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On the existence of solution for fractional differential equations of order $3 < \delta_1 \leq 4$

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Abstract

In this paper, we deal with a fractional differential equation of order $\delta_1 \in (3,4]$ with initial and boundary conditions, $\mathcal{D}^{\delta_1}\psi(x) = -\mathcal{H}(x,\psi(x))$, $\mathcal{D}^{\alpha_1}\psi(1) = 0 = \mathcal{I}^{3-\delta_1}\psi(0) = \mathcal{I}^{4-\delta_1}\psi(0)$, $\psi(1) = \frac{\Gamma(\delta_1-\alpha_1)}{\Gamma(\nu_1)}\mathcal{I}^{\delta_1-\alpha_1}\mathcal{H}(x,\psi(x))(1)$, where $x \in [0,1]$, $\alpha_1 \in (1,2]$, addressing the existence of a positive solution (EPS), where the fractional derivatives $\mathcal{D}^{\delta_1}$, $\mathcal{D}^{\alpha_1}$ are in the Riemann-Liouville sense of the order δ_1, α_1, respectively. The function $\mathcal{H} \in C([0,1] \times R, R)$ and $\mathcal{I}^{\delta_1-\alpha_1}\mathcal{H}(x,\psi(x))(1) = \frac{1}{\Gamma(\delta_1-\alpha_1)}\int_0^1 (1-z)^{\delta_1-\alpha_1-1}\mathcal{H}(z,\psi(z))\,dz$. To this aim, we establish an equivalent integral form of the problem with the help of a Green's function. We also investigate the properties of the Green's function in the paper which we utilize in our main result for the EPS of the problem. Results for the existence of solutions are obtained with the help of some classical results.

Keywords: existence of positive solutions; Green's function; Krasnosel'skiĭ theorem; Arzela-Ascoli theorem

1 Introduction

Fractional differential equations (FDEs) in different scientific fields have attracted the attention of scientists. Scientists are utilizing different and new mathematical tools for the study of FDEs. The study in applied scientific fields can be observed in fields like physics, biology, chemistry, economics, mechanics, aerodynamics, biophysics, *etc.* [1, 2].

In the study of FDEs, one can see valuable scientific work for the existence and uniqueness of solution (EUS), multiple positive solutions for the nonlinear boundary value problems (BVPs). This work is nowadays a lively research area and scientists are highly interested in it. Scientists have given good contributions to this area, some of their work can be studied in [3–7]. Here we highlight some useful and new important scientific work in FDEs. Work on the integro-differential equations as regards the existence of solutions can be studied in [4]. Baleanu *et al.* [5] considered the existence of a solution for a class of sequential FDEs in the Riemann-Liouville sense. Agarwal *et al.* [6] have considered a class of FDEs with two fractional derivatives for the existence of solutions in the Caputo sense. Agarwal *et al.* [7] studied a class of FDEs with sum boundary conditions. Abbas [8] studied a FDE of order $\alpha \in (m-1, m]$ in Caputo's sense for the EUS by using Schaefer's fixed point theorem and Hölder's inequality. Baleanu *et al.* [9] considered a finite difference inclusion of fractional order $2 < \gamma < 3$ for the existence of solutions. Wu and Liu [10] investigated a FDE of an m-point BVP at resonance in Caputo's sense by the use of a Leggett-Williams norm-type theorem. Xin and Zhao [11] have considered a Rayleigh equation for a periodic solution with the help of coincidence degree theory. Sitho *et al.* [12] have studied a class of hybrid fractional integro-differential equations. Naceri *et al.* [13] have considered a fourth order differential equation with deviating arguments for the existence of solutions with the help of upper and lower solutions and Schauder's fixed point theorem. Henderson and Luca [14] have considered a coupled system of a fractional order BVP in the Riemann-Liouville sense for the nonexistence of solutions.

From the study of the scientific work as discussed above we felt the need of exploration of the fractional differential equation (FDE) of order $\delta_1 \in (3,4]$:

$$\begin{aligned} &\mathcal{D}^{\delta_1}\psi(x) = -\mathcal{H}\big(x,\psi(x)\big), \\ &\mathcal{D}^{\alpha_1}\psi(1) = 0 = \mathcal{I}^{3-\delta_1}\psi(0) = \mathcal{I}^{4-\delta_1}\psi(0), \\ &\psi(1) = \frac{\Gamma(\delta_1-\alpha_1)}{\Gamma(\delta_1)}\mathcal{I}^{\delta_1-\alpha_1}\mathcal{H}\big(x,\psi(x)\big)(1), \end{aligned} \tag{1}$$

where $x \in [0,1]$. $\delta_1 \in (3,4]$, $\alpha_1 \in (1,2]$, for the existence of positive solution (EPS), where the fractional derivatives $\mathcal{D}^{\delta_1}$, $\mathcal{D}^{\alpha_1}$ are in the Riemann-Liouville sense of the order δ_1, α_1, respectively, and $\mathcal{H} : C([0,1] \times R, R)$ and $\mathcal{I}^{\delta_1-\alpha_1}\mathcal{H}(x,\psi(x))(1) = \frac{1}{\Gamma(\delta_1-\alpha_1)}\int_0^1 (1-z)^{\delta_1-\alpha_1-1}\mathcal{H}(z,\psi(z))\,dz$. To this aim, we establish an equivalent integral form of the problem with the help of a Green's function. We also investigate the properties of the Green's function in the paper which we utilize in our main result for the EPS of the problem. We use Arzela-Ascoli for the complete continuity of the integral operator and the Krasnosel'skiĭ fixed point theorem for the EPS.

Third order ordinary differential equations (TOODEs) are very much popular in the mathematical modeling of engineering problems. Fakhar and Kara [15] have given many examples of TOODEs related to boundary layer models of the type $\psi''' = -(\psi\psi'' - \psi'^2 - A(\psi' + \frac{1}{2}\eta\psi'') - M^2\psi')$, Blasius flow which is equivalent to the TOODE $2\psi''' = -\psi\psi''$, the Falkner-Skan equation $\psi''' = -(\psi\psi'' + \beta(1-\psi'^2))$, and many different classes of canonical Chazy equations. Mohammadyari *et al.* [16] have described a model of magneto hydrodynamics and have presented the analytical solution of the model by a differential transform method; the model is equivalent to the TOODE $\psi''' + \mathrm{Re}(\psi'^2 - \psi\psi'') - M^2\psi' = 0$ with conditions $\psi = 0$, $\psi'' = 0$, at $x = 0$ and $\psi = 1/2$, $\psi' = 0$ at $x = 1/2$. All these models are special cases of our proposed problem.

This paper is organized in four sections. The first section is a literature review including the most relevant and recent contributions. In the second section, we produce the equivalent integral form of the problem (1) with the help of a Green's function. Also some properties of the Green's function for the problem (1) are studied. In the third section we have our main theorem for the existence of solution of the problem (1) based on the Krasnosel'skiĭ fixed point theorem and the Arzela-Ascoli theorem. The final section presents the conclusion of the paper and future plans as regards the problem (1).

In this paper we will need the definitions of a fractional order integral and the fractional order derivative in the Riemann-Liouville sense and some basic results of fractional calculus. Some basic definitions and results are hereby given; for more details one may refer to the references.

Definition 1 If $\psi(x) \in L^1(a,b)$, the set of all integrable functions, and $\delta_1 > 0$, then the left Riemann-Liouville fractional integral, of order δ_1, is defined by

$$I_{0+}^{\delta_1}\psi(x) = \frac{1}{\Gamma(\delta_1)}\int_0^x (x-z)^{\delta_1-1}\psi(z)\,dz. \tag{2}$$

Definition 2 For $\delta_1 > 0$ the left Riemann-Liouville fractional derivative of order δ_1 is defined by

$$\mathcal{D}^{\delta_1}\psi(x) = \frac{1}{\Gamma(n-\delta_1)}D^n\int_0^x (x-z)^{n-\delta_1-1}\psi(z)\,dz, \tag{3}$$

where n is such that $n-1 < \delta_1 < n$ and $D = \frac{d}{dz}$.

Lemma 3 *For $\delta_1, \epsilon > 0$, such that $n-1 < \delta_1 < n$, the following relations hold*: $\mathcal{D}^{\delta_1}x^{\epsilon} = \frac{\Gamma(1+\epsilon)}{\Gamma(1+\epsilon-\delta_1)}x^{\epsilon-\delta_1}$, $\epsilon \geq n$ *and* $\mathcal{D}^{\delta_1}x^{\epsilon} = 0$ *if* $\epsilon \leq n-1$.

Lemma 4 *Let $a,b \geq 0$ and $\mathcal{H} \in L_1[p,q]$. Then $I_{0^+}^{a}I_{0^+}^{b}\mathcal{H}(x) = I_{0^+}^{a+b}\mathcal{H}(x) = I_{0^+}^{b}I_{0^+}^{a}\mathcal{H}(x)$ and $D^bI_{0^+}^{b}\mathcal{H}(x) = \mathcal{H}(x)$, for all $x \in [p,q]$.*

Lemma 5 *For $\epsilon \geq \delta_1 > 0$ and $\mathcal{H}(x) \in L_1[a,b]$, the following hold*:

$$D^{\delta_1}I_{a+}^{\epsilon}\mathcal{H}(x) = I_{a+}^{\epsilon-\delta_1}\mathcal{H}(x)$$

on the interval $[a,b]$, if $\mathcal{H} \in C[a,b]$.

2 Green's function and properties

Lemma 6 *For $z,x \in [0,1]$, the solution of* (1) *is equivalent to the solution of the following integral equation*:

$$\psi(x) = \int_0^1 \mathcal{K}(x,z)\mathcal{H}\big(z,\psi(z)\big)\,dz, \tag{4}$$

where $\mathcal{K}(x,z)$ is the Green's function given by

$$\mathcal{K}(x,z) = \frac{1}{\Gamma(\delta_1)}\begin{cases} -(x-z)^{\delta_1-1} + x^{\delta_1-1}(1-z)^{\delta_1-\alpha_1-1} + x^{\delta_1-2}(1-z)^{\delta_1-1}, & z \leq x, \\ x^{\delta_1-1}(1-z)^{\delta_1-\alpha_1-1} + x^{\delta_1-2}(1-z)^{\delta_1-1}, & x \leq z. \end{cases} \tag{5}$$

Proof Applying the operator $I_0^{\delta_1}$ on the differential equation in (1), we get the following equivalent integral form:

$$\psi(x) = -\mathcal{I}^{\delta_1}\mathcal{H}\big(x,\psi(x)\big) + c_1x^{\delta_1-1} + c_2x^{\delta_1-2} + c_3x^{\delta_1-3} + c_4x^{\delta_1-4}. \tag{6}$$

The initial conditions $\mathcal{I}^{3-\delta_1}\psi(0) = \mathcal{I}^{4-\delta_1}\psi(0) = 0$ in (1) imply that $c_3 = c_4 = 0$. Using the boundary conditions $\mathcal{D}^{\alpha_1}\psi(1) = 0$ and $\psi(1) = \frac{\Gamma(\delta_1-\alpha_1)}{\Gamma(\nu_1)}\mathcal{I}^{\delta_1-\alpha_1}\mathcal{H}(x,\psi(x))(1)$ on (6), we get

$$\begin{aligned} c_1 &= \int_0^1 \frac{(1-z)^{\delta_1-\alpha_1-1}\mathcal{H}(z,\psi(z))\,dz}{\Gamma(\delta_1)}, \\ c_2 &= \frac{\int_0^1 (1-z)^{\delta_1-1}\mathcal{H}(z,\psi(z))\,dz}{\Gamma(\delta_1)}. \end{aligned} \tag{7}$$

By substituting the values of c_1, c_2, c_3, c_4, in (6), we have

$$\begin{aligned}\psi(x) = &-\int_0^x \frac{(x-z)^{\delta_1-1}}{\Gamma(\delta_1)}\mathcal{H}\big(z,\psi(z)\big)\,dz \\ &+ x^{\delta_1-1}\int_0^1 \frac{(1-z)^{\delta_1-\alpha_1-1}\mathcal{H}(z,\psi(s))\,ds}{\Gamma(\delta_1)} \\ &+ x^{\delta_1-2}\int_0^1 \frac{(1-z)^{\delta_1-1}\mathcal{H}(z,\psi(z))\,dz}{\Gamma(\delta_1)} = \int_0^1 \mathcal{K}(x,z)\mathcal{H}\big(z,\psi(z)\big)\,ds,\end{aligned} \tag{8}$$

where $\mathcal{K}(x,z)$ is the Green's function which is given by (5). Thus, the proof is completed. □

Lemma 7 *For the Green's function $\mathcal{K}(x,z)$ given by* (5) *and $\mathcal{J}=[0,1]$, $\nu_1\in(3,4]$, $\alpha_1\in(1,2]$, the following are satisfied*:

(A_1) *$\mathcal{K}(x,z)$ is continuous and $\mathcal{K}(x,z)\geq 0$ for each $x,z\in\mathcal{J}$*;
(A_2) *$\max_{x\in\mathcal{J}}\mathcal{K}(x,z)=\mathcal{K}(1,z)$ for each $z\in\mathcal{J}$*;
(A_3) *$\min_{x\in[\frac{1}{3},1]}\mathcal{K}(x,z)\geq\lambda_0\mathcal{K}(1,z)$ for some $\lambda_0\in(0,1)$.*

Proof The continuity of the Green's function $\mathcal{K}(x,z)$ is obvious from the definition in (5). Consider $\mathcal{K}(x,z)$, for $x,z\in\mathcal{J}$ such that $x\geq z$. $z\leq\frac{z}{x}$ implies that $-(1-z)\leq-(1-\frac{z}{x})$ and $\delta_1-\alpha_1-1<\delta_1-1$ implies that $(1-z)^{\delta_1-\alpha_1-1}>(1-z)^{\delta_1-1}$. Thus

$$\begin{aligned}\mathcal{K}(x,z) &= \frac{-(x-z)^{\delta_1-1}}{\Gamma(\delta_1)}+\frac{x^{\delta_1-1}(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)}+\frac{x^{\delta_1-2}(1-z)^{\delta_1-1}}{\Gamma(\delta_1)} \\ &= -\frac{(1-\frac{z}{x})^{\delta_1-1}}{\Gamma(\delta_1)}x^{\delta_1-1}+\frac{(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)}x^{\delta_1-1}+\frac{(1-z)^{\delta_1-1}}{\Gamma(\delta_1)}x^{\delta_1-2} \\ &\geq \frac{-(1-z)^{\delta_1-1}}{\Gamma(\delta_1)}x^{\delta_1-1}+\frac{(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)}x^{\delta_1-1}+\frac{(1-z)^{\delta_1-1}}{\Gamma(\delta_1)}x^{\delta_1-2} \\ &= \big((1-z)^{\delta_1-\alpha_1-1}-(1-z)^{\delta_1-1}\big)\frac{x^{\delta_1-1}}{\Gamma(\delta_1)}+\frac{x^{\delta_1-2}(1-z)^{\delta_1-1}}{\Gamma(\delta_1)}\geq 0.\end{aligned} \tag{9}$$

From (5), for $x\leq z$ it is obvious that $\mathcal{K}(x,z)\geq 0$. This completes the proof of (A_1). For (A_2), we consider $z,x\in\mathcal{J}$, such that $x\geq z$. For $\delta_1\in(3,4]$, $\alpha_1\in(1,2]$, we have $\delta_1-\alpha_1-1\leq\delta_1-2$; this implies that $(1-z)^{\delta_1-\alpha_1-1}\geq(1-z)^{\delta_1-2}$ and

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{K}(x,z) &= \frac{-(\delta_1-1)(x-z)^{\delta_1-2}}{\Gamma(\delta_1)}+\frac{(\delta_1-1)x^{\delta_1-2}(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)} \\ &\quad+\frac{(\delta_1-2)x^{\delta_1-3}(1-z)^{\delta_1-1}}{\Gamma(\delta_1)} \\ &= (\delta_1-1)\left[\frac{-(x-z)^{\delta_1-2}+x^{\delta_1-2}(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)}\right] \\ &\quad+\frac{(\delta_1-2)x^{\delta_1-3}(1-z)^{\delta_1-1}}{\Gamma(\delta_1)} \\ &= (\delta_1-1)\left[\frac{-(1-\frac{z}{x})^{\delta_1-2}+(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)}\right]x^{\delta_1-2}\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\delta_1-2)(1-z)^{\delta_1-1}}{\Gamma(\delta_1)}x^{\delta_1-3}\\
&\geq(\delta_1-1)\left[\frac{-(1-z)^{\delta_1-2}+(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)}\right]x^{\delta_1-2}\\
&+\frac{(\delta_1-2)(1-z)^{\delta_1-1}}{\Gamma(\delta_1)}x^{\delta_1-3}\geq 0. \qquad (10)
\end{aligned}
$$

Hence, it follows that $\max_{x\in J}\mathcal{K}(x,z)=\mathcal{K}(1,z)=\frac{1}{\Gamma(\delta_1)}[(1-z)^{\delta_1-\alpha_1-1}+(1-z)^{\delta_1-1}]$ and $\min_{x\in[\frac{1}{3},1]}\mathcal{K}(x,z)=\mathcal{K}(\frac{1}{3},z)$. For the proof of ($A_3$), we utilize ($A_1$) and ($A_2$) in the following calculations. For $z\in(0,\frac{1}{3}]$, we have

$$
\begin{aligned}
\frac{\min_{x\in[\frac{1}{3},1]}\mathcal{K}(x,z)}{\max_{x\in[\frac{1}{3},1]}\mathcal{K}(x,z)}&=\frac{(-(\frac{1}{3}-z)^{\delta_1-1}+(\frac{1}{3})^{\delta_1-1}(1-z)^{\delta_1-\alpha_1-1}+(\frac{1}{3})^{\delta_1-2}(1-z)^{\delta_1-1})}{(-(1-z)^{\delta_1-1})+(1-z)^{\delta_1-\alpha_1-1}+(1-z)^{\delta_1-1}}\\
&=\frac{(-(\frac{1}{3}-z)^{\delta_1-1}+(\frac{1}{3})^{\delta_1-2}(1-z)^{\delta_1-\alpha_1-1}[\frac{1}{3}+(1-z)^{\delta_1}])}{((1-z)^{\delta_1-\alpha_1-1})}\\
&\geq\frac{(-(\frac{1}{3}-\frac{1}{3})^{\delta_1-1}+(\frac{1}{3})^{\delta_1-2}(1-\frac{1}{3})^{\delta_1-\alpha_1-1}[\frac{1}{3}+(1-\frac{1}{3})^{\alpha_1}])}{((1-\frac{1}{3})^{\delta_1-\alpha_1-1})}\\
&=\left(\frac{1}{3}\right)^{\delta_1-2}\left[\frac{1}{3}+\left(\frac{2}{3}\right)^{\alpha_1}\right]. \qquad (11)
\end{aligned}
$$

For $z\in(\frac{1}{3},1]$, we have

$$
\begin{aligned}
\frac{\min_{x\in[\frac{1}{3},1]}\mathcal{K}(x,z)}{\max_{x\in[\frac{1}{3},1]}\mathcal{K}(x,z)}&=\frac{(\frac{1}{3})^{\delta_1-2}(1-z)^{\delta_1-\alpha_1-1}[\frac{1}{3}+(1-z)^{\alpha_1}]}{(1-z)^{\delta_1-\alpha_1-1}[1+(1-z)^{\alpha_1}]}\\
&\geq\frac{(\frac{1}{3})^{\delta_1-2}[\frac{1}{3}+(1-\frac{1}{3})^{\alpha_1}]}{[1+(1-\frac{1}{3})^{\alpha_1}]}=\frac{(\frac{1}{3})^{\delta_1-2}[\frac{1}{3}+(\frac{2}{3})^{\alpha_1}]}{[1+(\frac{2}{3})^{\alpha_1}]}. \qquad (12)
\end{aligned}
$$

Choose

$$
\lambda_0=\min\left\{\left(\frac{1}{3}\right)^{\delta_1-2}\left[\frac{1}{3}+\left(\frac{2}{3}\right)^{\alpha_1}\right],\frac{(\frac{1}{3})^{\delta_1-2}[\frac{1}{3}+(\frac{2}{3})^{\alpha_1}]}{[1+(\frac{2}{3})^{\alpha_1}]}\right\}. \qquad (13)
$$

Therefore, in view of (11), (12), and (13), we have $\lambda_0\in(0,1)$ such that

$$
\min_{x\in[\frac{1}{3},1]}\mathcal{K}(x,z)\geq\lambda_0\max_{x\in\mathcal{J}}\mathcal{K}(x,z)=\lambda_0\mathcal{K}(1,z). \qquad (14)
$$

This completes the proof. □

3 Existence criterion

In this section, we address the existence of a positive solution of our problem (1). For this purpose, we get help from the Krasnosel'skiĭ result. The details of the result can be found in [2].

Lemma 8 [2] *Let $\mathcal{E}$ be a Banach space and $\mathcal{B}\subset\mathcal{E}$ be a cone. Assume that $\mathcal{Q}_1$, $\mathcal{Q}_2$ are open sets contained in $\mathcal{E}$ such that $0\in\mathcal{Q}_1$ and $\overline{\mathcal{Q}}_1\subset\mathcal{Q}_2$. Assume, further, that $\mathcal{F}:\mathcal{B}\cap(\overline{\mathcal{Q}}_2\setminus\mathcal{Q}_1)\to\mathcal{B}$ is a completely continuous operator. If either*

(B_1) $\|\mathcal{F}v\| \leq \|v\|$ *for* $v \in \mathcal{B} \cap \partial\mathcal{Q}_1$ *and* $\|\mathcal{F}v\| \geq \|v\|$ *for* $v \in \mathcal{B} \cap \partial\mathcal{Q}_2$, *or*

(B_2) $\|\mathcal{F}v\| \geq \|v\|$ *for* $v \in \mathcal{B} \cap \partial\mathcal{Q}_1$ *and* $\|\mathcal{F}v\| \leq \|v\|$ *for* $v \in \mathcal{B} \cap \partial\mathcal{Q}_2$,

then $\mathcal{F}$ *has at least one fixed point in* $\mathcal{B} \cap (\overline{\mathcal{Q}}_2 \setminus \mathcal{Q}_1)$.

Consider the Banach space $\mathcal{E} = \{\psi(x) : \psi(x) \in C(\mathcal{J}), \text{where } \mathcal{J} = [0,1]\}$, endowed with the norm $\|\psi(x)\| = \max_{x\in\mathcal{J}} |\psi(x)|$. We define an operator $\mathcal{F} : \mathcal{E} \to \mathcal{E}$ by

$$\begin{aligned}\mathcal{F}\psi(x) = &-\int_0^x \frac{(x-z)^{\delta_1-1}}{\Gamma(\delta_1)}\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \\ &+ x^{\delta_1-1}\int_0^1 \frac{(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)}\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \\ &+ x^{\delta_1-2}\int_0^1 \frac{(1-z)^{\delta_1-1}}{\Gamma(\delta_1)}\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz = \int_0^1 \mathcal{K}(x,z)\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz.\end{aligned} \tag{15}$$

Theorem 9 *Suppose that there are real constants* $k_2 > k_1 > 0$ *such that conditions* (C_1), (C_2) *hold*:

(C_1) *There exists a real number* $k_1 > 0$ *such that* $\mathcal{H}(x,\psi) \leq \xi k_1$ *whenever* $0 \leq \psi \leq k_1$.

(C_2) *There exists a real number* $k_2 > 0$ *such that* $\mathcal{H}(x,\psi) \geq \nu k_2$ *whenever* $\lambda_0 k_2 \leq \psi \leq k_2$, *where* λ_0 *is the constant defined by* (13).

Suppose also that $\mathcal{H}(x,\psi) \geq 0$ *and is continuous. Then the problem* (1) *has at least one positive solution.*

Proof We define the terms $\xi = [\int_0^1 \mathcal{K}(1,z)\,dz]^{-1}$ and $\nu = [\int_{\frac{1}{3}}^1 \mathcal{K}(\frac{2}{3},z)\,dz]^{-1}$. From Lemma 7 ($A_1$) the Green's function $\mathcal{K}(x,z)$ is continuous and nonnegative, and also $\mathcal{H}(x,\psi(x)) \in C(\mathcal{J} \times R, R)$, therefore the operator $\mathcal{F}$ is continuous. Let $\mathcal{S} = \{\psi(x) \in \mathcal{E} : \|\psi(x)\| \leq \Delta\}$ where $\Delta = \max_{x\in\mathcal{J}} \mathcal{H}(x,z) + 1$. For any $\psi(x) \in \mathcal{S}$, the operator $\mathcal{F}$, defined in (15), is

$$\begin{aligned}\bigl|\mathcal{F}\psi(x)\bigr| &= \biggl|-\int_0^x \frac{(x-z)^{\delta_1-1}}{\Gamma(\delta_1)}\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz + x^{\delta_1-1}\int_0^1 \frac{(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)}\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \\ &\quad + x^{\delta_1-2}\int_0^1 \frac{(1-z)^{\delta_1-1}}{\Gamma(\delta_1)}\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz\biggr| \\ &\leq \Delta\left(\frac{(x-z)^{\delta_1}}{\Gamma(\delta_1+1)}\bigg|_x^0 + x^{\delta_1-1}\frac{(1-z)^{\delta_1-\alpha_1}}{(\delta_1-\alpha_1)\Gamma(\delta_1)}\bigg|_1^0 + x^{\delta_1-2}\frac{(1-z)^{\delta_1}}{\Gamma(\delta_1+1)}\bigg|_1^0\right) \\ &= \Delta\left[\frac{x^{\delta_1}}{\Gamma(\delta_1+1)} + \frac{x^{\delta_1-1}}{(\delta_1-\alpha_1)\Gamma(\delta_1)} + \frac{x^{\delta_1-2}}{\Gamma(\delta_1+1)}\right] \\ &\leq \Delta\left[\frac{2}{\Gamma(\delta_1+1)} + \frac{1}{(\delta_1-\alpha_1)\Gamma(\delta_1)}\right] < \infty,\end{aligned} \tag{16}$$

and it is bounded. Next, for $\psi(x) \in \mathcal{S}$, $x_1, x_2 \in \mathcal{J}$, such that $x_2 > x_1$, we have

$$\begin{aligned}\bigl|\mathcal{F}\psi(x_2) - \mathcal{F}\psi(x_1)\bigr| = \biggl|&-\int_0^{x_2} \frac{(x_2-z)^{\delta_1-1}}{\Gamma(\delta_1)}\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \\ &+ \int_0^{x_1} \frac{(x_1-z)^{\delta_1-1}}{\Gamma(\delta_1)}\mathcal{H}\bigl(z,\psi(z)\bigr)\,ds\end{aligned}$$

$$
\begin{aligned}
&+ \left(x_2^{\delta_1-1} - x_1^{\delta_1-1}\right) \int_0^1 \frac{(1-z)^{\delta_1-\alpha_1-1}}{\Gamma(\delta_1)} \mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \\
&+ \left(x_2^{\delta_1-2} - x_1^{\delta_1-2}\right) \int_0^1 \frac{(1-z)^{\delta_1-1}}{\Gamma(\delta_1)} \mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \Bigr| \\
&\le \Delta\left(\frac{x_2^{\delta_1} - x_1^{\delta_1}}{\Gamma(\delta_1+1)} + \frac{x_2^{\delta_1-1} - x_1^{\delta_1-1}}{(\delta_1-\alpha_1)\Gamma(\delta_1)} + \frac{x_2^{\delta_1-2} - x_1^{\delta_1-2}}{\Gamma(\delta_1+1)}\right),
\end{aligned} \tag{17}
$$

that is, $\|\mathcal{F}\psi(x_2) - \mathcal{F}\psi(x_1)\| \to 0$ as $x_1 \to x_2$. With the help of (16)-(17) and the Arzela-Ascoli theorem, the operator $\mathcal{F}$ is completely continuous.

Consider a cone $\mathcal{B} = \{\psi(x) \in \mathcal{E} : \psi(x) \ge 0 \text{ and } \min_{x\in[\frac{1}{3},1]} \psi(x) \ge \lambda_0 \|\psi(x)\|\}$ in $\mathcal{E}$, then for any $\psi \in \mathcal{B}$, we have

$$
\begin{aligned}
\min_{x\in[\frac{1}{3},1]} (\mathcal{F}\psi)(x) &\ge \lambda_0 \int_0^1 \mathcal{K}(1,z)\mathcal{H}\bigl(z,\psi(z)\bigr)\,ds \\
&= \lambda_0 \max_{x\in\mathcal{J}} \int_0^1 \mathcal{K}(x,z)\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz = \lambda_0 \bigl\|\mathcal{F}\psi(x)\bigr\|,
\end{aligned} \tag{18}
$$

this implies that $\mathcal{F}\psi(x) \in \mathcal{B}$. Let $\mathcal{Q} = \{\psi(x) \in \mathcal{B} : \|\psi(x)\| < k_1\}$, we see that, for any $\psi(x) \in \partial\mathcal{Q}_1$, $\|\psi(x)\| = k_1$, so (C_1) is satisfied for all $\psi \in \partial\mathcal{Q}_1$. So, for $\psi(x) \in \mathcal{B} \cap \partial\mathcal{Q}_1$, we get

$$
\begin{aligned}
\bigl\|\mathcal{F}\psi(x)\bigr\| &= \max_{x\in\mathcal{J}} \int_0^1 \mathcal{K}(x,z)\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \\
&\le \xi k_1 \int_0^1 \mathcal{K}(x,z)\,dz = k_1,
\end{aligned} \tag{19}
$$

by (19), we get $\|\mathcal{F}\psi(x)\| \le \|\psi(x)\|$ for $\psi \in \mathcal{B} \cap \partial\mathcal{Q}_1$. Assume $\mathcal{Q}_2 = \{\psi \in \mathcal{B} : \|\psi(x)\| < k_2\}$, for $\psi \in \partial\mathcal{Q}_2$, we have $\|\psi(x)\| = k_2$, this implies that the condition (C_2) is satisfied for $\psi \in \mathcal{B} \cap \partial\mathcal{Q}_2$; further, we have

$$
\begin{aligned}
\mathcal{F}\left(\psi\left(\frac{2}{3}\right)\right) &= \int_0^1 \mathcal{K}\left(\frac{2}{3},z\right)\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \ge \int_{\frac{1}{3}}^1 \mathcal{K}\left(\frac{2}{3},z\right)\mathcal{H}\bigl(z,\psi(z)\bigr)\,dz \\
&\ge \nu k_2 \int_{\frac{1}{3}}^1 \mathcal{K}\left(\frac{2}{3},z\right)\,ds = k_2.
\end{aligned} \tag{20}
$$

Thus, (20) yields $\|\mathcal{F}(\psi(x))\| \ge \|\psi(x)\|$ for $\psi \in \mathcal{B} \cap \partial\mathcal{Q}_2$. Therefore, with the help of Lemma 3, the operator $\mathcal{F}$ has a fixed point, say ψ_0, such that $k_1 \le \|\psi_0\| \le k_2$. This completes the proof. □

4 Illustrative example

Example 1 Consider the problem for $x, z \in (0,1]$ and $\psi(x) \ge 0$

$$
\mathcal{D}^{\delta_1}\psi(x) = \left(\int_{\frac{1}{3}}^1 \mathcal{K}\left(\frac{2}{3},z\right)dz\right)^{-1}\left(\int_0^1 \mathcal{K}(1,z)\,dz\right)^{-1} \frac{1 + 2\max_{x\in(0,1]}|\psi(x)|}{2}, \tag{21}
$$

with the conditions as defined in (1).

We assume $k_2 = \max_{x\in(0,1]}|\psi(x)|$, then from (21) we have

$$
\mathcal{H}\bigl(x,\psi(x)\bigr) = \left(\int_{\frac{1}{3}}^1 \mathcal{K}\left(\frac{2}{3},z\right)dz\right)^{-1}\left(\int_0^1 \mathcal{K}(1,z)\,dz\right)^{-1}\left(\frac{1 + 2\psi(x)}{2}\right) \ge \nu\psi(x),
$$

for $\nu = (\int_{\frac{1}{3}}^{1} \mathcal{K}(\frac{2}{3}, z)\,dz)^{-1}$ and $\lambda_0\psi(x) \le \psi(x) \le \max_{x\in(0,1]} |\psi(x)| = k_2$, where λ_0 is defined by (13). We also have

$$\mathcal{H}(x, \psi(x)) = \left(\int_{\frac{1}{3}}^{1} \mathcal{K}\left(\frac{2}{3}, z\right) dz\right)^{-1} \left(\int_{0}^{1} \mathcal{K}(1, z)\,dz\right)^{-1} \left(\frac{1 + 2\psi(x)}{2}\right) \le \xi k_1,$$

for $k_1 = (\int_{\frac{1}{3}}^{1} \mathcal{K}(\frac{2}{3}, z)\,dz)^{-1}(\frac{1+3\max_{x\in(0,1]} |\psi(x)|}{2})$. Here $0 \le \psi \le k_1$ is obvious. Therefore the assumptions (C_1), (C_2) are satisfied and hence by Theorem 9, we find that the problem (21) has a solution.

5 Conclusion

In this paper, we have utilized the Krasnosel'skiĭ fixed point theorem along with the Arzela-Ascoli theorem for the existence of a solution of the problem (1). For this, we have produced the equivalent integral form of the problem (1) using the Green's function in Lemma 6, then we discussed some properties of the Green's function in Lemma 7. These properties of the Green's function, the Arzela-Ascoli theorem, and Krasnosel'skiĭ fixed point theorem were then utilized in Theorem 9 for the existence of a solution of the problem (1). These results can be utilized for further studies of the problem (1) in q-difference equations, p-Laplacian BVPs, hybrid FDEs for the existence and multiplicity, and many other aspects.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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