac{1}{\frac{\sqrt{(2v\beta_1+3\beta_3\beta_4)^2}}{2\Lambda\beta_1^3\beta_2 - 2\Lambda\beta_1\beta_2^3}(\eta + \eta_0)}\Big)^2. \tag{4.34}$$

Set 6: When $\Lambda \in R$ and $\Delta = 0$,

$$u_{14}(x,y,z,w,t) = \frac{2v\beta_1 + 3\beta_3\beta_4 - 2\sqrt{(2v\beta_1 + 3\beta_3\beta_4)^2}}{6\beta_1\beta_2} + \frac{(2v\beta_1 + 3\beta_3\beta_4)^2}{4\Lambda^2\beta_1^2\beta_2^2(\beta_1^2 - \beta_2^2)}\Big(\Lambda(\eta+\eta_0)\Big)^2, \tag{4.35}$$

where $\chi = sgn(\Lambda)$, $\eta = \beta_1\frac{x^\alpha}{\alpha} + \beta_2\frac{y^\alpha}{\alpha} + \beta_3\frac{z^\alpha}{\alpha} + \beta_4\frac{w^\alpha}{\alpha} - v\frac{t^\alpha}{\alpha}$.

5. Results and discussion

Herein study, we have successfully built various solitons solutions for the (4+1)-dimensional Fokas equation applying SSM and new EHFM. The proposed methods are measured most efficient techniques in this field and that are not applied to this model previous. To analyze physically, 3D and 2D graphs of some selected results are added with suitable parameters. These obtained solutions determine their applications in propagation to carry information since solitons have the competency to mobile long spaces without drop and shifting their systems. In this study, we added only certain figures (see Figures 1–7) to dodge overloading the document. Absolutely the achieved results are fresh and different from that reported results.

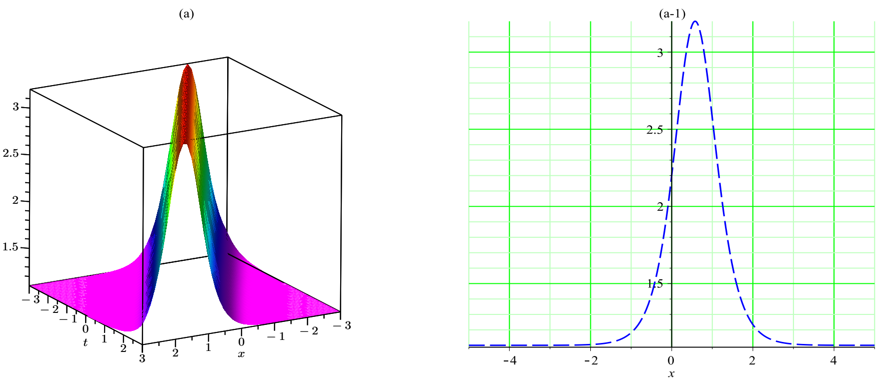

Figure 1. (a) 3D graph of (4.4) with $F_0 = 1.75$, $\beta_1 = 1.65$, $\beta_2 = 1.74$, $\beta_3 = 1.91$, $\beta_4 = 1.65$, $p = 0.98$, $q = 0.95$, $\alpha = 0.99$, $\delta = 0.8$, $v = 1.2$. (a-1) 2D plot of (4.4) with $t = 1$.

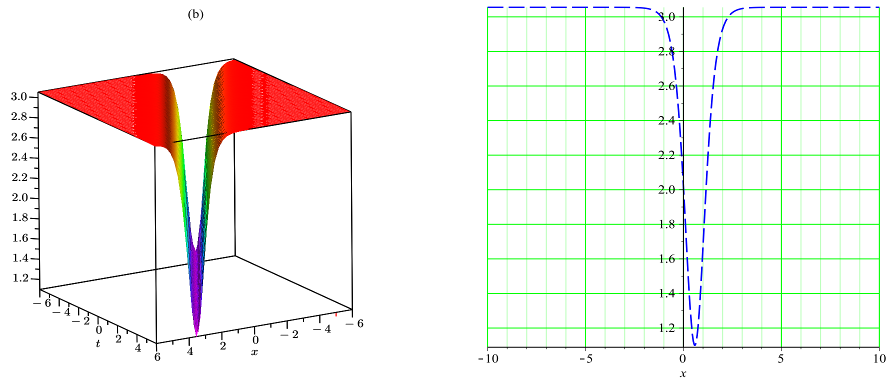

Figure 2. (b) 3D graph of (4.8) with $F_0 = 1.5$, $\beta_1 = 1.5$, $\beta_2 = 1.64$, $\beta_3 = 1.81$, $\beta_4 = 1.55$, $p = 0.98$, $q = 0.95$, $\alpha = 0.99$, $\delta = -0.8$, $v = 1.1$. (b-1) 2D plot of (4.8) with $t = 1$.

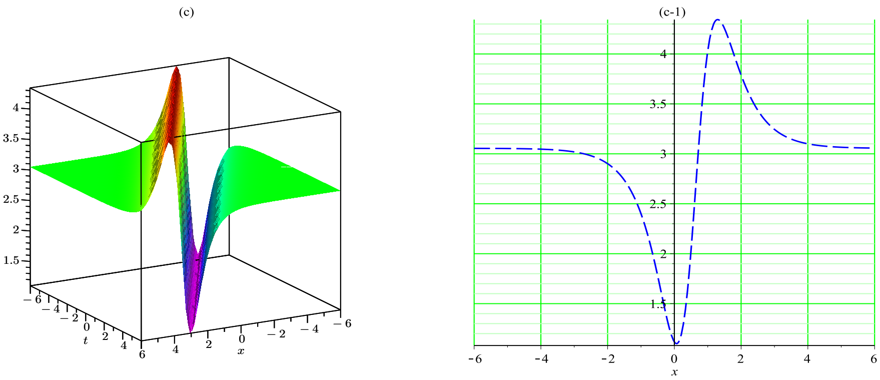

Figure 3. (c) 3D graph of (4.10) with $F_0 = 1.77$, $\beta_1 = 1.95$, $\beta_2 = 1.24$, $\beta_3 = 1.51$, $\beta_4 = 1.65$, $p = 0.98$, $q = 0.95$, $\alpha = 0.99$, $\delta = -0.65$, $v = 1.4$. (c-1) 2D plot of (4.10) with $t = 1$.

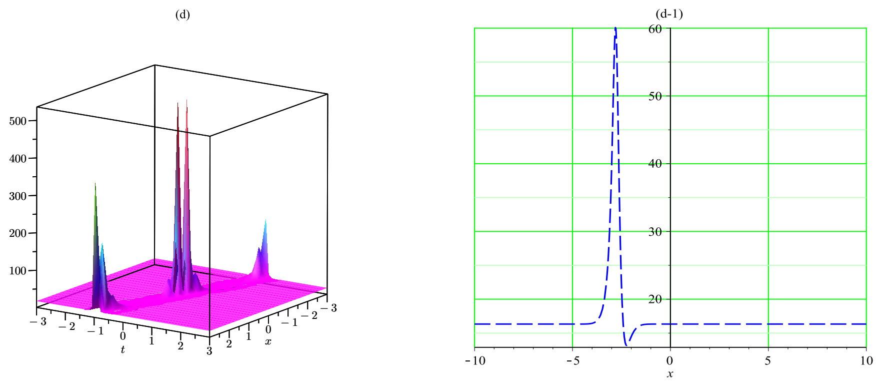

Figure 4. (d) 3D graph of (4.17) with $F_0 = 1.37$, $\beta_1 = 1.75$, $\beta_2 = 2.44$, $\beta_3 = 1.71$, $\beta_4 = 1.55$, $p = 0.98$, $q = 0.95$, $\alpha = 0.99$, $\delta = -0.75$, $v = 1.6$. (d-1) 2D graph of (4.17) with $t = 1$.

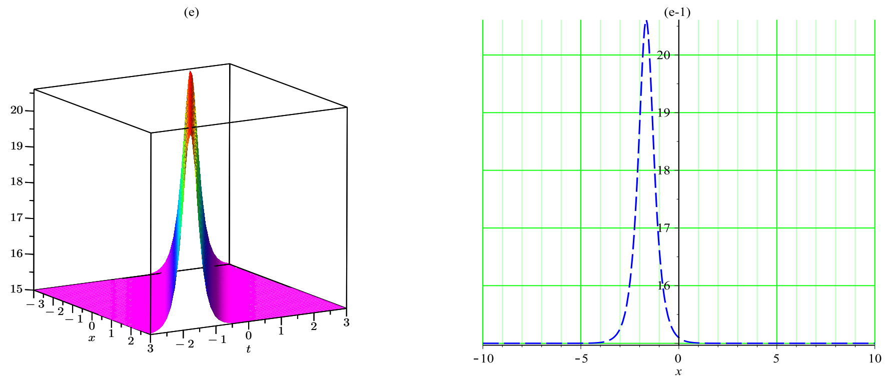

Figure 5. (e) 3D graph of (4.22) with $F_0 = 1.4,\ \beta_2 = 0.45,\ \beta_2 = 2.54,\ \beta_3 = 1.91,\ \beta_4 = 1.15,\ p = 0.98,\ q = 0.95,\ \alpha = 0.99,\ \delta = -0.65,\ v = 0.6$. (e-1) 2D plot of (4.22) with $t = 1$.

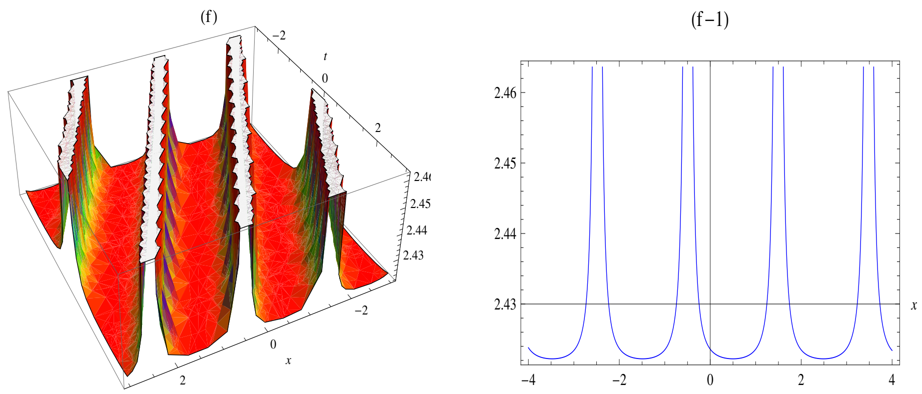

Figure 6. (f) 3D plot of (4.30) with $F_0 = 1.2,\ \beta_1 = 1.79,\ \beta_2 = 1.47,\ \beta_3 = 1.74,\ \beta_4 = 1.65,\ p = 0.98,\ q = 0.95,\ \alpha = 0.99,\ \chi = 1.2,\ \delta = -0.7,\ v = 1.2$. (f-1) 2D plot of (4.30) as $t = 1$.

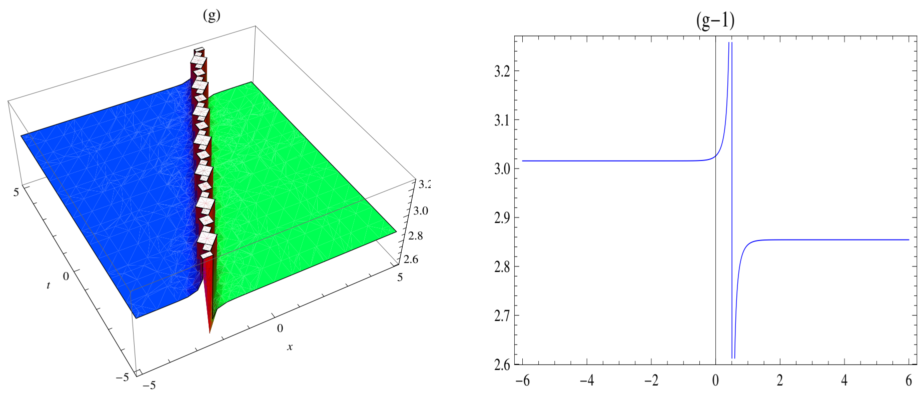

Figure 7. (g) 3D graph of (4.32) with $F_0 = 1.5,\ \beta_1 = 1.25,\ \beta_2 = 1.34,\ \beta_3 = 2.85,\ \beta_4 = 1.95,\ p = 0.98,\ q = 0.95,\ \alpha = 0.99,\ \delta = -0.45,\ \chi = 1.2,\ v = 0.77$. (g-1) 2D plot of (4.32) with $t = 1$.

6. Conclusions

We have constructed novel multi solitons solutions for the nonlinear fractional-order Fokas equation by applying the SSM and new EHFM. From results we obtained multi solitons solutions such as dark, singular, bright and periodic solitons. By the purpose of describe the behavior of achieved solutions of the model, we plotted some particular solutions by assigning the suitable values to the parameters involved. The obtained results may have much inspiration in various fields of physical sciences. From obtained results, we can know that the under study techniques are proficient, consistent and beneficial for rescuing the exact solutions of nonlinear FPDEs in a wide range. Also, these results are supportive to learn the dynamics of nonlinear waves in optics, hydrodynamics, solid state physics, and plasma. In future, we anticipate that this study is a step towards the solutions of such kinds of higher dimensional problems by using the proposed methods in this study. This work may be extended for a novel fractional order Fokas dynamical model in future.

Acknowledgments

The work in this paper has been supported by the Polish National Science Centre under the grant OPUS 14 No. 2017/27/B/ST8/01330.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. B. A. Malomed, Optical solitons and vortices in fractional media: A mini-review of recent results, *Photonics*, **8** (2021), 353. http://dx.doi.org/10.3390/photonics8090353
2. L. Zeng, J. Shi, X. Lu, Y. Cai, Q. Zhu, H. Chen, et al., Stable and oscillating solitons of PT-symmetric couplers with gain and loss in fractional dimension, *Nonlinear Dyn.*, **103** (2021), 1831–1840. https://doi.org/10.1007/s11071-020-06180-7
3. L. Zeng, M. R. Belic, D. Mihalache, Q. Wang, J. Chen, J. Shi, et al., Solitons in spin-orbit-coupled systems with fractional spatial derivatives, *Chaos Solitons Fract.*, **152** (2021), 111406. https://doi.org/10.1016/j.chaos.2021.111406
4. H. Bulut, T. A. Sulaiman, H. M. Baskonus, H. Rezazadeh, M. Eslami, M. Mirzazadeh, Optical solitons and other solutions to the conformable space-time fractional Fokas-Lenells equation, *Optik*, **172** (2018), 20–27. https://doi.org/10.1016/j.ijleo.2018.06.108
5. R. Khalil, M. Al Forani, A. Yousef, M. Sababheh, A new denition of fractional derivatives, *J. Comput. Appl. Math.*, **264** (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002
6. Z. Bin, (G'/G)-expansion method for solving fractional partial differential equations in the theory of mathematical physics, *Commun. Theor. Phys.*, **58** (2012), 623. https://doi.org/10.1088/0253-6102/58/5/02

7. S. Nestor, G. Betchewe, M. Inc, S. Y. Doka, Exact traveling wave solutions to the higher-order nonlinear Schrödinger equation having Kerr nonlinearity form using two strategic integrations, *Eur. Phys. J. Plus*, **135** (2020), 380. https://doi.org/10.1140/epjp/s13360-020-00384-x
8. H. Triki, C. Bensalem, A. Biswas, Q. Zhou, M. Ekici, S. P. Moshokoa, et al., W-shaped and bright optical solitons in negative indexed materials, *Chaos Solitons Fract.*, **123** (2019), 101–107. https://doi.org/10.1016/j.chaos.2019.04.003
9. M. Yousuf, K. M. Furati, A. Q. M. Khaliq, High-order timestepping methods for two-dimensional Riesz fractional nonlinear reaction-diffusion equations, *Comput. Math. Appl.*, **80** (2020), 204–226. https://doi.org/10.1016/j.camwa.2020.03.010
10. K. M. Furati, M. Yousuf, A. Q. M. Khaliq, Fourth-order methods for space fractional reaction-diffusion equations with nonsmooth data, *Int. J. Comput. math.*, **95** (2018), 1240–1256. https://doi.org/10.1080/00207160.2017.1404037
11. S. S. Alzahrani, A. Q. M. Khaliq, T. A. Biala, K. M. Furati, Fourth-order time stepping methods with matrix transfer techniques for space-fractional reaction-diffusion equations, *Appl. Numer. Math.*, **146** (2019), 123–144. https://doi.org/10.1016/j.apnum.2019.07.006
12. H. P. Bhatt, A. Q. M. Khaliq, K. M. Furati, Efficient high-order compact exponential time differencing method for spacefractional reaction-diffusion systems with nonhomogeneous boundary conditions, *Numer. Algor.*, **83** (2019), 1373–1397. https://doi.org/10.1007/s11075-019-00729-3
13. M. A. Zahid, S. Sarwar, M. Arshad, A. M. Arshad, New solitary wave solutions of generalized space-time fractional fifth order Laxs and Sawada Kotera KdV type equations in mathematical physics, *J. Adv. Phys.*, **7** (2018), 342–349. https://doi.org/10.1166/jap.2018.1447
14. S. Sarwar, S. Alkhalaf, S. Iqbal, M. A. Zahid, A note on optimal homotopy asymptotic method for the solutions of fractional order heat-and wave-like partial differential equations, *Comput. Math. Appl.*, **70** (2015), 942–953. https://doi.org/10.1016/j.camwa.2015.06.017
15. S. Sarwar, M. A. Zahid, S. Iqbal, Mathematical study of fractional order biological population models using Optimal homotopy asymptotic method, *Int. J. Biomath.*, **9** (2016), 1650081. https://doi.org/10.1142/S1793524516500819
16. S. Sarwar, M. M. Rashidi, Approximate solution of two term fractional order diffusion, wave-diffusion and telegraph models arising in mathematical physics using optimal homotopy asymptotic method, *Waves Random Complex Media*, **26** (2016), 365–382. https://doi.org/10.1080/17455030.2016.1158436
17. S. Sarwar, S. Iqbal, Stability analysis, dynamical behavior and analytical solutions of nonlinear fractional differential system arising in chemical reaction, *Chinese J. Phys.*, **56** (2018), 374–384. https://doi.org/10.1016/j.cjph.2017.11.009
18. S. Sarwar, M. A. Zahid, S. Iqbal, Mathematical study of fractional order biological model using optimal homotopy asymptotic method, *Int. J. Biomath.*, **9** (2016), 1650081. https://doi.org/10.1142/S1793524516500819
19. S. Sarwar, S. Iqbal, Exact solution of non-linear fractional order Klein-Gordon partial differential equations using optimal homotopy asymptotic method, *Nonlinear Sci. Lett. A*, **8** (2017), 340–348.

20. J. Wang, R. Zhang, L. Yang, Solitary waves of nonlinear barotropic-baroclinic coherent structures, *Phys. Fluids*, **32** (2020), 096604. https://doi.org/10.1063/5.0025167

21. A. S. Fokas, Integrable nonlinear evolution partial differential equations in (4+2) and (3+1)-dimensions, *Phys. Rev. Lett.*, **96** (2006), 190201. https://doi.org/10.1103/PhysRevLett.96.190201

22. S. T. Demiray, H. Bulut, Investigation of dark and bright soliton solutions of some nonlinear evolution equations, *ITM Web Conf.*, **22** (2018), 01056. https://doi.org/10.1051/itmconf/20182201056

23. A. Davey, K. Stewartson, On three-dimensional packets of surface waves, *Proc. R. Soc. Lond. Ser. A, Math. Phys. Eng. Sci.*, **338** (1974), 101–110. https://doi.org/10.1098/rspa.1974.0076

24. S. Sarwar, New soliton wave structures of nonlinear (4+1)-dimensional Fokas dynamical model by using different methods, *Alex. Eng. J.*, **60** (2021), 795–803. https://doi.org/10.1016/j.aej.2020.10.009

25. S. Zhang, H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, *Phys. Lett. A*, **375** (2011), 1069–1073. https://doi.org/10.1016/j.physleta.2011.01.029

26. J. Lee, R. Sakthivel, L. Wazzan, Exact traveling wave solutions of a higher-dimensional nonlinear evolution equation, *Mod. Phys. Lett. B*, **24** (2010), 1011–1021. https://doi.org/10.1142/S0217984910023062

27. B. Zheng, C. Wen, Exact solutions for fractional partial differential equations by a new fractional sub-equation method, *Adv. Differ. Equ.*, **2013** (2013), 199. https://doi.org/10.1186/1687-1847-2013-199

28. J. H. Choi, H. Kim, Soliton solutions for the space-time nonlinear partial differential equations with fractional-orders, *Chinese J. Phys.*, **55** (2017), 556–565. https://doi.org/10.1016/j.cjph.2016.10.019

29. B. Zheng, Exp-function method for solving fractional partial differential equations, *Sci. World J.*, **2013** (2013), 465723. https://doi.org/10.1155/2013/465723

30. Y. Zhao, Y. He, The extended fractional ($G0 = G$)-expansion method and its applications to a space-time fractional Fokas equation, *Math. Probl. Eng.*, **2017** (2017), 8251653. https://doi.org/10.1155/2017/8251653

31. F. Meng, A new approach for solving fractional partial differential equations, *J. Appl. Math.*, **2013** (2013), 256823. https://doi.org/10.1155/2013/256823

32. D. Lu, A. R. Seadawy, J. Wang, M. Arshad, U. Farooq, Soliton solutions of generalized third-order nonlinear Schrödinger equation by two mathematical methods and their stability, *Pramana*, **93** (2019), 44. https://doi.org/10.1007/s12043-019-1804-5

33. M. Arshad, D. Lu, M. U. Rehman, I. Ahmed, A. M. Sultan, Optical solitary wave and elliptic function solutions of Fokas-Lenells equation in presence of perturbation terms and its modulation instability, *Phys. Scripta*, **94** (2019), 105202.

34. A. M. Sultan, D. Lu, M. Arshad, H. U. Rehman, M. S. Saleem, Soliton solutions of higher order dispersive cubic-quintic nonlinear Schrödinger equation and its applications, *Chinese J. Phys.*, **67** (2019), 405–413. https://doi.org/10.1016/j.cjph.2019.10.003

35. D. Lu, C. Yue, M. Arshad, Traveling wave solutions of spacetime fractional generalized fifth order KdV equation, *Adv. Math. Phys.*, **2017** (2017), 6743276. https://doi.org/10.1155/2017/6743276

36. M. Arshad, A. R. Seadawy, D. Lu, J. Wang, Travelling wave solutions of generalized coupled Zakharov-Kuznetsov and dispersive long wave equations, *Results Phys.*, **6** (2016), 1136–1145. https://doi.org/10.1016/j.rinp.2016.11.043

37. M. Arshad, A. R. Seadawy, D. Lu, Exact bright-dark solitary wave solutions of the higher-order cubic-quintic nonlinear Schrödinger equation and its stability, *Optik*, **138** (2017), 40–49. https://doi.org/10.1016/j.ijleo.2017.03.005

38. M. Arshad, D. Lu, J. Wang, (N+1)-dimensional fractional reduced differential transform method for fractional order partial differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **48** (2017), 509–519. https://doi.org/10.1016/j.cnsns.2017.01.018

39. A. A. Omar, Application of residual power series method for the solution of time-fractional Schrödinger equations in one-dimensional space, *Fund. Inform.*, **166** (2019), 87–110. https://doi.org/10.3233/FI-2019-1795

40. C. Q. Dai, Y. Y. Wang, Coupled spatial periodic waves and solitons in the photovoltaic photorefractive crystals, *Nonlinear Dyn.*, **102** (2020), 1733–1741. https://doi.org/10.1007/s11071-020-05985-w

41. C. Y. Ma, B. Shiri, G. C. Wu, D. Baleanu, New fractional signal smoothing equations with short memory and variable order, *Optik*, **218** (2020), 164507. https://doi.org/10.1016/j.ijleo.2020.164507

42. B. H. Wang, Y. Y. Wang, C. Q. Dai, Y. X. Chen, Dynamical characteristic of analytical fractional solitons for the space-time fractional Fokas-Lenells equation, *Alex. Eng. J.*, **59** (2020), 4699–4707. https://doi.org/10.1016/j.aej.2020.08.027

43. M. S. Osman, H. I. Abdel-Gawad, Multi-wave solutions of the (2+1)-dimensional Nizhnik-Novikov-Veselov equations with variable coefficients, *Eur. Phys. Jour. Plus*, **130** (2015), 215. https://doi.org/10.1140/epjp/i2015-15215-1

44. K. K. Ali, M. S. Osman, M. Abdel-Aty, New optical solitary wave solutions of Fokas-Lenells equation in optical fiber via Sine-Gordon expansion method, *Alex. Eng. J.*, **59** (2020), 1191–1196. https://doi.org/10.1016/j.aej.2020.01.037

45. I. Siddique, M. M. M. Jaradat, A. Zafar, K. Bukht Mehdi, M. S. Osman, Exact traveling wave solutions for two prolific conformable M-Fractional differential equations via three diverse approaches, *Results Phys.*, **28** (2021), 104557. https://doi.org/10.1016/j.rinp.2021.104557

46. H. F. Ismael, S. S. Atas, H. Bulut, M. S. Osman, Analytical solutions to the M-derivative resonant Davey-Stewartson equations, *Mod. Phys. Lett. B*, **35** (2021), 2150455. https://doi.org/10.1142/S0217984921504558

47. H. Rezazadeh, M. Inc, D. Baleanu, New solitary wave solutions for variants of (3+1)-dimensional Wazwaz-Benjamin-Bona-Mahony equations, *Front. Phys.*, **8** (2020), 332. https://doi.org/10.3389/fphy.2020.00332

48. F. Meng, Q. Feng, A new rractional Subequation method and its applications for space-time fractional partial differential equations, *J. Appl. Math.*, **2013** (2013), 481729. https://doi.org/10.1155/2013/481729
49. Y. Huang, Y. Shang, The extended hyperbolic function method for generalized forms of nonlinear heat conduction and Huxley equations, *J. Appl. Math.*, **2012** (2012), 769843. https://doi.org/10.1155/2012/769843
50. Y. Shang, The extended hyperbolic function method and exact solutions of the long-short wave resonance equations, *Chaos, Solitons Fract.*, **36** (2008), 762–771. https://doi.org/10.1016/j.chaos.2006.07.007
51. Y. Shang, Y. Huang, W. Yuan, The extended hyperbolic functions method and new exact solutions to the Zakharov equations, *Appl. Math. Comput.*, **200** (2008), 110–122. https://doi.org/10.1016/j.amc.2007.10.059
52. S. Nestor, A. Houwe, G. Betchewe, M. Inc, S. Y. Doka, A series of abundant new optical solitons to the conformable space-time fractional perturbed nonlinear Schrödinger equation, *Phys. Scripta*, **95** (2020), 085108.
53. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, **1** (2015), 1–13.
54. M. Caputo, Linear models of dissipation whose Q is almost frequency independent, *Geophys. J. Int.*, **13** (1967), 529–539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x
55. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769.
56. R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, A new definition of fractional derivative, *J. Comput. Appl. Math.*, **264** (2014), 65–70. https://doi.org/10.1016/j.cam.2014.01.002
57. C. S. Liu, Counter examples on Jumarie's two basic fractional calculus formulae, *Commun. Nonlinear Sci. Numer. Simul.*, **22** (2015), 92–94. https://doi.org/10.1016/j.cnsns.2014.07.022
58. C. S. Liu, Counterexamples on Jumarie's three basic fractional calculus formulae for non-differentiable continuous functions, *Chaos, Solitons, Fractals*, **109** (2018), 219–222. https://doi.org/10.1016/j.chaos.2018.02.036

AIMS Press