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OPTIMAL SYSTEM AND SYMMETRY REDUCTION OF THE $(1+1)$ DIMENSIONAL SAWADA-KOTERA EQUATION

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Abstract: We study the nonlinear fifth order $(1+1)$ dimensional Sawada-Kotera equation using Lie symmetry group. For this equation Lie point symmetry operators and optimal system are obtained. We determine the corresponding invariant solutions and reduced equations using obtained infinitesimal generators.

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1. Introduction

Nonlinear phenomena are very important in applied mathematics and physics. They appear in various scientific fields such as biology, signal processing, viscoelastic materials, fluid mechanics, optical fiber, and so on. Some years ago, researchers have provided many methods for obtaining the numerical and analytical solutions of nonlinear equations, such as tanh function method, extended tanh-function method [1, 2], $(\frac{G'}{G})$-expansion method [3, 4], Sine-cosine method [5, 6], Simplest equation method[7] and so on. In this paper we study the Sawada-Kotera equation, namely

$$u_t + 5u^2 u_x + 5u_x u_{xx} + 2u u_{xxx} + u_{xxxxx} = 0. \tag{1}$$

Sawada and Kotera proposed it more than thirty years ago [8]. Because many various properties are satisfied for this equation, much effort has been made about its exact solutions.

For example, Fuchssteiner and Oevel studied its biHamiltonian structure [9], and a Darboux transformation was obtained for this system [10, 11], Liu and Dai[12] accomplished the Hirota's bilinear method to obtain exact solutions of the same equation. Feng and Zheng [13] investigated this equation to establish traveling wave solutions via the $(\frac{G'}{G})$-expansion method. In [14], Wazwaz implemented the extended tanh method for constructing analytical solutions of the same equation.

The outline of the paper is as follows. In Section 2 we discuss the methodology of Lie symmetry analysis of the Sawada-Kotera equation. Then in Section 3 we describe the classical symmetries of the Sawada-Kotera equation and we obtain the Lie point symmetries of this equation. In Section 4 we explain the Group Invariant solutions. In Section 5 the optimal system are obtained of the one-dimensional subalgebras of the Sawada-Kotera equation. In Section 6 we obtain symmetry reduction and differential invariants for the Sawada-Kotera equation. Finally, concluding remarks are summarized in Section 7.

2. Symmetry Group Analysis

Nonlinear partial differential equations occur in most phenomenon in nature and finding exact solutions of these equations is very important. We know that some of these equations are very difficult to solve, but much effort has been made to construct the analytical solutions for them. Symmetry is one of the most important methods of differential equations to obtain the exact solutions.

The invariance of the differential equation under infinitesimal operators is an essential concept of the Lie theory.

Sophus Lie, introduced Lie groups to solve differential equations in the nineteenth century, when he discovered that the differential equations are invariant under the continuous groups of transformations. The differential equations can be reduced to simpler equations using symmetries of them. In the last century, many researchers have developed applications of the Lie group method, some of the mathematicians who have studied in this field are Ibragimov [15], Ovsiannikov [16], Olver [17], Baumann [18] and Bluman [19].

Now let us consider a system of partial differential equations as follows[20]:

$$\Lambda_\nu(x, u^{(n)}) = 0, \quad \nu = 1, 2, \ldots, \ell, \tag{2}$$

where $u = (u^1, u^2, \ldots, u^q), x = (x^1, x^2, \ldots, x^p)$, $u^{(n)}$ denotes all the derivatives of u of all orders from 0 to n.

The one-parameter Lie group of infinitesimal transformations of the system (2) is given by

$$\begin{aligned} x^{*i} &= x^i + \varepsilon\xi^i(x, u) + O(\varepsilon^2); \quad i = 1, 2, \ldots, p, \\ u^{*\alpha} &= u^\alpha + \varepsilon\eta^\alpha(x, u) + O(\varepsilon^2); \quad \alpha = 1, 2, \ldots, q, \end{aligned} \tag{3}$$

where ε is the group parameter. The Lie algebra of (2) is spanned by vector field

$$X = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{4}$$

The n-th order prolongation of X is given by:

$$X^{(n)} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} + \sum_{\alpha=1}^{q}\sum_{J} \eta^\alpha_J(x, u^{(n)})\frac{\partial}{\partial u^\alpha_J} \tag{5}$$

Where $J = (i_1, \ldots, i_k), 1 \leq i_k \leq p, 1 \leq k \leq n$, and the sum is over all $J's$ of order $0 < \#J \leq n$. If $\#J = k$, the coefficient η^α_J of $\frac{\partial}{\partial u^\alpha_J}$ will depend only on k-th and lower order derivatives of u, and

$$\eta^J_\alpha(x, u^{(n)}) = D_J(\eta_\alpha - \sum_{i=1}^{p} \xi^i u^\alpha_i) + \sum_{i=1}^{p} \xi^i u^\alpha_{J,i}, \tag{6}$$

where $u^\alpha_i = \frac{\partial u^\alpha}{\partial x^i}$ and $u^\alpha_{J,i} = \frac{\partial u^\alpha_J}{\partial x^i}$.

3. Classical Symmetries of the Sawada-Kotera Equation

The one-parameter Lie group of infinitesimal transformations of the Sawada-Kotera Equation in x, t, u is given by

$$\begin{aligned} x^* &= x + \varepsilon\xi^1(x,t,u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\xi^2(x,t,u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\eta(x,t,u) + O(\varepsilon^2). \end{aligned}$$

where ε is the group parameter, and the Lie algebra of (1) is spanned by vector field of the form:

$$X = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{7}$$

The fifth prolongation of X is given by

$$\begin{aligned} X^{(5)} = X \quad &+ \quad \eta^x\frac{\partial}{\partial u_x} + \eta^t\frac{\partial}{\partial u_t} + \eta^{2x}\frac{\partial}{\partial u_{2x}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{2t}\frac{\partial}{\partial u_{2t}} \\ &+ \quad \eta^{3x}\frac{\partial}{\partial u_{3x}} + \ldots + \eta^{3t}\frac{\partial}{\partial u_{3t}} + \eta^{4x}\frac{\partial}{\partial u_{4x}} + \ldots + \eta^{4t}\frac{\partial}{\partial u_{4t}} \\ &+ \quad \eta^{5x}\frac{\partial}{\partial u_{5x}} + \ldots + \eta^{5t}\frac{\partial}{\partial u_{5t}} \end{aligned} \tag{8}$$

with coefficients

$$\eta^J = D_J(\eta - \sum_{i=1}^{2}\xi^i u_i^\alpha) + \sum_{i=1}^{2}\xi^i u_{J,i}^\alpha, \tag{9}$$

where $J = (i_1, \ldots, i_k)$, $1 \le i_k \le 2, 1 \le k \le 5$, and the sum is over all $J's$ of order $0 < \#J \le 5$. Using the fifth prolongation $(X^{(5)})$ to Eq.(1)

$$X^{(5)}(u_t + 5u^2u_x + 5u_xu_{xx} + 2uu_{xxx} + u_{xxxxx})|_{(1)=0} = 0, \tag{10}$$

we can obtain ξ^1, ξ^2 and η.

Theorem 1. *The Lie algebra of Eq.(1)(Sawada-Kotera equation) is generated by the vector field*

$$X = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{11}$$

where

$$\xi^1 = \frac{1}{5}c_1\,x + c_3$$

$$\xi^2 = c_1\, t + c_2$$
$$\eta = -\frac{2}{5} c_1\, u$$

and c_1, c_2, c_3 *are arbitrary constants.*

Proof. we Expand the Eq.(10) *and then we solve the obtained set of linear differential equations using the Maple. this completes the proof.*

Corollary 2. *Lie point symmetries of the Sawada-Kotera equation are*

$$X_1 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_3 = \frac{\partial}{\partial t}.$$

The commutation relations between these vector fields are shown in the table 1, and the entry in the i-th row and j-th column is determined as $[X_i, X_j] = X_iX_j - X_jX_i, i, j = 1, 2, 3.$

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-X_2$	$-5X_3$
X_2	X_2	0	0
X_3	$5X_3$	0	0

Table 1: Commutation relations between vector fields.

4. Group Invariant Solutions

For attain the generated group transformations by the X_i for $i = 1, 2, 3,$ we should to solve the follow system of three ODEs

$$\frac{dx^*}{d\varepsilon} = \xi_i^1(x^*(\varepsilon), t^*(\varepsilon), u^*(\varepsilon)), x^*(0) = x, \tag{12}$$
$$\frac{dt^*}{d\varepsilon} = \xi_i^2(x^*(\varepsilon), t^*(\varepsilon), u^*(\varepsilon)), t^*(0) = t, \quad i = 1, 2, 3, \tag{13}$$
$$\frac{du^*}{d\varepsilon} = \eta_i(x^*(\varepsilon), t^*(\varepsilon), u^*(\varepsilon)), u^*(0) = u. \tag{14}$$

Exponentiation the infinitesimal symmetries of equation (1), we get the one-parameter groups $G_i(\varepsilon)$ generated by X_i for i = 1,2,3.

$$G_1(\varepsilon) : (x, t, u) \to (xe^{\varepsilon}, te^{5\varepsilon}, ue^{-2\varepsilon}) \tag{15}$$

$$G_2(\varepsilon) : (x, t, u) \to (x + \varepsilon, t, u) \tag{16}$$

$$G_3(\varepsilon) : (x, t, u) \to (x, t + \varepsilon, u) \tag{17}$$

Theorem 3. *If $u = f(x,t)$ is a solution of the Sawada-Kotera equation, so are the functions*

$$G_1(\varepsilon)f(x,t) = f(xe^{-\varepsilon}, te^{-5\varepsilon})e^{-2\varepsilon},$$
$$G_2(\varepsilon)f(x,t) = f(x - \varepsilon, t),$$
$$G_3(\varepsilon)f(x,t) = f(x, t - \varepsilon).$$

Proof. A symmetry group of (2) is a local group of transformations G with the property that for solution $u = f(x)$ of (2) and defined $g.f$ for $g \in G$, then $u = g.f(x)$ is also a solution of (2)[17].

For the one-parameter group

$$G_1(\varepsilon)f(x,t) = f(xe^{-\varepsilon}, te^{-5\varepsilon})e^{-2\varepsilon}.$$

If $f(x,t)$ is any function, then its transform by $G_1(\varepsilon)$ is

$$\tilde{u} = e^{-2\varepsilon}u = e^{-2\varepsilon}f(x,t),$$

which now be written in form

$$(\tilde{x}, \tilde{t}) = G_1(\varepsilon).(x,t) = (xe^{\varepsilon}, te^{5\varepsilon}),$$

therefore

$$\tilde{u} = e^{-2\varepsilon}f(\tilde{x}e^{-\varepsilon}, \tilde{t}e^{-5\varepsilon}).$$

The proof for G_2, G_3 are similar above. □

5. Optimal System of Sawada-Kotera Equation

In this paper we want to divide the set of all invariant solutions of a given differential equation into equivalence classes. If one solution can be mapped to the other solution by a point symmetry of the PDE, then these solutions are equivalent. Classification greatly simplifies the problem of determining all invariant solutions. We need only to find one invariant solution from each class, then the whole class can be constructed by applying the symmetries. This strategy minimizes the effort needed to obtain invariant solutions[20].

Definition 4. The solutions $u = f(x)$ and $u = \bar{f}(x)$ are equivalent if a symmetry maps one to the other. Similarly, the symmetry maps X to $\bar{X}$, so these generators are regarded as equivalent. It is important to classify invariant solutions by classifying the associated symmetry generators. Having done this, one generator from each class is used to obtain the desired set of invariant solutions. An optimal system of generators is a set consisting of exactly one generator from each class.[21].

To obtain an optimal system of subgroups, we can obtain an optimal system of subalgebras. Because these are evidently equivalent. The optimal system for one-dimensional subalgebras is the same as classification of the orbits for the adjoint representation. An adjoint representation $Ad(exp(\varepsilon X_i))$ is defined by the Lie series

$$Ad(exp(\varepsilon.X_i).X_j) = X_j - \varepsilon[X_i, X_j] + \frac{\varepsilon^2}{2}[X_i, [X_i, X_j]] - ..., \qquad (18)$$

where $[X_i, X_j]$ is the commutator for the Lie algebra, ε is a parameter, and i, j = 1, 2, 3 [17]. We list all the adjoint representations of the generators of algebra Sawada-Kotera equation in Table 2, where the (i, j)-th entry indicates $Ad(exp(\varepsilon Xi))Xj$.

$Ad(exp(\varepsilon.X_i).X_j)$	X_1	X_2	X_3
X_1	X_1	$e^{\varepsilon}X_2$	$e^{5\varepsilon}X_3$
X_2	$X_1 - \varepsilon X_2$	X_2	X_3
X_3	$X_1 - 5\varepsilon X_3$	X_2	X_3

Table 2: The adjoint representation of the Sawada-Kotera equation.

Theorem 5. *An optimal system of one-dimensional Lie algebras of the Sawada-Kotera equation is provided by*

$$(i):\ X_1, \quad (ii):\ X_2, \quad (iii):\ X_2 + X_3, \quad (iv):\ X_2 - X_3 \qquad (19)$$

Proof. Consider the symmetry algebra of the Eq.(1) whose adjoint representation was determined in table 2 and let $F_i^s(X) = Ad(exp(\varepsilon X_i)X)$ is a linear map, for i = 1, 2, 3. The matrices $M_i^{\varepsilon}, i = 1, 2, 3$, with respect to basis $\{X_1, X_2, v_3\}$ are

$$M_1^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & e^{-5\varepsilon} \end{pmatrix}, \quad M_2^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$M_3^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5\varepsilon & 0 & 1 \end{pmatrix}. \qquad (20)$$

Let us consider

$$X = a_1X_1 + a_2X_2 + a_3X_3, \tag{21}$$

is a nonzero vector. Our task is to simplify it by utilizing convenient adjoint maps. Assume first that $a_1 \neq 0$. If necessary, We can suppose $a_1 = 1$. According to table 2, if we accomplish on such a X by $Ad(exp(\frac{1}{5}a_3X_3))X$, we can make the coefficient of X_3 vanish:

$$X^{'a} = Ad(exp(\frac{1}{5}a_3X_3))X = X_1 + a_2X_2$$

Then we accomplish on $X^{'a}$ using $Ad(exp(a_2X_2))X^{'a}$ to vanish the coefficient X_2, so that X is equivalent to $X^{''a} = X_1$ under the adjoint representation. The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_1 = 0$. If $a_2 \neq 0$, we put $a_2 = 1$, and then we have

$$X^{'b} = X_2 + a_3X_3.$$

We can further act on $X^{'b}$ by the group generated by X_1, this has the net effect of scaling the coefficient of X_2, X_3:

$$X^{''b} = Ad(exp(\varepsilon X_1))X^{'b} = a_2e^{\varepsilon}X_2 + a_3e^{5\varepsilon}X_3.$$

This is a scalar multiple of $X^{'''b} = X_2 + a_3'e^{4\varepsilon}X_3$, so, depending on the sign of a_3', we can make the coefficient of X_3 either +1, -1 or 0. Thus any one-dimensional subalgebra spanned by X with $a_1 = 0$, $a_2 \neq 0$ is equivalent to one spanned by either $X_2, X_2 - X_3, X_2 + X_3$. The further simplifications are not possible. Then an optimal system of the Sawada-Kotera equation is given by

$$(i):\ X_1, \quad (ii):\ X_2, \quad (iii):\ X_2 + X_3, \quad (iv):\ X_2 - X_3. \tag{22}$$

□

6. Symmetry Reduction and Differential Invariants for the Sawada-Kotera Equation

In this section we derive the symmetries and corresponding reductions of the Sawada-Kotera equation. The Sawada-Kotera equation is expressed in terms of (x, t, u), so we should to used a specific coordinates for this equation in order to reduce it. For this purpose, we should to reduce the number of independent variables of the Sawada-Kotera equation. We consider the reduction in the number of independent variables as a way to convert an original PDE into a simpler PDE or an ODE. we can convert the Sawada-Kotera in terms of (x, t, u) into an ODE in terms of (r, z). Now using the chain rule and the new obtained coordinate, we can obtain the reduced equation. For the operator,

$$(a): \ X_1 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u},$$

we have

$$\frac{dx}{x} = \frac{dt}{5t} = \frac{du}{-2u}, \tag{23}$$

the similarity variables are as follows:

$$r = \frac{t}{x^5}, \quad z = x^2 u, \tag{24}$$

a solution of our equation in this case is

$$z = f(r) \Rightarrow x^2 u = f(r) \Rightarrow u = \frac{1}{x^2} f(r), \tag{25}$$

we substitute it into (1) to determine the form of the function $f(r)$. We obtain that $f(r)$ has to satisfy the equation:

$$\begin{aligned} &-720f(r) - 180f^2(r) + (1 - 7640r)f'(r) - 10f(r)f^2(r) \\ &-25rf'(r)f^2(r) + 2750r^2 f''(r) - 2550rf(r)f'(r) - 1250r^2 f'^2(r) \\ &-625r^3 f'(r)f''(r) - 172500r^3 f^{(3)}(r) - 3000r^2 f(r)f''(r) \\ &-625r^3 f(r)f^{(3)}(r) - 43750r^4 f^{(4)}(r) - 3125r^5 f^{(5)}(r) = 0. \end{aligned} \tag{26}$$

Similar as above, for

$$(b): \ X_2 = \frac{\partial}{\partial x},$$

we obtain:

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}, \tag{27}$$

$$r = t, \quad z = u, \tag{28}$$

$$z = f(r) \Rightarrow u = f(r), \tag{29}$$

when substituting it into (1), we perceive that $f(r)$ should satisfy the equation

$$f'(r) = 0. \tag{30}$$

$$(c): \ X_3 = \frac{\partial}{\partial t},$$

The characteristic equation of this case is:

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}, \tag{31}$$

$$r = x, \quad z = u, \tag{32}$$

$$z = f(r) \Rightarrow u = f(r), \tag{33}$$

with the perceived form of $f(r)$ when substituting it into (1). After substituting we obtain that $f(r)$ has to be a solution of the following equation:

$$5f^2(r)f'(r) + 5f'(r)f''(r) + 5f(r)f^{(3)}(r) + f^{(5)}(r) = 0. \tag{34}$$

$$(d) : For \ X_2 + X_3 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t},$$

we have

$$\frac{dx}{1} = \frac{dt}{1} = \frac{du}{0}, \tag{35}$$

$$r = t - x, \quad z = u, \tag{36}$$

$$z = f(r) \Rightarrow u = f(r), \tag{37}$$

when substituting it into (1), we perceive that $f(r)$ should satisfy the equation:

$$f'(r)(1 - 5f^2(r) - 5f''(r)) - 5f(r)f^{(3)}(r) - f^{(5)}(r) = 0. \tag{38}$$

$$(e) : Lastly, \ X_2 - X_3 = \frac{\partial}{\partial x} - \frac{\partial}{\partial t},$$

for this operator, the characteristic equation and independent invariants are:

$$\frac{dx}{1} = \frac{dt}{-1} = \frac{du}{0}, \tag{39}$$

$$r = -t - x, \quad z = u, \tag{40}$$

$$z = f(r) \Rightarrow u = f(r), \tag{41}$$

the form of the function $f(r)$ is obtained with substituting it into (1), then it should satisfy the equation:

$$-f'(r)(1 + 5f^2(r) + 5f''(r)) - 5f(r)f^{(3)}(r) - f^{(5)}(r) = 0. \tag{42}$$

7. Conclusion

In this work, we used the Lie group method to find the Lie point symmetries of the Sawada-Kotera equation, which concludes to similarity variables. We utilized the corresponding invariant solutions to reduce the number of independent variables in the Sawada-Kotera equation. Also, we have obtained an optimal system for the Sawada-Kotera equation.

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