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On Some Generalizations of Integral Inequalities in n Independent Variables and Their Applications

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Abstract: Throughout this article, generalizations of some Grónwall–Bellman integral inequalities for two real-valued unknown functions in n independent variables are introduced. We are looking at some novel explicit bounds of a particular class of Young and Pachpatte integral inequalities. The results in this paper can be utilized as a useful way to investigate the uniqueness, boundedness, continuousness, dependence and stability of nonlinear hyperbolic partial integro-differential equations. To highlight our research advantages, several implementations of these findings will be presented. Young's method, which depends on a Riemann method, will follow to prove the key results. Symmetry plays an essential role in determining the correct methods for solving dynamic inequalities.

Keywords: integral inequalities; hyperbolic partial differential equation; Young's technique

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1. Introduction

Gronwall–Bellman's inequality [1] in the integral form states the following: Let u and f be continuous and nonnegative functions defined on $[a,b]$, and let u_0 be a nonnegative constant. Then, the inequality

$$u(t) \leq u_0 + \int_a^t f(s)u(s)ds, \quad \text{for all} \quad t \in [a,b], \tag{1}$$

implies that

$$u(t) \leq u_0 \exp\left(\int_a^t f(s)ds\right), \quad \text{for all} \quad t \in [a,b].$$

Baburao G. Pachpatte [2] proved the discrete version of Equation (1). In particular, he proved that if $u(n)$, $a(n)$, and $\gamma(n)$ are nonnegative sequences defined for $n \in \mathbb{N}_0$, $a(n)$ is non-decreasing for $n \in \mathbb{N}_0$, and if

$$u(n) \leq a(n) + \sum_{s=0}^{n-1} \gamma(n)u(n), n \in \mathbb{N}_0, \tag{2}$$

then

$$u(n) \leq a(n) \prod_{s=0}^{n-1} [1 + \gamma(n)], n \in \mathbb{N}_0.$$

The authors of [3] studied the following result:

$$\Psi(u(\ell,t)) \leq a(\ell,t) + \int_0^{\theta(\ell)}\int_0^{\vartheta(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(u(\varsigma,\eta))\varpi(u(\varsigma,\eta)) + \int_0^{\varsigma} \Im_2(\chi,\eta)\zeta(u(\chi,\eta))\varpi(u(\chi,\eta))d\chi\Big] d\eta d\varsigma,$$

where u, f, $\Im \in C(I_1 \times I_2, \mathbb{R}_+)$, and $a \in C(\zeta, \mathbb{R}_+)$ are non-decreasing functions, $I_1, I_2 \in \mathbb{R}$, $\theta \in C^1(I_1, I_1)$, and $\vartheta \in C^1(I_2, I_2)$ are non-decreasing with $\theta(\ell) \leq \ell$ on I_1, $\vartheta(t) \leq t$ on I_2, $\Im_1$, $\Im_2 \in C(\zeta, \mathbb{R}_+)$, and Ψ, ζ, $\varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\{\Psi, \zeta, \varpi\}(u) > 0$ for $u > 0$, and $\lim_{u\to+\infty} \Psi(u) = +\infty$.

Additionally, Anderson [4] studied the following result:

$$\omega(u(t,s)) \leq a(t,s) + c(t,s) \int_{t_0}^{t} \int_{s}^{\infty} \omega'(u(\tau,\eta))[d(\tau,\eta)w(u(\tau,\eta)) + b(\tau,\eta)]\nabla\eta\Delta\tau, \quad (3)$$

where u, a, c, and d are nonnegative continuous functions defined for $(t,s) \in \mathbb{T} \times \mathbb{T}$, b is a nonnegative continuous function for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$ and $\omega \in C^1(\mathbb{R}_+, \mathbb{R}_+)$, with $\omega' > 0$ for $u > 0$.

Wendroff's inequality, see [5], states the following: Let $\psi(\xi,\rho)$ and $\phi(\xi,\rho)$ be nonnegative and continuous functions where $\xi, \rho \in \mathbb{R}_+$. If

$$\psi(\xi,\rho) \leq A_1(\xi) + A_2(\rho) + \int_0^{\xi}\int_0^{\rho} \phi(\theta,t)\psi(\theta,t)d\theta dt, \quad (4)$$

holds for $\xi, \rho \in \mathbb{R}_+$, where $A_1(\xi)$ and $A_2(\rho)$ are continuous and positive functions on ξ, $\rho \in \mathbb{R}_+$, and the derivatives $A_1'(\xi)$ and $A_2'(\rho)$ on $\xi, \rho \in \mathbb{R}_+$ are nonnegative, then

$$\psi(\xi,\rho) \leq E(\xi,\rho)\exp\left(\int_0^{\xi}\int_0^{\rho} \phi(\theta,t)d\theta dt\right),$$

on $\xi, \rho \in \mathbb{R}_+$, where

$$E(\xi,\rho) = \frac{[A_2(\rho) + A_1(0)][A_1(\xi) + A_2(0)]}{A_1(0) + A_2(0)},$$

on $\xi, \rho \in \mathbb{R}_+$.

Subsequently, some new Wendroff-type inequalities were developed (see, for example, [6,7]) to provide natural and effective means to further develop the theory of integral and partial integro-differential equations.

Wendroff's inequality (Inequality (4)) has gained significant attention, and numerous articles have been published in the literature involved various extensions, generalizations, and applications [5–22].

For example, Bondge and Pachpatte [7] investigated some simple Wendroff-type inequalities with n independent variables as follows: Let $\psi(t)$, $P(t)$, and $Q(t)$ be continuous and nonnegative functions defined on Ω and $\Psi i(\xi i) > 0$ and $\Psi' i(\xi i) \geq 0$ for $1 \leq i \leq n$ be continuous functions defined for $\xi i \geq \xi i^0$:

(i) If

$$\psi(\xi) \leq \sum_{i=1}^{n} \Psi_i(\xi_i) + \int_{\xi^0}^{\xi} P(\rho)\psi(\rho)d\rho,$$

for $\xi \in \Omega$, then

$$\psi(\xi) \leq E(\xi)\exp\left(\int_{\xi^0}^{\xi} P(\rho)d\rho\right),$$

for $\xi \in \Omega$, where

$$E(\xi) = \frac{\left[\sum_{i=1}^{n} \Psi_i(\varsigma_i) + \Psi_1(\varsigma_1^0) - \Psi_1(\xi_1)\right]\left[\sum_{i=1}^{n} \Psi_i(\xi_i) + \Psi_2(\xi_2^0) - \Psi_2(\xi_2)\right]}{\left[\sum_{i=1}^{n} \Psi_i(\xi_i) + \Psi_1(\xi_1^0) + \Psi_2(\xi_2)\right]}. \tag{5}$$

(ii) If

$$\psi(\xi) \le \sum_{i=1}^{n} \Psi_i(\xi_i) + \int_{\xi^0}^{\xi} P(\rho)\psi(\rho)d\rho + \int_{\xi^0}^{\xi} P(\rho)\psi(\rho)\left(\int_{\xi^0}^{\rho} Q(\theta)\psi(\theta)d\theta\right)d\rho,$$

for $\xi \in \Omega$, then

$$\psi(\xi) \le \sum_{i=1}^{n} \Psi_i(\xi_i) + \int_{\xi^0}^{\xi} P(\rho)E(\rho)(\rho)\exp\left(\int_{\xi^0}^{\rho} [P(\theta)Q(\theta)]d\theta\right)d\rho,$$

for $\xi \in \Omega$, where $E(\xi)$ is defined by Equation (5).

An extension of Snow's technique of n independent variables was performed by Young [15]. His inequality has several valuable applications in the theory of integro-differential and partial differential equations with n independent variables. He considered that $\phi(\xi)$, $\Psi(\xi) \ge 0$, and $\Phi(\xi)$ are continuous functions on $\Omega \subset \mathbb{R}^n$. Let $\nu(\tau;\xi)$ be a solution of the characteristic initial value problem

$$\begin{aligned} (-1)^n \nu_{\tau_1\ldots\tau_n}(\tau;\xi) &= 0 \quad \text{in} \quad \Omega, \\ \nu(\tau;\xi) &= 1 \quad \text{on} \quad \tau_i = \varsigma_i, \quad i = 1,\ldots,n \end{aligned}$$

In addition, let D^+ be a connected subdomain of Ω containing ξ such that $\nu \ge 0$ for all $\tau \in D^+$. If $D \subset D^+$ and

$$\phi(\xi) \le \Psi(\xi) + \int_{\xi^0}^{\xi} \Phi(\tau)\phi(\tau)d\tau,$$

then

$$\phi(\xi) \le \Psi(\xi) + \int_{\xi^0}^{\xi} \Psi(\tau)\Phi(\tau)\nu(\tau)d\tau.$$

Motivated by the inequalities mentioned above, we prove more general integral inequalities with n independent variables by using Young's technique. The proposed general integral inequalities can be employed in the analysis of many problems in the theory of integral and partial differential equations, which could easily be considered powerful tools. Symmetry plays an essential role in determining the correct methods for solving dynamic inequalities.

2. Auxiliary Results

First, we state and prove two important lemmas, and we will use them to prove the main results of this paper. To prove the following lemma, Bellman's technique (see, for instance, [16]) will be applied.

Lemma 1. *Let $\phi(\xi)$ and $\nu(\xi)$ be real-valued, positive, and continuous functions. In addition, let all derivatives of $\phi(\xi)$ be positive on Ω with $\phi(\xi) = 1$ on $\xi_i = \xi_i^o$. If the inequality*

$$D\phi(\xi) \le \nu(\xi)\phi(\xi), \tag{6}$$

holds, then

$$\phi(\xi) \le \exp\left(\int_{\xi^o}^{\xi} \nu(t)dt\right). \tag{7}$$

Proof. The inequality (6) leads to

$$\frac{\phi(\xi)D\phi(\xi)}{\phi^2(\xi)} \leq \nu(\xi).$$

Thus, by the assumptions on $f(\varsigma)$ and its derivatives, we have

$$\frac{\phi(\xi)D\phi(\xi)}{\phi^2(\xi)} \leq \nu(\xi) + \frac{(D_n\phi(\xi))(D_1 \dots D_{n-1}\phi(\xi))}{f^2(\xi)},$$

which implies that

$$D_n\left(\frac{D_1 \dots D_{n-1}\phi(\xi)}{\phi(\xi)}\right) \leq \nu(\xi). \tag{8}$$

Integrate both sides of the inequality (8) with respect to the component ξ_n from ξ_n^o to ξ_n to obtain

$$\frac{D_1 \dots D_{n-1}\phi(\xi)}{\phi(\xi)} \leq \int_{\xi_n^o}^{\xi_n} \nu(\xi_1, \dots, \xi_{n-1}, t_n)dt_n.$$

Therefore, and by the assumptions $\phi(\xi)$ and its derivatives, we can write the following inequality:

$$\frac{\phi(\xi)D_1 \dots D_{n-1}\phi(\xi)}{\phi^2(\xi)} \leq \int_{\xi_n^o}^{\xi_n} \nu(\xi_1, \dots, \xi_{n-1}, t_n)dt_n + \frac{(D_{n-1}\phi(\xi))(D_1 \dots D_{n-2}\phi(\xi))}{\xi^2(\xi)},$$

which yields

$$D_{n-1}\left(\frac{D_1 \dots D_{n-2}\phi(\xi)}{\phi(\xi)}\right) \leq \int_{\xi_n^o}^{\xi_n} \nu(\xi_1, \dots, \xi_{n-1}, t_n)dt_n. \tag{9}$$

Now, integrate both sides of inequality (9) with respect to the component ξ_{n-1} from ξ_{n-1}^o to ξ_{n-1} to obtain

$$\frac{D_1 \dots D_{n-2}\phi(\xi)}{\phi(\xi)} \leq \int_{\xi_{n-1}^o}^{\xi_{n-1}} \int_{\xi_n^o}^{\xi_n} \nu(\xi_1, \dots \xi_{n-2}, t_{n-1}, t_n)dt_n dt_{n-1}.$$

We can continue this way until reaching

$$\frac{D_1\phi(\xi)}{\phi(\xi)} \leq \int_{\xi_2^o}^{\xi_2} \dots \int_{\xi_n^o}^{\xi_n} \nu(\xi_1, t_2, \dots, t_n)dt_n \dots dt_2. \tag{10}$$

Integrating both sides of inequality (10) with respect to the component ξ_1 from ξ_1^o to ξ_1 gives

$$\log\left(\frac{\phi(\xi)}{\phi(\xi_1^o, \xi_2, \dots, \xi_n)}\right) \leq \int_{\xi^o}^{\xi} \nu(t)dt,$$

which implies inequality (7). This proves the lemma. □

To prove the following lemma, Young's technique (see, for instance, [17]) will be applied:

Lemma 2. *Let $K(\xi)$, $\omega(\xi)$, and $\Lambda(\xi)$ be real-valued nonnegative differentiable functions on Ω. Moreover, suppose that $K(\xi)$ and all its derivatives with respect to $\xi_1, \dots, \xi_n$ up to an order $n-1$ vanish at $\xi_i = \xi_i^o$ for $i = 1, \dots, n$. Let $v(\theta; \xi)$ be the solution of the following characteristic initial value problem:*

$$(-1)^n \frac{\partial^n v(\theta;\xi)}{\partial\theta_1 \dots \partial\theta_n} - \Lambda(\theta)v(\theta;\xi) = 0, \ in\ \Omega,$$
$$v(\theta;\xi) = 1 \ on\ \theta_i = \xi_i, \ i = 1, \dots, n. \tag{11}$$

If the inequality

$$DK(\xi) \le \omega(\xi) + \Lambda(\xi)K(\xi), \tag{12}$$

holds, then

$$K(\xi) \le \int_{\xi^o}^{\xi} \omega(\theta)v(\theta;\varsigma)d\theta. \tag{13}$$

Proof. The inequality (12) implies that

$$\mathcal{L}[K(\xi)] \le \omega(\xi), \ \textit{where}\ \mathcal{L} \equiv D - \Lambda(\xi). \tag{14}$$

If $z(\xi)$ is a function that is continuously differentiable n times in the parallelepiped $\xi^o < t < \xi$ (denoted by $\mathcal{D}$), then

$$z\mathcal{L}[K] - K\mathcal{L}_1[z] = \sum_{j=1}^{n}(-1)^{j-1}D_j[(D_0 \dots D_{j-1}z)(D_{j+1}\dots D_n D_{n+1}K)], \tag{15}$$

where $\mathcal{L}_1 \equiv (-1)^n D - \Lambda(\xi)$ and $D_0 = D_{n+1} = I$ is the identity operator. Integrating both sides of Equation (15) over $\mathcal{D}$ and taking into account that $K(\xi)$ and all of its derivatives with respect to $\xi_1, \dots, \xi_n$ up to the order $n-1$ vanish at $\theta_i = \xi_i^o$ for $i = 1, \dots, n$ produces

$$\int_{\mathcal{D}}(z\mathcal{L}[K] - K\mathcal{L}_1[z])d\theta = \sum_{j=1}^{n}(-1)^{j-1}\int_{\theta_j=\xi_j}(D_1\dots D_{j-1}z)(D_{j+1}\dots D_n K)d\theta', \tag{16}$$

where $d\theta' = d\theta_1 \dots d\theta_{j-1}d\theta_{j+1}\dots d\theta_n$. Now, we choose $z(\xi)$ to be the function $v(\theta;\xi)$ that satisfies the IVP (11). Since $v(\theta;\xi) = 1$ on $\theta_j = \xi_j$ for $j = 1, \dots, n$, it follows that

$$D_1 \dots D_{j-1}v(\theta;\xi) = 0, \ \textit{on}\ \ \theta_j = \xi_j,\ j = 2, \dots, n.$$

Therefore, Equation (16) becomes

$$\begin{aligned}\int_{\mathcal{D}} v\mathcal{L}[K]d\theta &= \int_{\theta_1=\xi_1} D_2 \dots D_n K\, d\theta' \\ &= K(\xi).\end{aligned} \tag{17}$$

The continuity of v along with the fact that $v = 1$ on $\theta = \xi$ leads to the existence of a domain Ω^+ containing ξ for which $v \ge 0$. We multiply both sides of inequality (14) by v and then use Equation (17) to obtain inequality (13). Tis proves the lemma. □

Now, we are ready to state and prove our main results.

3. Results and Discussion

In this section, the main results of this paper are stated and proven in Theorems 1–3. This is accomplished by using Lemmas 1 and 2:

Theorem 1. *Let $t_i(\varsigma), b_i(\varsigma), q_i(\varsigma), e_i(\varsigma), f_i(\varsigma), g_i(\varsigma)$, and $h_i(\varsigma)$ be nonnegative, real-valued continuous functions on Ω and $a_i(\varsigma)$ be positive, nondecreasing, and continuous functions on Ω, where $i = 1, 2$. Assume that the system*

$$\begin{aligned}t_i(\varsigma) \le a_i(\varsigma) &+ \int_{\varsigma^o}^{\varsigma} b_i(v)t_1(v)ds + \int_{\varsigma^o}^{\varsigma} q_i(v)t_2(v)ds + \int_{\varsigma^o}^{\varsigma} e_i(v)\left(\int_{\varsigma^o}^{v} f_i(c)t_1(c)dt\right)ds \\ &+ \int_{\varsigma^o}^{\varsigma} g_i(v)\left(\int_{\varsigma^o}^{v} h_i(c)t_2(c)dt\right)ds,\end{aligned} \tag{18}$$

is satisfied for all $\varsigma \in \Omega$ with $\varsigma \geq \varsigma^o$. Then, we have

$$t_i(\varsigma) \leq a_i(\varsigma)\left(1+\int_{\varsigma^o}^{\varsigma}\left(\phi_i(v)\eta(v)ds+\rho_i(v)\int_{\varsigma^o}^{v}\psi_i(c)\eta(c)dt\right)ds\right), \tag{19}$$

where
$\phi_1(\varsigma)=b_1(\varsigma)+\frac{a_2(\varsigma)}{a_1(\varsigma)}q_1(\varsigma)$, $\phi_2(\varsigma)=q_2(\varsigma)+\frac{a_1(\varsigma)}{a_2(\varsigma)}b_2(\varsigma)$, $\phi(\varsigma)=\sum_{i=1}^{2}\phi_i(\varsigma)$,
$\psi_1(\varsigma)=f_1(\varsigma)+\frac{a_2(\varsigma)}{a_1(\varsigma)}h_1(\varsigma)$, $\psi_2(\varsigma)=h_2(\varsigma)+\frac{a_1(\varsigma)}{a_2(\varsigma)}f_2(\varsigma)$, $\psi(\varsigma)=\sum_{i=1}^{2}\psi_i(\varsigma)$,
$\rho_i(\varsigma)=e_i(\varsigma)+g_i(\varsigma)$, *and* $\eta(\varsigma)=2+2\int_{\varsigma^o}^{\varsigma}\phi(v)\exp\left(\int_{\varsigma^o}^{v}(\phi(c)+\psi(c))dt\right)ds$.

Proof. Assuming that the functions $a_i(\varsigma),\ i=1,2$ are positive and nondecreasing functions, this allows us to rewrite the system in (18) in the following form:

$$\frac{t_i(\varsigma)}{a_i(\varsigma)} \leq \mathcal{T}_i(\varsigma), \tag{20}$$

where

$$\begin{aligned}\mathcal{T}_i(\varsigma) = 1+&\int_{\varsigma^o}^{\varsigma} b_i(v)\frac{t_1(v)}{a_i(v)}ds+\int_{\varsigma^o}^{\varsigma} q_i(v)\frac{t_2(v)}{a_i(v)}ds+\int_{\varsigma^o}^{\varsigma} e_i(v)\left(\int_{\varsigma^o}^{v} f_i(c)\frac{t_1(c)}{a_i(c)}dt\right)ds\\ &+\int_{\varsigma^o}^{\varsigma} g_i(v)\left(\int_{\varsigma^o}^{v} h_i(c)\frac{t_2(c)}{a_i(c)}dt\right)ds;\\ &\mathcal{T}_i(\varsigma)=1 \quad on \quad \varsigma_i=\varsigma_i^o, i=1,\dots,n.\end{aligned} \tag{21}$$

For $i=1$, we differentiate both sides of Equation (21) and then use inequality (20) to obtain

$$\begin{aligned}D\mathcal{T}_1(\varsigma) \leq b_1(\varsigma)\mathcal{T}_1(\varsigma)+q_1(\varsigma)\frac{a_2(\varsigma)}{a_1(\varsigma)}\mathcal{T}_2(\varsigma)+e_1(\varsigma)\int_{\varsigma^o}^{\varsigma} f_1(v)\mathcal{T}_1(v)ds\\ +g_1(\varsigma)\int_{\varsigma^o}^{\varsigma} h_1(\varsigma)\frac{a_2(v)}{a_1(v)}\mathcal{T}_2(v)ds.\end{aligned} \tag{22}$$

Since all functions are nonnegative, the inequality (22) takes the following form:

$$D\mathcal{T}_1(\varsigma) \leq \phi_1(\varsigma)\mathcal{T}(\varsigma)+\rho_1(\varsigma)\int_{\varsigma^o}^{\varsigma}\psi_1(v)\mathcal{T}(v)ds, \tag{23}$$

where $\mathcal{T}(\varsigma)=\sum_{i=1}^{2}\mathcal{T}_i(\varsigma)$. Similarly, for $\mathcal{T}_2(\varsigma)$, we have

$$D\mathcal{T}_2(\varsigma) \leq \phi_2(\varsigma)\mathcal{T}(\varsigma)+\rho_2(\varsigma)\int_{\varsigma^o}^{\varsigma}\psi_2(v)\mathcal{T}(v)ds. \tag{24}$$

Adding inequalities (23) and (24) gives

$$D\mathcal{T}(\varsigma) \leq \phi(\varsigma)\mathcal{T}(\varsigma)+\rho(\varsigma)\int_{\varsigma^o}^{\varsigma}\psi(v)\mathcal{T}(v)ds. \tag{25}$$

Adding and subtracting $\rho(\varsigma)\mathcal{T}(\varsigma)$ to the right-hand side of inequality (25) yields

$$D\mathcal{T}(\varsigma) \leq (\phi(\varsigma)-\rho(\varsigma))\mathcal{T}(\varsigma)+\rho(\varsigma)\mathcal{Z}(\varsigma), \tag{26}$$

where

$$\begin{aligned}&\mathcal{Z}(\varsigma)=\mathcal{T}(\varsigma)+\int_{\varsigma^o}^{\varsigma}\psi(v)\mathcal{T}(v)ds, \quad \textit{which implies that}\\ &\mathcal{Z}(\varsigma)\geq\mathcal{T}(\varsigma), \quad \mathcal{Z}(\varsigma)\geq\int_{\varsigma^o}^{\varsigma}\psi(v)\mathcal{T}(v)ds, \quad and \quad \mathcal{Z}(\varsigma^o)=\mathcal{T}(\varsigma^o)=2.\end{aligned} \tag{27}$$

Using inequality (27) permits writing inequality (26) in the following form:

$$D\mathcal{T}(\varsigma) \leq \phi(\varsigma)\mathcal{Z}(\varsigma). \tag{28}$$

On the other hand, the relation (27) along with the inequality (28) allows us to write

$$D\mathcal{Z}(\varsigma) \leq (\phi(\varsigma) + \psi(\varsigma))\mathcal{Z}(\varsigma). \tag{29}$$

We can apply Lemma 1 on the inequality (29) to obtain

$$\mathcal{Z}(\varsigma) \leq 2\exp\left(\int_{\varsigma^o}^{\varsigma} [\phi(v) + \psi(v)]ds\right). \tag{30}$$

By using the upper bound in inequality (30) on $\mathcal{Z}(\varsigma)$ in the inequality (28) and then integrating both sides of the resulting inequality with respect to ς from ς^o to ς, we obtain

$$\mathcal{T}(\varsigma) \leq \eta(\varsigma). \tag{31}$$

Utilizing the inequality (31) in the inequality (23) gives

$$D\mathcal{T}_1(\varsigma) \leq \phi_1(\varsigma)\eta(\varsigma) + \rho_1(\varsigma)\int_{\varsigma^o}^{\varsigma} \psi_1(v)\eta(v)ds,$$

which, by integration with respect to ς from ς^o to ς, yields

$$\mathcal{T}_1(\varsigma) \leq 1 + \int_{\varsigma^o}^{\varsigma}\left(\phi_1(v)\eta(v) + \rho_1(v)\int_{\varsigma^o}^{v} \psi_1(c)\eta(c)dt\right)ds. \tag{32}$$

Similarly, we have the following upper bound for $\mathcal{T}_2(\varsigma)$:

$$\mathcal{T}_2(\varsigma) \leq 1 + \int_{\varsigma^o}^{\varsigma}\left(\phi_2(v)\eta(v) + \rho_2(v)\int_{\varsigma^o}^{v} \psi_2(c)\eta(c)dt\right)ds. \tag{33}$$

Embedding these upper bounds from inequalities (32) and (33) on $\mathcal{T}_1(\varsigma)$ and $\mathcal{T}_2(\varsigma)$, respectively, into the inequality (20) produces the system's solution (inequality (19)). This proves the theorem. □

Remark 1. *If $n = 2$ (i.e., we are dealing with functions in two variables), and $\varsigma^o = 0$, then Theorem 1 yields that for $i = 1, 2$, if the system*

$$\begin{aligned} t_i(\varsigma, \Im) \leq a_i(\varsigma, \Im) &+ \int_0^{\varsigma}\int_0^{\Im} b_i(v,c)t_1(v,c)dtds + \int_0^{\varsigma}\int_0^{\Im} q_i(v,c)t_2(v,c)dtds \\ &+ \int_0^{\varsigma}\int_0^{\Im} e_i(v,c)\left(\int_0^{v}\int_0^{t} f_i(r,\theta)t_1(r,\theta)d\theta dr\right)dtds \\ &+ \int_0^{\varsigma}\int_0^{\Im} g_i(v,c)\left(\int_0^{v}\int_0^{t} h_i(r,\theta)t_2(r,\theta)d\theta dr\right)dtds, \end{aligned} \tag{34}$$

holds, then

$$t_i(\varsigma, \Im) \leq a_i(\varsigma, \Im)\left(1 + \int_0^{\varsigma}\int_0^{\Im}\left(\phi_i(v,c)\eta(v,c) + \rho_i(v,c)\int_0^{v}\int_0^{t} \psi_i(r,\theta)\eta(r,\theta)d\theta dr\right)dtds\right), \tag{35}$$

where

$\phi_1(v,c) = b_1(v,c) + \frac{a_2(v,c)}{a_1(v,c)}q_1(v,c)$, $\phi_2(v,c) = q_2(v,c) + \frac{a_1(v,c)}{a_2(v,c)}b_2(v,c)$, $\phi(\varsigma, \Im) = \sum_{i=1}^{2}\phi_i(\varsigma, \Im)$, $\psi_1(v,c) = f_1(v,c) + \frac{a_2(v,c)}{a_1(v,c)}h_1(v,c)$, $\psi_2(v,c) = h_2(v,c) + \frac{a_1(v,c)}{a_2(v,c)}f_2(v,c)$, $\psi(\varsigma, \Im) = \sum_{i=1}^{2}\psi_i(\varsigma, \Im)$, $\rho_i(\varsigma, \Im) = e_i(\varsigma, \Im) + g_i(\varsigma, \Im)$, *and*

$\eta(\varsigma, \Im) = 2 + 2\int_0^{\varsigma}\int_0^{\Im}\phi(v,c)\exp\left(\int_0^{v}\int_0^{t}(\phi(r,\theta) + \psi(r,\theta))d\theta dr\right)dtds.$

Theorem 2. *Suppose that* $t_i(\varsigma), a_i(\varsigma), e_i(\varsigma), g_i(\varsigma), f_i(\varsigma), h_i(\varsigma), Db_i(\varsigma)$, *and* $Dc_i(\varsigma)$, *where* $i = 1, 2$, *are real-valued, nonnegative, continuous, and non-decreasing functions defined on* Ω. *Assume that the system*

$$t_i(\varsigma) \le a_i(\varsigma) + \int_{\varsigma^o}^{\varsigma} b_i(\varsigma, v)t_1(v)ds + \int_{\varsigma^o}^{\varsigma} c_i(\varsigma, v)t_2(v)ds + \int_{\varsigma^o}^{\varsigma} e_i(v)\left(\int_{\varsigma^o}^{v} f_i(\varsigma, c)t_1(c)dt\right)ds + \int_{\varsigma^o}^{\varsigma} g_i(v)\left(\int_{\varsigma^o}^{v} h_i(\varsigma, c)t_2(c)dt\right)ds;\ i = 1, 2, \tag{36}$$

is satisfied. Then, we have

$$t_i(\varsigma) \le a_i(\varsigma) + \int_{\varsigma^o}^{\varsigma}\Bigg[\phi_i(v) + \int_{\varsigma^o}^{v} D\beta_i(\varsigma, c)\tau(\varsigma, c)dt + \beta_i(\varsigma, v)\tau(\varsigma, v) + \rho_i(v)\int_{\varsigma^o}^{v} \sigma_i(\varsigma, c)\tau(\varsigma, c)dt + \int_{\varsigma^o}^{v}\left[\rho_i(c)\left(\int_{\varsigma^o}^{t} D\sigma_i(\varsigma, r)\tau(\varsigma, r)dr\right)dt\right]\Bigg]ds;\ i = 1, 2, \tag{37}$$

where

$$\begin{aligned}\phi_i(\varsigma) =& a_1(\varsigma)b_i(\varsigma) + a_2(\varsigma)c_i(\varsigma) + \int_{\varsigma^o}^{\varsigma} a_1(v)Db_i(\varsigma, v)ds + \int_{\varsigma^o}^{\varsigma} a_2(v)Dc_i(\varsigma, v)ds \\ &+ \int_{\varsigma^o}^{\varsigma} e_i(v)\left(\int_{\varsigma^o}^{v} a_1(c)Df_i(\varsigma, c)dt\right)ds + e_i(\varsigma)\int_{\varsigma^o}^{\varsigma} a_1(v)f_i(\varsigma, v)ds \\ &+ \int_{\varsigma^o}^{\varsigma} g_i(v)\left(\int_{\varsigma^o}^{v} a_2(c)Dh_i(\varsigma, c)dt\right)ds + g_i(\varsigma)\int_{\varsigma^o}^{\varsigma} a_2(v)h_i(\varsigma, v)ds,\end{aligned}$$

$\beta_i(\varsigma) = b_i(\varsigma) + c_i(\varsigma), \rho_i(\varsigma) = e_i(\varsigma) + g_i(\varsigma), \sigma_i(\varsigma) = f_i(\varsigma) + h_i(\varsigma)$, $\beta(\varsigma) = \sum_{i=1}^{2} \beta_i(\varsigma), \rho(\varsigma) = \sum_{i=1}^{2} \rho_i(\varsigma), \sigma(\varsigma) = \sum_{i=1}^{2} \sigma_i(\varsigma), \phi(\varsigma) = \sum_{i=1}^{2} \phi_i(\varsigma)$,

$\tau(\varsigma, \vartheta) = \int_{\varsigma^o}^{\vartheta} Y(\varsigma, \varrho)\psi_2(\varrho)d\varrho$; $Y(\varsigma; v)$ *is the solution to the following characteristic initial value problem:*

$$\begin{aligned}(-1)^n \frac{\partial^n Y(\varsigma; v)}{\partial v_1 \dots \partial v_n} - [\beta(\varsigma) - D\beta - \rho(\varsigma)\sigma(\varsigma)]Y(\varsigma; v) = 0,\ in\ \Omega, \\ Y(\varsigma; v) = 1\ on\ v_i = \varsigma_i,\ i = 1, \dots, n,\end{aligned} \tag{38}$$

$\psi_2 = \phi(\varsigma) + (D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\psi_1(\varsigma) + \int_{\varsigma^o}^{\varsigma} \rho(v)D\sigma(\varsigma, v)\psi_1(v)ds$,

$\psi_1(\varsigma) = \int_{\varsigma^o}^{\varsigma}\left(\phi(v) + \rho(v)D\sigma(\varsigma, v)\int_{\varsigma^o}^{v} \phi(c)v(\varsigma; c)dt\right)w(\varsigma; v)ds$ *such that* $v(\varsigma; v)$ *is the solution to the initial value problem*

$$\begin{aligned}(-1)^n \frac{\partial^n v(\varsigma; v)}{\partial v_1 \dots \partial v_n} - [\beta(\varsigma) + D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma) + \rho(\varsigma)D\sigma(\varsigma) + 2]v(\varsigma; v) = 0,\ in\ \Omega, \\ v(\varsigma; v) = 1\ on\ v_i = \varsigma_i,\ i = 1, \dots, n.\end{aligned} \tag{39}$$

Additionally, $w(\varsigma; v)$ *is the solution to the initial value problem*

$$\begin{aligned}(-1)^n \frac{\partial^n w(\varsigma; v)}{\partial v_1 \dots \partial v_n} - [\beta(\varsigma) + D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma) - \rho(\varsigma)D\sigma(\varsigma) + 1]w(\varsigma; v) = 0,\ in\ \Omega, \\ w(\varsigma; v) = 1\ on\ v_i = \varsigma_i,\ i = 1, \dots, n.\end{aligned} \tag{40}$$

Proof. Start with

$$\zeta_i(\varsigma) = \int_{\varsigma^o}^{\varsigma} b_i(\varsigma, v)t_1(v)ds + \int_{\varsigma^o}^{\varsigma} c_i(\varsigma, v)t_2(v)ds + \int_{\varsigma^o}^{\varsigma} e_i(v)\left(\int_{\varsigma^o}^{v} f_i(\varsigma, c)t_1(c)dt\right)ds$$
$$+ \int_{\varsigma^o}^{\varsigma} g_i(v)\left(\int_{\varsigma^o}^{v} h_i(\varsigma, c)t_2(c)dt\right)ds;$$
$$\text{and} \quad \zeta(\varsigma) = \sum_{j=1}^{2} \zeta_j(\varsigma); \; i = 1, 2. \tag{41}$$

Thus, the system given in (36) takes the form

$$t_i(\varsigma) \leq a_i(\varsigma) + \zeta_i(\varsigma); \; i = 1, 2. \tag{42}$$

Since all functions are nonnegative and non-decreasing, relation (41), along with Equation (42), gives the following inequalities:

$$D\zeta_i(\varsigma) \leq \phi_i(\varsigma) + \zeta_1(\varsigma)b_i(\varsigma) + \int_{\varsigma^o}^{\varsigma} \zeta_1(v)Db_i(\varsigma, v)ds + \zeta_2(\varsigma)c_i(\varsigma) + \int_{\varsigma^o}^{\varsigma} \zeta_2(v)Dc_i(\varsigma, v)ds$$
$$+ e_i(\varsigma)\int_{\varsigma^o}^{\varsigma} \zeta_1(v)f_i(\varsigma, v)ds + \int_{\varsigma^o}^{\varsigma} e_i(v)\left(\int_{\varsigma^o}^{v} \zeta_1(c)Df_i(c)dt\right)ds$$
$$+ g_i(\varsigma)\int_{\varsigma^o}^{\varsigma} \zeta_2(v)h_i(\varsigma, v)ds + \int_{\varsigma^o}^{\varsigma} g_i(v)\left(\int_{\varsigma^o}^{v} \zeta_2(c)Dh_i(c)dt\right)ds$$
$$\leq \phi_i(\varsigma) + \zeta(\varsigma)\beta_i(\varsigma) + \int_{\varsigma^o}^{\varsigma} \zeta(v)D\beta_i(\varsigma, v)ds + \rho_i(\varsigma)\int_{\varsigma^o}^{\varsigma} \zeta(v)\sigma_i(\varsigma, v)ds$$
$$+ \int_{\varsigma^o}^{\varsigma} \rho_i(v)\left(\int_{\varsigma^o}^{v} \zeta(c)D\sigma_i(c)dt\right)ds; \qquad i = 1, 2. \tag{43}$$

Adding the inequalities in (43) (i.e., for the cases where $i = 1$ and $i = 2$) gives

$$D\zeta(\varsigma) \leq \phi(\varsigma) + \zeta(\varsigma)\beta(\varsigma) + \int_{\varsigma^o}^{\varsigma} \zeta(v)D\beta(\varsigma, v)ds + \rho(\varsigma)\int_{\varsigma^o}^{\varsigma} \zeta(v)\sigma(\varsigma, v)ds$$
$$+ \int_{\varsigma^o}^{\varsigma} \rho(v)\left(\int_{\varsigma^o}^{v} \zeta(c)D\sigma(\varsigma, c)dt\right)ds. \tag{44}$$

Clearly, all functions in the inequality (44) are nonnegative and non-decreasing as well. Therefore, the inequality (44) can be written as

$$D\zeta(\varsigma) \leq \phi(\varsigma) + \zeta(\varsigma)\beta(\varsigma) + (D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\int_{\varsigma^o}^{\varsigma} \zeta(v)ds + \int_{\varsigma^o}^{\varsigma} \rho(v)D\sigma(\varsigma, v)\left(\int_{\varsigma^o}^{v} \zeta(c)dt\right)ds. \tag{45}$$

Adding $(D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\zeta(\varsigma)$ to both sides of inequality (45) produces

$$D\zeta(\varsigma) + (D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\zeta(\varsigma)$$
$$\leq \phi(\varsigma) + \zeta(\varsigma)\beta(\varsigma) + (D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\mathcal{K}_1(\varsigma) + \int_{\varsigma^o}^{\varsigma} \rho(v)D\sigma(\varsigma, v)\left(\int_{\varsigma^o}^{v} \zeta(c)dt\right)ds, \tag{46}$$

where $\mathcal{K}_1(\varsigma) = \zeta(\varsigma) + \int_{\varsigma^o}^{\varsigma} \zeta(v)ds$. This definition of $\mathcal{K}_1(\varsigma)$ implies that

$$\mathcal{K}_1(\varsigma) \geq \zeta(\varsigma),\, \mathcal{K}_1(\varsigma) \geq \int_{\varsigma^o}^{\varsigma} \zeta(v)ds, \text{ and } \mathcal{K}_1(\varsigma^o) = \zeta(\varsigma^o) = 0. \tag{47}$$

From inequalities (47) and (46), we obtain

$$D\zeta(\varsigma) \leq \phi(\varsigma) + \mathcal{K}_1(\varsigma)\beta(\varsigma) + (D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\mathcal{K}_1(\varsigma) + \int_{\varsigma^o}^{\varsigma} \rho(v)D\sigma(\varsigma, v)\mathcal{K}_1(v)ds,$$

which again, by the fact that $\zeta(\varsigma) \leq \mathcal{K}_1(\varsigma)$ and that the functions $\rho(\varsigma)$ and $D\sigma(\varsigma)$ are nonnegative and non-decreasing, implies

$$\begin{aligned} D\mathcal{K}_1(\varsigma) &\leq \phi(\varsigma) + (\beta(\varsigma) + D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\mathcal{K}_1(\varsigma) + \int_{\varsigma^o}^{\varsigma} \rho(v)D\sigma(\varsigma, v)\mathcal{K}_1(v)ds + \mathcal{K}_1(\varsigma) \\ &\leq \phi(\varsigma) + (\beta(\varsigma) + D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma) + 1)\mathcal{K}_1(\varsigma) + \rho(\varsigma)D\sigma(\varsigma)\int_{\varsigma^o}^{\varsigma} \mathcal{K}_1(v)ds. \end{aligned} \tag{48}$$

By adding $\rho(\varsigma)D\sigma(\varsigma)\mathcal{K}_1(\varsigma)$ to both sides of the inequality (48), we have

$$\begin{aligned} D\mathcal{K}_1(\varsigma) + \rho(\varsigma)D\sigma(\varsigma)\mathcal{K}_1(\varsigma) \leq \phi(\varsigma) + (\beta(\varsigma) + D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma) + 1)\mathcal{K}_1(\varsigma) \\ + \rho(\varsigma)D\sigma(\varsigma)\mathcal{K}_2(\varsigma), \end{aligned} \tag{49}$$

where $\mathcal{K}_2(\varsigma) = \mathcal{K}_1(\varsigma) + \int_{\varsigma^o}^{\varsigma} \mathcal{K}_1(v)ds$. This definition of $\mathcal{K}_2(\varsigma)$ together with the inequality (49) leads to

$$D\mathcal{K}_2(\varsigma) - (\beta(\varsigma) + D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma) + \rho(\varsigma)D\sigma(\varsigma) + 2)\mathcal{K}_2(\varsigma) \leq \phi(\varsigma). \tag{50}$$

We can apply Lemma 2 to inequality (50) to find

$$\mathcal{K}_2(\varsigma) \leq \int_{\varsigma^o}^{\varsigma} \phi(v)v(\varsigma; v)ds, \tag{51}$$

where $v(\varsigma; v)$ is the solution to the initial value problem (39).

We can substitute this bound (inequality (51)) onto $\mathcal{K}_2(\varsigma)$ in the inequality (49) to have

$$\begin{aligned} D\mathcal{K}_1(\varsigma) - (\beta(\varsigma) + D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma) - \rho(\varsigma)D\sigma(\varsigma) + 1)\mathcal{K}_1(\varsigma) \\ \leq \phi(\varsigma) + \rho(\varsigma)D\sigma(\varsigma)\int_{\varsigma^o}^{\varsigma} \phi(v)v(\varsigma; v)ds. \end{aligned} \tag{52}$$

By applying Lemma 2 to inequality (52), we obtain

$$\mathcal{K}_1(\varsigma) \leq \psi_1(\varsigma), \tag{53}$$

where $\psi_1(\varsigma) = \int_{\varsigma^o}^{\varsigma} \Big(\phi(v) + \rho(v)D\sigma(\varsigma, v)\int_{\varsigma^o}^{v} \phi(c)v(\varsigma; c)dt\Big) w(\varsigma; v)ds$;

$w(\varsigma; v)$ is the solution to the initial value problem in (40). We can use the bound in inequality (53) on $\mathcal{K}_1(\varsigma)$ along with inequality (47) in the inequality (46) to obtain

$$\begin{aligned} D\zeta(\varsigma) - [\beta(\varsigma) - D\beta(\varsigma) - \rho(\varsigma)\sigma(\varsigma)]\zeta(\varsigma) \\ \leq \phi(\varsigma) + (D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\psi_1(\varsigma) + \int_{\varsigma^o}^{\varsigma} \rho(v)D\sigma(\varsigma, v)\psi_1(v)ds. \end{aligned} \tag{54}$$

An application of Lemma 2 on the inequality (54) gives

$$\zeta(\varsigma) \leq \int_{\varsigma^o}^{\varsigma} \psi_2(v)Y(\varsigma; v)ds, \tag{55}$$

where $\psi_2(\varsigma) = \phi(\varsigma) + (D\beta(\varsigma) + \rho(\varsigma)\sigma(\varsigma))\psi_1(\varsigma) + \int_{\varsigma^o}^{\varsigma} \rho(v)D\sigma(\varsigma, v)\psi_1(v)ds$, and $Y(\varsigma; v)$ is the solution to the initial value problem in (38).

We can use the upper bound in (55) on $\zeta(\varsigma)$ in the inequality (43) to obtain

$$\begin{aligned}\zeta_i(\varsigma) \le \int_{\varsigma^o}^{\varsigma}\Big[&\phi_i(v)+\beta_i(v)\tau(\varsigma,v)+\int_{\varsigma^o}^{v} D\beta_i(\varsigma,c)\tau(\varsigma,c)dt\\ &+\rho_i(v)\int_{\varsigma^o}^{v}\sigma_i(\varsigma,c)\tau(\varsigma,c)dt\\ &+\int_{\varsigma^o}^{v}\rho_i(c)\Big(\int_{\varsigma^o}^{t} D\sigma_i(\varsigma,r)\tau(\varsigma,r)dr\Big)dt\Big]ds;\ i=1,2. \end{aligned} \tag{56}$$

Substituting from inequality (56) in the inequality (42) produces inequality (37). This completes the proof. □

Remark 2. *Let us consider the following system with $n=2$ (i.e., we are dealing with functions in $\mathbb{R}^2$, and $\varsigma^o=0$). For $i=1,2$, if*

$$\begin{aligned} t_i(\varsigma,\Im) \le a_i(\varsigma,\Im) &+ \int_0^{\varsigma}\int_0^{\Im} b_i(v,c)t_1(v,c)dtds + \int_0^{\varsigma}\int_0^{\Im} c_i(v,c)t_2(v,c)dtds\\ &+\int_0^{\varsigma}\int_0^{\Im} e_i(v,c)\Big(\int_0^{v}\int_0^{t} f_i(r,\theta)t_1(r,\theta)d\theta dr\Big)dtds\\ &+\int_0^{\varsigma}\int_0^{\Im} g_i(v,c)\Big(\int_0^{v}\int_0^{t} h_i(r,\theta)t_2(r,\theta)d\theta dr\Big)dtds, \end{aligned}$$

then

$$t_i(\varsigma,\Im) \le a_i(\varsigma,\Im) + \int_0^{\varsigma}\int_0^{\Im}\Big[\phi_i(v,c)+\beta_i(v,c)\tau(v,c)+\rho_i(v,c)\int_0^{v}\int_0^{t}\sigma_i(r,\theta)\tau(r,\theta)d\theta dr\Big]dtds, \tag{57}$$

where

$$\begin{aligned}\phi_i(\varsigma,\Im) = a_1(\varsigma,\Im)b_i(\varsigma,\Im)+a_2(\varsigma,\Im)c_i(\varsigma,\Im)+e_i(\varsigma,\Im)\int_0^{\varsigma}\int_0^{\Im} a_1(v,c)f_i(v,c)dtds\\ + g_i(\varsigma,\Im)\int_0^{\varsigma}\int_0^{\Im} a_2(v,c)h_i(v,c)dtds,\end{aligned}$$

$\beta_i(\varsigma,\Im) = b_i(\varsigma,\Im)+c_i(\varsigma,\Im)$, $\rho_i(\varsigma,\Im) = e_i(\varsigma,\Im)+g_i(\varsigma,\Im)$, $\sigma_i(\varsigma,\Im) = f_i(\varsigma,\Im)+h_i(\varsigma,\Im)$, $\beta(\varsigma,\Im)=\sum_{i=1}^{2}\beta_i(\varsigma,\Im)$, $\rho(\varsigma,\Im)=\sum_{i=1}^{2}\rho_i(\varsigma,\Im)$, $\sigma(\varsigma,\Im)=\sum_{i=1}^{2}\sigma_i(\varsigma,\Im)$, $\phi(\varsigma,\Im)=\sum_{i=1}^{2}\phi_i(\varsigma,\Im)$, $\tau(\varsigma,\Im;v,c)=\int_0^{v}\int_0^{t} Y(\varsigma,\Im;r,\theta)\psi(r,\theta)d\theta dr$; *$Y(\varsigma,\Im;v,c)$ is the solution to the following characteristic initial value problem:*

$$\begin{aligned}\frac{\partial^2 Y(\varsigma,\Im;v,c)}{\partial v\partial t} - [\beta(\varsigma)-\rho(\varsigma)\sigma(\varsigma)]Y(\varsigma,\Im;v,c)=0,\ in\ \Omega,\\ Y(\varsigma,\Im;v,c)=1\ on\ v=\varsigma,\ t=\Im,\end{aligned}$$

$\psi(\varsigma,\Im)=\phi(\varsigma,\Im)+\rho(\varsigma,\Im)\sigma(\varsigma,\Im)\int_0^{\varsigma}\int_0^{\Im}\phi(v,c)w(\varsigma,\Im;v,c)dtds$, *and* $w(\varsigma,\Im;v,c)$ *is the solution to the initial value problem*

$$\begin{aligned}\frac{\partial^2 w(\varsigma,\Im;v,c)}{\partial v\partial t} - [\beta(\varsigma)+\rho(\varsigma)\sigma(\varsigma)+1]w(\varsigma,\Im;v,c)=0,\ in\ \Omega,\\ w(\varsigma,\Im;v,c)=1\ on\ v=\varsigma,\ t=\Im.\end{aligned}$$

Theorem 3. *Let $t_i(\varsigma), p_i(\varsigma), q_i(\varsigma), c_i(\varsigma), f_i(\varsigma), g_i(\varsigma)$, and $h_i(\varsigma)$ be real-valued, positive, continuous functions on Ω, and let $a_i(\varsigma)$ be positive, continuous, non-decreasing functions on Ω; $i=1,2$. In addition, let $H(\alpha)$ be positive, continuous, non-decreasing function satisfying $t^{-1}H(\alpha)\le H(t^{-1}\alpha)$, where $\alpha\ge 0$. Assume that the system*

$$t_i(\varsigma) \le a_i(\varsigma) + \int_{\varsigma^o}^{\varsigma} p_i(v)H(t_1(v))ds + \int_{\varsigma^o}^{\varsigma} q_i(v)H(t_2(v))ds$$
$$+ \int_{\varsigma^o}^{\varsigma} e_i(v)\left(\int_{\varsigma^o}^{v} f_i(c)H(t_1(c))dt\right)ds + \int_{\varsigma^o}^{\varsigma} g_i(v)\left(\int_{\varsigma^o}^{v} h_i(c)H(t_2(c))dt\right)ds, \tag{58}$$

is satisfied for all $\varsigma \in \Omega$ *with* $\varsigma \ge \varsigma^o$. *Then, for* $\varsigma^o \le \varsigma \le \varsigma^*$, *we have*

$$t_i(\varsigma) \le a_i(\varsigma)\left[1 + 2\int_{\varsigma^o}^{\varsigma}\left[\phi_i(v)H(\gamma(v))ds + \rho_i(v)\int_{\varsigma^o}^{v}\psi_i(c)H(\gamma(c))dt\right]ds\right], \tag{59}$$

where $\phi_1(\varsigma) = p_1(\varsigma) + q_1(\varsigma)\frac{a_2(\varsigma)}{a_1(\varsigma)}$, $\phi_2(\varsigma) = p_2(\varsigma)\frac{a_1(\varsigma)}{a_2(\varsigma)} + q_2(\varsigma)$, $\phi(\varsigma) = \sum_{i=1}^{2}\phi_i(\varsigma)$, $\psi_1(\varsigma) = f_1(\varsigma) + h_1(\varsigma)\frac{a_2(\varsigma)}{a_1(\varsigma)}$, $\psi_2(\varsigma) = f_2(\varsigma)\frac{a_1(\varsigma)}{a_2(\varsigma)} + h_2(\varsigma)$, $\psi(\varsigma) = \sum_{i=1}^{2}\psi_i(\varsigma)$, $\rho_i(\varsigma) = e_i(\varsigma) + g_i(\varsigma)$, $\rho(\varsigma) = \sum_{i=1}^{2}\rho_i(\varsigma)$, *and* $G(r) = \int_{r^o}^{r}\frac{ds}{H(v)}$, *while* ς^* *is chosen so that* $G(2) + 2\int_{\varsigma^o}^{\varsigma}(\phi(v) + \rho(v)\int_{\varsigma^o}^{\varsigma}\psi(c)(c)dt)ds \in Dom(G^{-1})$ *and*
$\gamma(\varsigma) = G^{-1}\left(G(2) + 2\int_{\varsigma^o}^{\varsigma}\left(\phi(v) + \rho(v)\int_{\varsigma^o}^{\varsigma}\psi(c)dt\right)ds\right)$.

Proof. Utilizing the assumptions on $a_i(\varsigma)$, where $i = 1,2$, and $H(\alpha)$ allows us to write the system in (58) as follows:

$$\frac{t_i(\varsigma)}{a_i(\varsigma)} \le \mathcal{T}_i(\varsigma); \quad i = 1,2, \tag{60}$$

where

$$\mathcal{T}_1(\varsigma) = 1 + \int_{\varsigma^o}^{\varsigma} p_1(v)H\left(\frac{t_1(v)}{a_1(v)}\right)ds + \int_{\varsigma^o}^{\varsigma} q_1(v)\frac{a_2(v)}{a_1(v)}H\left(\frac{t_2(v)}{a_2(v)}\right)ds$$
$$+ \int_{\varsigma^o}^{\varsigma} e_1(v)\left(\int_{\varsigma^o}^{v} f_1(c)H\left(\frac{t_1(c)}{a_1(c)}\right)dt\right)ds + \int_{\varsigma^o}^{\varsigma} g_1(v)\left(\int_{\varsigma^o}^{v} h_1(c)\frac{a_2(c)}{a_1(c)}H\left(\frac{t_2(c)}{a_2(c)}\right)dt\right)ds;$$
$$\textit{where} \quad \mathcal{T}_1 = 1 \quad \textit{on} \quad \varsigma_i = \varsigma_i^o, \quad i = 1,\dots,n, \tag{61}$$

and

$$\mathcal{T}_2(\varsigma) = 1 + \int_{\varsigma^o}^{\varsigma} p_2(v)\frac{a_1(v)}{a_2(v)}H\left(\frac{t_1(v)}{a_1(v)}\right)ds + \int_{\varsigma^o}^{\varsigma} q_2(v)H\left(\frac{t_2(v)}{a_2(v)}\right)ds$$
$$+ \int_{\varsigma^o}^{\varsigma} e_2(v)\left(\int_{\varsigma^o}^{v} f_2(c)\frac{a_1(c)}{a_2(c)}H\left(\frac{t_1(c)}{a_1(c)}\right)dt\right)ds + \int_{\varsigma^o}^{\varsigma} g_2(v)\left(\int_{\varsigma^o}^{v} h_2(c)H\left(\frac{t_2(c)}{a_2(c)}\right)dt\right)ds;$$
$$\textit{where} \quad \mathcal{T}_2 = 1 \quad \textit{on} \quad \varsigma_i = \varsigma_i^o, \quad i = 1,\dots,n. \tag{62}$$

Now, from relations (60) and (61), we have

$$D\mathcal{T}_1(\varsigma) \le p_1(\varsigma)H(\mathcal{T}_1(\varsigma)) + q_1(\varsigma)\frac{a_2(\varsigma)}{a_1(\varsigma)}H(\mathcal{T}_2(\varsigma)) + e_1(\varsigma)\left(\int_{\varsigma^o}^{\varsigma} f_1(v)H(\mathcal{T}_1(v))ds\right)$$
$$+ g_1(\varsigma)\left(\int_{\varsigma^o}^{\varsigma} h_1(v)\frac{a_2(v)}{a_1(v)}H(\mathcal{T}_2(v))ds\right). \tag{63}$$

Since all functions are positive and H is a non-decreasing function, the inequality (63) can be written as follows:

$$D\mathcal{T}_1(\varsigma) \le 2\phi_1(\varsigma)H(\mathcal{T}(\varsigma)) + 2\rho_1(\varsigma)\left(\int_{\varsigma^o}^{\varsigma}\psi_1(v)H(\mathcal{T}(v))ds\right), \tag{64}$$

where $\phi_1(\varsigma), \rho_1(\varsigma)$, and $\psi_1(\varsigma)$ are as defined in the statement of this theorem and $\mathcal{T}(\varsigma) = \sum_{i=1}^{2}\mathcal{T}_i(\varsigma)$. Similarly, by relations (60) and (62), we obtain

$$D\mathcal{T}_2(\varsigma) \leq 2\phi_2(\varsigma)H(\mathcal{T}(\varsigma)) + 2\rho_2(\varsigma)\left(\int_{\varsigma^o}^{\varsigma} \psi_2(v)H(\mathcal{T}(v))ds\right), \tag{65}$$

where $\phi_2(\varsigma)$, $\rho_2(\varsigma)$ and $\psi_2(\varsigma)$ are as defined in the statement of this theorem.

Now, adding the inequalities in (64) and (65) yields

$$D\mathcal{T}(\varsigma) \leq 2\phi(\varsigma)H(\mathcal{T}(\varsigma)) + 2\rho(\varsigma)\left(\int_{\varsigma^o}^{\varsigma} \psi(v)H(\mathcal{T}(v))ds\right), \tag{66}$$

where $\phi(\varsigma)$, $\rho(\varsigma)$ and $\psi(\varsigma)$ are as defined in the statement of this theorem. Since H is a non-decreasing function and $\varsigma^o \leq v \leq \varsigma$, then $H(\mathcal{T}(v)) \leq H(\mathcal{T}(\varsigma))$, which causes inequality (66) to take the form

$$D\mathcal{T}(\varsigma) \leq 2\left[\phi(\varsigma) + \rho(\varsigma)\int_{\varsigma^o}^{\varsigma} \psi(v)ds\right]H(\mathcal{T}(\varsigma)). \tag{67}$$

By applying Lemma 1, we find from inequality (67) that

$$G(\mathcal{T}(\varsigma)) - G(2) \leq 2\int_{\varsigma^o}^{\varsigma}\left(\phi(v) + \rho(v)\int_{\varsigma^o}^{v} \psi(c)dt\right)ds,$$

which implies that

$$\mathcal{T}(\varsigma) \leq G^{-1}\left(G(2) + 2\int_{\varsigma^o}^{\varsigma}\left(\phi(v) + \rho(v)\int_{\varsigma^o}^{v} \psi(c)dt\right)ds\right),$$

In other words, we have

$$\mathcal{T}(\varsigma) \leq \gamma(\varsigma), \tag{68}$$

where $\gamma(\varsigma)$ is as given in the statement of the theorem. We can use the bound in (68) on $\mathcal{T}(\varsigma)$ in inequality (64) to obtain

$$D\mathcal{T}_1(\varsigma) \leq 2\phi_1(\varsigma)H(\gamma(\varsigma)) + 2\rho(\varsigma)\left(\int_{\varsigma^o}^{\varsigma} \psi_1(v)H(\gamma(v))ds\right). \tag{69}$$

Integrating both sides of inequality (69) with respect to ς from ς^o to ς produces

$$\mathcal{T}_1(\varsigma) \leq 1 + 2\int_{\varsigma^o}^{\varsigma}\left[\phi_1(v)H(\gamma(v)) + \rho_1(v)\left(\int_{\varsigma^o}^{v} \psi_1(c)H(\gamma(c))\right)dt\right]ds. \tag{70}$$

Similarly, we can obtain from inequalities (65) and (68) that

$$\mathcal{T}_2(\varsigma) \leq 1 + 2\int_{\varsigma^o}^{\varsigma}\left[\phi_2(v)H(\gamma(v)) + \rho_2(v)\left(\int_{\varsigma^o}^{v} \psi_2(c)H(\gamma(c))\right)dt\right]ds. \tag{71}$$

Using inequalities (70) and (71) in inequality (60) leads to inequality (59). This completes the proof. □

4. Applications

This section presents some applications of the results proven in this paper. The theorems proven in this paper cover a wide range of previously proven results. This aim can be attained by limiting some of the functions in our results. For instance, if the functions $e_i(\varsigma)$ and $g_i(\varsigma)$ vanish, then consequently, the functions $\psi_i(\varsigma), \psi(\varsigma), \rho(i)(\varsigma)$, and $\rho(\varsigma)$ vanish as well, and then Theorem 3 states the following. If the system

$$t_i(\varsigma) \leq a_i(\varsigma) + \int_{\varsigma^o}^{\varsigma} p_i(v)H(t_1(v))ds + \int_{\varsigma^o}^{\varsigma} q_i(v)H(t_2(v))ds,$$

is satisfied for all $\varsigma \in \Omega$ with $\varsigma \geq \varsigma^o$, then for $\varsigma^o \leq \varsigma \leq \varsigma^*$, we have

$$t_i(\varsigma) \leq a_i(\varsigma)\left[1 + 2\int_{\varsigma^o}^{\varsigma} \phi_i(v)H(\gamma(v))ds\right],$$

where $\phi_1(\varsigma) = p_1(\varsigma) + q_1(\varsigma)\frac{a_2(\varsigma)}{a_1(\varsigma)}$, $\phi_2(\varsigma) = p_2(\varsigma)\frac{a_1(\varsigma)}{a_2(\varsigma)} + q_2(\varsigma)$, $\phi(\varsigma) = \sum_{i=1}^{2}\phi_i(\varsigma)$, and $G(r) = \int_{r^o}^{r}\frac{ds}{H(v)}$, while ς^* is chosen so that $G(2) + 2\int_{\varsigma^o}^{\varsigma}\phi(v)ds \in Dom(G^{-1})$ and $\gamma(\varsigma) = G^{-1}\Big(G(2) + 2\int_{\varsigma^o}^{\varsigma}\phi(v)ds\Big)$.

On the other hand, we can apply Remarks 1 and 2 in order to obtain some upper bounds for the systems of integral inequalities. In what follows, we present two applications:

1. Consider the following system of integral inequalities in two unknown functions $t_1(\varsigma, \Im)$ and $t_2(\varsigma, \Im)$:

$$\begin{aligned} t_1(\varsigma,\Im) \leq xy &+ \int_0^{\varsigma}\int_0^{\Im} st_1(v,c)dtds - \int_0^{\varsigma}\int_0^{\Im} tu_2(v,c)dtds \\ &+ \int_0^{\varsigma}\int_0^{\Im}\frac{1}{st}\left(\int_0^{v}\int_0^{t}\theta\, t_2(r,\theta)d\theta dr\right)dtds, \end{aligned}$$

and

$$t_2(\varsigma,\Im) \leq xy + \int_0^{\varsigma}\int_0^{\Im} tt_1(v,c)dtds + \int_0^{\varsigma}\int_0^{\Im}\frac{1}{t^2}\left(\int_0^{v}\int_0^{t}(-r-\theta)t_1(r,\theta)d\theta dr\right)dtds.$$

Comparing this system with the system given in Remark 1 indicates that

$$\begin{aligned} &\phi_1(\varsigma,\Im) = \varsigma - \Im, \phi_2(\varsigma,\Im) = \Im, \phi(\varsigma,\Im) = \varsigma, \psi_1(\varsigma,\Im) = \Im,\ \psi_2(\varsigma,\Im) = -(\varsigma+\Im), \\ &\psi(\varsigma,\Im) = -\varsigma, \rho_1(\varsigma,\Im) = \frac{1}{xy},\ \rho_2(\varsigma,\Im) = \frac{1}{\Im^2},\ \rho(\varsigma,\Im) = \frac{\varsigma+\Im}{xy^2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \eta(\varsigma,\Im) &= 2 + 2\int_0^{\varsigma}\int_0^{\Im} v\exp\left(\int_0^{v}\int_0^{t}(r-r)d\theta dr\right)dtds \\ &= 2 + \varsigma^2\Im, \end{aligned}$$

which leads to

$$\begin{aligned} t_1(\varsigma,\Im) &\leq xy\left[1 + \int_0^{\varsigma}\int_0^{\Im}\left((v-c)(2+v^2c) + \frac{1}{st}\int_0^{v}\int_0^{t}\theta(2+r^2\theta)d\theta dr\right)dtds\right] \\ &= xy\left[1 + \int_0^{\varsigma}\int_0^{\Im}\left(2v + v^3t - t - \frac{8}{9}v^2t^2\right)dtds\right] \\ &= xy\left[1 + xy\left(\varsigma + \left(\frac{\varsigma^3-4}{8}\right)\Im - \frac{8}{81}\varsigma^2\Im^2\right)\right], \end{aligned}$$

and

$$\begin{aligned} t_2(\varsigma,\Im) &\leq xy\left[1 + \int_0^{\varsigma}\int_0^{\Im}\left(t(2+v^2c) - \frac{1}{t^2}\int_0^{v}\int_0^{t}(r+\theta)(2+r^2\theta)d\theta dr\right)dtds\right] \\ &= xy\left[1 + \int_0^{\varsigma}\int_0^{\Im}\left(2t + v^2t^2 - \left(\frac{v^2}{t} + v + \frac{v^4}{8} + \frac{v^3t}{9}\right)\right)dtds\right] \\ &= xy\left[1 + xy\left(\frac{-\varsigma}{2}\left(1+\frac{\varsigma^3}{20}\right) + \left(1-\frac{\varsigma^3}{72}\right)\Im + \frac{1}{9}\varsigma^2\Im^2\right) - \frac{\varsigma^3}{3}\ln\Im\right];\ \Im > 0. \end{aligned}$$

2. Consider the following system of integral inequalities in two unknown functions $t_1(\varsigma,\Im)$ and $t_2(\varsigma,\Im)$:

$$t_1(\varsigma,\Im) \leq \frac{-1}{2}\int_0^{\varsigma}\int_0^{\Im} t_1(v,c)dtds - \int_0^{\varsigma}\int_0^{\Im}\left(\frac{1}{2}+v\right)t_2(v,c)dtds - \int_0^{\varsigma}\int_0^{\Im}\frac{1}{2}\left(\int_0^{v}\int_0^{t}\frac{1}{\theta}t_2(r,\theta)d\theta dr\right)dtds,$$

and

$$t_2(\varsigma,\Im) \leq \Im + \int_0^{\varsigma}\int_0^{\Im} vt_1(v,c)dtds + \frac{1}{2}\int_0^{\varsigma}\int_0^{\Im} t_2(v,c)dtds + \int_0^{\varsigma}\int_0^{\Im}\frac{1}{2t+2}\left(\int_0^{v}\int_0^{t} t_1(r,\theta)d\theta dr\right)dtds.$$

Comparing this system with the system given in Remark 2 indicates that

$$\begin{aligned}
&\phi_1(\varsigma,\Im) = -\Im\left(\frac{1}{2}+\varsigma\right) - \frac{1}{2}xy, \phi_2(\varsigma,\Im) = \frac{1}{2}\Im, \phi(\varsigma,\Im) = -\frac{3}{2}xy,\\
&\beta_1(\varsigma,\Im) = -(1+\varsigma),\ \beta_2(\varsigma,\Im) = \varsigma+\frac{1}{2},\ \beta(\varsigma,\Im) = -\frac{1}{2},\\
&\rho_1(\varsigma,\Im) = -\frac{1}{2}, \rho_2(\varsigma,\Im) = \frac{1}{2(\Im+1)}, \rho(\varsigma,\Im) = -\frac{\Im}{2(\Im+1)},\\
&\sigma_1(\varsigma,\Im) = \frac{1}{\Im}, \sigma_2(\varsigma,\Im) = 1,\ \sigma(\varsigma,\Im) = \frac{\Im+1}{\Im}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\beta-\rho\sigma = 0,\ \beta+\rho\sigma+1 = 0,\ \text{this leads to } \frac{\partial^2 Y}{\partial v\partial t} = 0 \text{ so } Y = 1,\\
&\text{in addition } \frac{\partial^2 w}{\partial v\partial t} = 0,\ \textit{i.e.},\ w = 1,\ \text{and } \tau(\varsigma,\Im) = \frac{3}{8}\left(\frac{\varsigma^3\Im^3}{18} - \varsigma^2\Im^2\right).
\end{aligned}$$

Thus, the inequalities in (57) lead to

$$\begin{aligned}
t_1(\varsigma,\Im) &\leq \int_0^{\varsigma}\int_0^{\Im}\left[-\frac{t}{2} - \frac{3st}{2} + \frac{3}{8}\left(v^2t^2 - \frac{v^3t^3}{18} + v^3t^2 - \frac{v^4t^3}{18}\right) - \frac{3}{16}\int_0^{v}\int_0^{t}\left(\frac{r^3\theta^2}{18} - r^2\theta\right)d\theta dr\right]dtds\\
&= \frac{xy^2}{8}\left[-2-3\varsigma+\left(\frac{13\varsigma+16}{48}\right)\varsigma^2\Im - \left(\frac{5\varsigma+6}{576}\right)\varsigma^3\Im^2\right],
\end{aligned}$$

and

$$\begin{aligned}
t_2(\varsigma,\Im) &\leq \Im + \int_0^{\varsigma}\int_0^{\Im}\left[\frac{t}{2} + \frac{3}{8}\left(v+\frac{1}{2}\right)\left(\frac{v^3t^3}{18} - v^2t^2\right) + \frac{3}{16(t+1)}\int_0^{v}\int_0^{t}\left(\frac{r^3\theta^3}{18} - r^2\theta^2\right)d\theta dr\right]dtds\\
&= \frac{\Im}{16}\left[16 - \frac{\varsigma^4}{2}\left(\frac{\varsigma+40}{16(15)}\right) + \left(\frac{\varsigma^4}{16(15)4} + \frac{\varsigma^3}{24} + 4\right)xy - \left(\frac{\varsigma^3}{40} + 19\varsigma^2 + 12\varsigma\right)\frac{\varsigma^2\Im^2}{2(18)} + \left(\frac{33\varsigma^2}{20} + \varsigma\right)\frac{\varsigma^3\Im^3}{48(16)}\right] + \frac{\varsigma^4}{2(16)}\left(\frac{\varsigma}{40}+1\right)\ln(1+\Im);\ \Im > -1.
\end{aligned}$$

5. Discussion

By applying Young's method, which depends on the Riemann method, we proved additional generalizations of the integral inequality in n independent variables. Some applications of the results proven in this paper are presented.

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