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Original Article

Orthonormal piecewise Vieta-Lucas functions for the numerical solution of the one- and two-dimensional piecewise fractional Galilei invariant advection-diffusion equations

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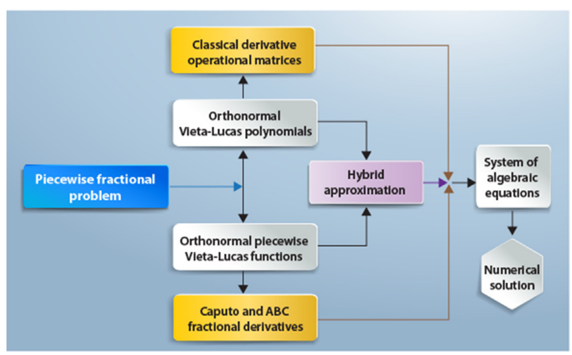
HIGHLIGHTS

- A new kind of piecewise fractional derivative is defined.
- The one- and two dimensional piecewise fractional Galilei invariant advection–diffusion equations are defined.
- The orthonormal piecewise Vieta-Lucas (VL) functions as a new family of basis functions are defined.
- Fractional derivatives in the Caputo and ABC senses of these functions are computed.
- Two hybrid methods based on the orthonormal VL polynomials and orthonormal piecewise VL functions are established for the introduced problems.
- The accuracy of the proposed methods is shown in several numerical examples.

GRAPHICAL ABSTRACT



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ABSTRACT

Introduction: Recently, a new family of fractional derivatives called the piecewise fractional derivatives has been introduced, arguing that for some problems, each of the classical fractional derivatives may not be able to provide an accurate statement of the consideration problem alone. In defining this kind of derivatives, several types of fractional derivatives can be used simultaneously.
Objectives: This study introduces a new kind of piecewise fractional derivative by employing the Caputo type distributed-order fractional derivative and ABC fractional derivative. The one- and two-dimensional piecewise fractional Galilei invariant advection–diffusion equations are defined using this piecewise fractional derivative.
Methods: A new class of basis functions called the orthonormal piecewise Vieta-Lucas (VL) functions are defined. Fractional derivatives of these functions in the Caputo and ABC senses are computed. These functions are utilized to construct two numerical methods for solving the introduced problems under non-

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equations

local boundary conditions. The proposed methods convert solving the original problems into solving systems of algebraic equations.
Results: The accuracy and convergence order of the proposed methods are examined by solving several examples. The obtained results are investigated, numerically.
Conclusion: This study introduces a kind of piecewise fractional derivative. This derivative is employed to define the one- and two-dimensional piecewise fractional Galilei invariant advection–diffusion equations. Two numerical methods based on the orthonormal VL polynomials and orthonormal piecewise VL functions are established for these problems. The numerical results obtained from solving several examples confirm the high accuracy of the proposed methods.

Introduction

During the last decades, fractional derivatives have received much attention due to their wide applications in accurate modeling of dynamic systems [1]. The most important reasons for these applications are the memory and inheritance properties of fractional derivatives, as well as their greater degree of freedom than conventional derivatives. Fractional derivatives are classified into singular and non-singular derivatives relative to the kernel function in their definition. The Caputo and Riemann–Liouville fractional derivatives can be mentioned as the most famous singular fractional derivatives [1]. In contrast, the fractional derivative presented by Atangana and Baleanu [2] and the fractional derivative introduced by Caputo and Fabrizio [3] can be mentioned as the most common non-singular fractional derivatives. Each of the introduced definitions can be used appropriately and as needed for different problems. The interested reader can find some of the uses of these derivatives in [4–9]. Meanwhile, various numerical and analytical methods have been proposed to solve problems involving these derivatives. For instance, see [10–13]. By integrating fractional derivatives with respect to the order of the derivative in a given interval, another family of fractional derivatives called the distributed-order fractional derivatives is produced [14,15]. Therefore, the distributed-order fractional form of the singular and non-singular fractional derivatives can be defined. In recent years, many applications of this type of derivatives have been reported in various authorities. For instance, see [16–19]. Recently, a new family of fractional derivatives called the piecewise fractional derivatives has been introduced, arguing that for some problems, each of the fractional derivatives listed above may not be able to provide an accurate statement of the consideration problem alone [20]. In defining this kind of derivatives, several types of fractional derivatives can be used simultaneously. Some of the research done in this field can be seen in [20–25]. In this work, we define another type of piecewise fractional derivatives using the Caputo distributed-order fractional derivative and the non-singular fractional derivative provided by Atangana and Baleanu in the Caputo sense (ABC). We also use this type of fractional derivative to define a piecewise fractional form of the one- and two-dimensional Galilei invariant advection–diffusion equations. We remind that the Galilei invariant advection–diffusion equations model the evolution of various phenomena in engineering and science [26]. During the last years, several numerical methods have been used to solve different forms of the fractional Galilei invariant advection–diffusion equations. For instance, see [27–31].

There are two important points that should be considered when choosing basis functions to construct a suitable numerical method for solving fractional differential equations. First, fractional differentiating and integrating of these functions should be easily possible. Second, a numerical method constructed based on these functions should have good accuracy, which requires that these functions be able to approximate the functions in the problem, as well as the solution of the problem with good accuracy. According to the above points, if we are dealing with a problem whose solution is a piecewise function, the basis functions of polynomials, despite the simplicity of calculating their fractional derivatives and integrals, can not be a good basis. In such cases, it is better to use piecewise polynomials as basis functions. In recent years, such basis functions have been widely used to solve various fractional problems. For instance, see [32–36]. In this study, we define the orthonormal piecewise Vieta-Lucas (VL) functions as a new family of the basis functions and employ them to solve the one- and two-dimensional piecewise fractional Galilei invariant advection–diffusion equations. Two formulas for computing fractional derivatives of these functions in the Caputo and ABC senses are presented. We use a collocation method by employing these functions and their fractional derivatives together with the Gauss–Legendre integration technique for converting the problems under consideration into algebraic systems of equations. We evaluate the accuracy of the methods numerically by solving some examples.

Preliminaries

Here, we briefly study some concepts regarding piecewise fractional derivative used in this work.

Definition 1. ([37]) The gamma function is given by

$$\Gamma(z) = \int_0^\infty s^{z-1}e^{-s}ds, z \in \mathbb{C},\ \mathscr{R}(z) > 0. \tag{1}$$

Note that for all $n \in \mathbb{N}$, we have $\Gamma(n) = (n-1)!$.

Definition 2. ([1]) The Mittag–Leffler functions are defined by

$$\mathbf{E}_\mu(\tau) = \sum_{j=0}^{\infty} \frac{\tau^j}{\Gamma(j\mu+1)}, \tag{2}$$

and

$$\mathbf{E}_{\mu,\nu}(\tau) = \sum_{j=0}^{\infty} \frac{\tau^j}{\Gamma(j\mu+\nu)}, \tag{3}$$

where $\tau \in \mathbb{C}$ and $\mu,\ \nu \in \mathbb{R}^+$.

Definition 3. ([1]) Assume $f(\tau)$ is a differential function over $[a, \tau_b]$ and $0 < \alpha \leqslant 1$ is a real number. The Riemann–Liouville fractional derivative of order α of $f(\tau)$ is given by

$$ {}^{RL}_{\ a}D^\alpha_\tau f(\tau) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_a^\tau (\tau - s)^{-\alpha} f(s) ds, & 0 < \alpha < 1, \\ f'(\tau), & \alpha = 1. \end{cases} \tag{4}$$

Definition 4. ([1]) Let $f(\tau)$ is a differential function over $[a, \tau_b]$ and $0 < \alpha \leqslant 1$ is a real number. The Caputo fractional derivative of order α of $f(\tau)$ is given by

$$ {}_a^CD_\tau^\alpha f(\tau) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_a^\tau (\tau - s)^{-\alpha} f'(s) ds, & 0 < \alpha < 1, \\ f'(\tau), & \alpha = 1. \end{cases} \quad (5) $$

Property 1. ([1]) For the differentiable function f, and $0 < \alpha < 1$, we have the following relation between the above two fractional derivatives:

$$ {}_a^{RL}D_\tau^\alpha f(\tau) = {}_a^CD_\tau^\alpha f(\tau) + \frac{f(a)}{\Gamma(1-\alpha)} \tau^{-\alpha}. \quad (6) $$

Property 2. ([1]) For $0 < \alpha < 1$ and $k \in \mathbb{N} \cup \{0\}$, we have

$$ {}_a^CD_\tau^\alpha (\tau - a)^k = \begin{cases} 0, & k = 0, \\ \frac{k! (\tau-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}, & k = 1, 2, \ldots. \end{cases} \quad (7) $$

Definition 5. ([2]) Suppose that $f(\tau)$ is a differential function over $[a, \tau_b]$ and $0 < \beta \leqslant 1$ is a real number. The ABC fractional derivative of order β of $f(\tau)$ is given by

$$ {}_a^{ABC}D_\tau^\beta f(\tau) = \begin{cases} \frac{AB(\beta)}{1-\beta} \int_a^\tau \mathbf{E}_\beta \left(\frac{-\alpha(\tau-s)^\beta}{1-\beta} \right) f'(s) ds, & 0 < \beta < 1, \\ f'(\tau), & \beta = 1, \end{cases} \quad (8) $$

where $AB(\beta) = 1 - \beta + \beta/\Gamma(\beta)$.

Property 3. For $0 < \beta < 1$ and $k \in \mathbb{N} \cup \{0\}$, we have

$$ {}_a^{ABC}D_\tau^\beta (\tau - a)^k = \begin{cases} 0, & k = 0, \\ \frac{AB(\beta)\, k! (\tau-a)^k}{1-\beta} \mathbf{E}_{\beta,k+1} \left(\frac{-\beta (\tau-a)^\beta}{1-\beta} \right), & k = 1, 2, \ldots. \end{cases} \quad (9) $$

Definition 6. ([38]) Let $f(\tau)$ is a differential function over $[a, \tau_b]$ and $\rho(\alpha) \geqslant 0$ where $\rho \neq 0, \alpha \in [0, 1]$ and $\int_0^1 \rho(\alpha) d\alpha = c_0 > 0$. The Caputo distributed-order fractional differentiation of $f(\tau)$ is defined by

$$ {}_a^CD_\tau^{\rho(\alpha)} f(\tau) = \int_0^1 \rho(\alpha) {}_a^CD_\tau^\alpha f(\tau) d\alpha, \quad (10) $$

where $\rho(\alpha)$ is the distribution of order α. Note that for $\alpha = 0$, we have ${}_a^CD_t^\alpha f(\tau) = f(\tau)$. Moreover, for $\rho(\alpha) = \delta(\alpha - \lambda)$, where δ is the Dirac delta function and $0 < \lambda < 1$, we have

$$ {}_a^CD_t^{\rho(\alpha)} f(\tau) = \int_0^1 \delta(\alpha - \lambda) {}_a^CD_\tau^\alpha f(\tau) d\alpha = {}_a^CD_\tau^\lambda f(\tau). \quad (11) $$

Definition 7. Let the assumptions of Definitions 5 and 6 be valid. Then, using the Caputo distributed-order fractional derivative and ABC fractional derivative the following kind of piecewise fractional derivative can be defined:

$$ {}_a^PD_{\tau;\tau_1}^{\rho(\alpha),\beta} f(\tau) = \begin{cases} {}_a^CD_\tau^{\rho(\alpha)} f(\tau), & a \leqslant \tau < \tau_1, \\ {}_a^{ABC}D_\tau^\beta f(\tau), & \tau_1 \leqslant \tau \leqslant \tau_b, \end{cases} \quad (12) $$

where $\tau_1 \in (0, \tau_b)$ and ${}_a^CD_t^{\rho(\alpha)} f(\tau)$ and ${}_a^{ABC}D_\tau^\beta f(\tau)$ are defined respectively in Definitions 6 and 5.

Basis functions and approximation

In this section, we first review the one variable VL polynomials and then introduce the orthonormal piecewise VL functions.

One variable Vieta-Lucas polynomials

The one variable VL polynomials can be defined over $[0, \tau_b]$ as [39]

$$ \hat{\psi}_{\tau_b,i}(\tau) = \begin{cases} 2, & i = 0, \\ \sum_{k=0}^{i} (-1)^{i-k} \frac{4^k (2i)(i+k-1)!}{(2k)!(i-k)!} \left(\frac{\tau}{\tau_b} \right)^k, & i \geqslant 1. \end{cases} \quad (13) $$

The orthogonal property of these polynomials is as

$$ \left\langle \hat{\psi}_{\tau_b,i}(\tau), \hat{\psi}_{\tau_b,j}(\tau) \right\rangle_{\omega_{\tau_b}} = \int_0^{\tau_b} \omega_{\tau_b}(\tau) \hat{\psi}_{\tau_b,i}(\tau) \hat{\psi}_{\tau_b,j}(\tau) d\tau = \begin{cases} 4\pi, & i = j = 0, \\ 2\pi, & i = j \neq 0, \\ 0, & i \neq j, \end{cases} \quad (14) $$

where $\omega_{\tau_b}(\tau) = \frac{1}{\sqrt{\tau_b \tau - \tau^2}}$ is the weight function. The orthonormal form of the above polynomials with respect to the weight function $\omega_{\tau_b}(\tau)$ can be defined over $[0, \tau_b]$ as

$$ \psi_{\tau_b,i}(\tau) = \sum_{k=0}^{i} a_{ik}^{(\tau_b)} \tau^k, i = 0, 1, \ldots, \quad (15) $$

where

$$ a_{ik}^{(\tau_b)} = \begin{cases} \frac{1}{\sqrt{\pi}}, & i = 0, \\ \frac{1}{\sqrt{2\pi}} \frac{(-1)^{i-k} (2i)(i+k-1)!}{(2k)!(i-k)!} \left(\frac{4}{\tau_b} \right)^k, & i \geqslant 1. \end{cases} $$

A function $f(\tau) \in L^2_{\omega_{\tau_b}}[0, \tau_b]$ can be approximated by the orthonormal one variable VL polynomials as

$$ f(\tau) \simeq \sum_{i=0}^{\widehat{N}} f_i \psi_{\tau_b,i}(\tau) \triangleq \mathbf{F}^T \Psi_{\tau_b,\widehat{N}}(\tau), \quad (16) $$

where

$$ \mathbf{F} = \left[f_0 f_1 \ldots f_{\widehat{N}} \right]^T, $$

with

$$ f_i = \left\langle \psi_{\tau_b,i}(\tau), f(\tau) \right\rangle_{\omega_{\tau_b}} = \int_0^{\tau_b} \omega_{\tau_b}(\tau) \psi_{\tau_b,i}(\tau) f(\tau) d\tau, $$

and

$$ \Psi_{\tau_b,\widehat{N}}(\tau) = \left[\psi_{\tau_b,0}(\tau)\ \psi_{\tau_b,1}(\tau)\ \ldots\ \psi_{\tau_b,\widehat{N}}(\tau) \right]^T. \quad (17) $$

Error upper bound of the one variable Vieta-Lucas polynomials expansion

In the sequel, we derive a formula for the error upper bound of the one variable VL polynomials expansion.

Theorem 1. Assume that $f \in C^{\widehat{N}+1}[0, \tau_b]$ and $X_{\tau_b,\widehat{N}} = span\left\{ \psi_{\tau_b,0}(\tau), \psi_{\tau_b,1}(\tau), \ldots, \psi_{\tau_b,\widehat{N}}(\tau) \right\}$. If $\mathbf{F}^T \Psi_{\tau_b,\widehat{N}}(\tau)$ is the best approximation of $f(\tau)$ in X. Then, we have

$$ \left\| f(\tau) - \mathbf{F}^T \Psi_{\tau_b,\widehat{N}}(\tau) \right\|_{L^2_{\omega_{\tau_b}}[0,\tau_b]} \leqslant \frac{\sqrt[4]{\pi} \overline{L} \tau_b^{\widehat{N}+1}}{\left(\widehat{N}+1 \right)!} \sqrt{\frac{\Gamma\left(2\widehat{N} + \frac{5}{2} \right)}{\left(2\widehat{N} + 2 \right)!}}, \quad (18) $$

where $\overline{L} = \sup_{\tau \in [0,\tau_b]} \left| f^{\left(\widehat{N}+1 \right)}(\tau) \right|$.

Proof. Since the set $\left\{1, \tau, \tau^2, \ldots, \tau^{\widehat{N}}\right\}$ is a basis for the polynomials space of degree $\widehat{N}$, we can define

$$f_0(\tau) = f(0) + \tau f'(0) + \frac{\tau^2}{2!} f''(0) + \ldots + \frac{\tau^{\widehat{N}}}{\widehat{N}!} f^{(\widehat{N})}(0).$$

Based on the Taylor series expansion, there is a $\bar{\tau} \in [0, \tau_b]$ such that

$$|f(\tau) - f_0(\tau)| = \left| \frac{\tau^{\widehat{N}+1}}{\left(\widehat{N}+1\right)!} f^{\left(\widehat{N}+1\right)}(\bar{\tau}) \right|.$$

Since $\mathbf{F}^T \Psi_{\tau_b,\widehat{N}}(\tau)$ is the best approximation of $f(\tau)$ in $X_{\tau_b,\widehat{N}}$, from the above result, we get

$$\begin{aligned}
\left\| f(\tau) - \mathbf{F}^T \Psi_{\tau_b,\widehat{N}}(\tau) \right\|_{L^2_{\omega_{\tau_b}}[0,\tau_b]} &\leqslant \| f(\tau) - f_0(\tau) \|_{L^2_{\omega_{\tau_b}}[0,\tau_b]} \\
&= \left(\int_0^{\tau_b} \omega_{\tau_b}(\tau) |f(\tau) - f_0(\tau)|^2 d\tau \right)^{1/2} \\
&= \frac{1}{\left(\widehat{N}+1\right)!} \left(\int_0^{\tau_b} \omega_{\tau_b}(\tau) \tau^{2\left(\widehat{N}+1\right)} \left| f^{\left(\widehat{N}+1\right)}(\bar{\tau}) \right|^2 d\tau \right)^{1/2} \\
&\leqslant \frac{\sup\limits_{\tau \in [0,\tau_b]} \left| f^{\left(\widehat{N}+1\right)}(\tau) \right| \sqrt[4]{\pi} \tau_b^{\widehat{N}+1}}{\left(\widehat{N}+1\right)!} \sqrt{\frac{\Gamma\left(2\widehat{N}+\frac{5}{2}\right)}{\left(2\widehat{N}+2\right)!}},
\end{aligned}$$

which completes the proof.

Operational matrices of derivative

Here, two matrix relationships regarding the first- and second-order derivatives of the orthonormal one variable VL polynomials are expressed.

Theorem 2. The first-order derivative of the vector $\Psi_{\tau_b,\widehat{N}}(\tau)$ defined in (17) can be represented as

$$\frac{d\Psi_{\tau_b,\widehat{N}}(\tau)}{d\tau} = \mathbf{D}^{(1)}_{\tau_b,\widehat{N}} \Psi_{\tau_b,\widehat{N}}(\tau), \tag{19}$$

where $\mathbf{D}^{(1)}_{\tau_b,\widehat{N}}$ is an $\left(\widehat{N}+1\right) \times \left(\widehat{N}+1\right)$ matrix as

$$\mathbf{D}^{(1)}_{\tau_b,\widehat{N}} = \mathbf{A}_{\tau_b,\widehat{N}} \hat{\mathbf{D}}^{(1)}_{\widehat{N}} \mathbf{A}^{-1}_{\tau_b,\widehat{N}},$$

with

$$\left[\mathbf{A}_{\tau_b,\widehat{N}}\right]_{ij} = \begin{cases} \frac{1}{\sqrt{\pi}}, & i = j = 1, \\ \frac{1}{\sqrt{2\pi}} \frac{(-1)^{i-j}(2i-2)(i+j-3)!}{(2j-2)!(i-j)!} \left(\frac{4}{\tau_b}\right)^{j-1}, & 2 \leqslant i \leqslant \widehat{N}+1, 1 \leqslant j \leqslant i, \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

$$\left[\mathbf{A}^{-1}_{\tau_b,\widehat{N}}\right]_{ij} = \begin{cases} \sqrt{\pi}, & i = j = 1, \\ \frac{\tau_b^{i-1}\Gamma\left(i-\frac{1}{2}\right)}{(i-1)!}, & 2 \leqslant i \leqslant \widehat{N}+1, j = 1, \\ \frac{\tau_b^{i-1}\sqrt{2}(i-1)!\Gamma\left(i-\frac{1}{2}\right)}{(i+j-2)!(i-j)!}, & 2 \leqslant i \leqslant \widehat{N}+1, 2 \leqslant j \leqslant i, \\ 0, & \text{otherwise}, \end{cases} \tag{21}$$

and

$$\left[\hat{\mathbf{D}}^{(1)}_{\widehat{N}}\right]_{ij} = \begin{cases} i-1, & 2 \leqslant i \leqslant \widehat{N}+1, 1 \leqslant j \leqslant i-1, i = j+1, \\ 0, & \text{otherwise}. \end{cases} \tag{22}$$

Proof. From (15), it is obvious that $\Psi_{\tau_b,\widehat{N}}(\tau)$ can be represented as

$$\Psi_{\tau_b,\widehat{N}}(\tau) = \mathbf{A}_{\tau_b,\widehat{N}} \begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \vdots \\ \tau^{\widehat{N}} \end{pmatrix},$$

where the elements of the matrix $\mathbf{A}_{\tau_b,\widehat{N}}$ can be computed using relation (20). We have

$$\frac{d\Psi_{\tau_b,\widehat{N}}(\tau)}{d\tau} = \mathbf{A}_{\tau_b,\widehat{N}} \frac{d}{d\tau} \begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \vdots \\ \tau^{\widehat{N}} \end{pmatrix} = \mathbf{A}_{\tau_b,\widehat{N}} \begin{pmatrix} 0 \\ 1 \\ 2\tau \\ \vdots \\ \widehat{N}\tau^{\widehat{N}-1} \end{pmatrix} = \mathbf{A}_{\tau_b,\widehat{N}} \hat{\mathbf{D}}^{(1)}_{\widehat{N}} \begin{pmatrix} 1 \\ \tau \\ \tau^2 \\ \vdots \\ \tau^{\widehat{N}} \end{pmatrix},$$

where the elements of the matrix $\hat{\mathbf{D}}^{(1)}_{\widehat{N}}$ can be computed using relation (22). Thus, from the above two relations, we obtain

$$\frac{d\Psi_{\tau_b,\widehat{N}}(\tau)}{d\tau} = \left(\mathbf{A}_{\tau_b,\widehat{N}} \hat{\mathbf{D}}^{(1)}_{\widehat{N}} \mathbf{A}^{-1}_{\tau_b,\widehat{N}}\right) \Psi_{\tau_b,\widehat{N}}(\tau) \triangleq \mathbf{D}^{(1)}_{\tau_b,\widehat{N}} \Psi_{\tau_b,\widehat{N}}(\tau),$$

where the elements of the matrix $\mathbf{A}^{-1}_{\tau_b,\widehat{N}}$ can be computed using relation (21). Then, the desired result is obtained. □

Corollary 3. From the above Theorem, we obtain the following relation for the second-order derivative of the vector $\Psi_{\tau_b,\widehat{N}}(\tau)$:

$$\frac{d^2\Psi_{\tau_b,\widehat{N}}(\tau)}{d\tau^2} = \mathbf{D}^{(1)}_{\tau_b,\widehat{N}} \times \mathbf{D}^{(1)}_{\tau_b,\widehat{N}} \Psi_{\tau_b,\widehat{N}}(\tau) \triangleq \mathbf{D}^{(2)}_{\tau_b,\widehat{N}} \Psi_{\tau_b,\widehat{N}}(\tau). \tag{23}$$

As a numerical example, for $\widehat{N} = 5$, we have

$$\mathbf{D}^{(1)}_{\tau_b,5} = \frac{1}{\tau_b} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 6\sqrt{2} & 0 & 12 & 0 & 0 & 0 \\ 0 & 16 & 0 & 16 & 0 & 0 \\ 10\sqrt{2} & 0 & 20 & 0 & 20 & 0 \end{bmatrix},$$

$$\mathbf{D}^{(2)}_{\tau_b,5} = \frac{1}{\tau_b^2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 16\sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 96 & 0 & 0 & 0 & 0 \\ 128\sqrt{2} & 0 & 192 & 0 & 0 & 0 \\ 0 & 480 & 0 & 320 & 0 & 0 \end{bmatrix}.$$

Orthonormal piecewise Vieta-Lucas functions

The orthonormal piecewise VL functions can be defined over $[0, \tau_b]$ as

$$\varphi_{\tau_b,nm}(\tau) = \begin{cases} \sqrt{N}\psi_{\tau_b,m}(N\tau - n\tau_b), & \tau \in \left[\frac{n\tau_b}{N}, \frac{(n+1)\tau_b}{N}\right], \\ 0, & \text{otherwise}, \end{cases} \tag{24}$$

where $N, M \in \mathbb{Z}^+, n = 0, 1, \ldots, N-1$ and $m = 0, 1, \ldots, M-1$. These functions are orthonormal with respect to the weigh functions

$$w_{\tau_b,n}(\tau)=\begin{cases}\frac{1}{\sqrt{\tau_b(N\tau-n\tau_b)-(N\tau-n\tau_b)^2}}, & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ 0, & otherwise,\end{cases} \tag{25}$$

where $n=0,1,\ldots,N-1$. These piecewise functions can be applied for approximating any function $g(\tau)\in L^2_{w_{\tau_b,n}}[0,\tau_b]$ as

$$g(\tau)\simeq\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}g_{nm}\varphi_{\tau_b,nm}(\tau)\triangleq\mathbf{G}^T\Phi_{\tau_b,NM}(\tau), \tag{26}$$

where

$$\mathbf{G}=\left[g_{00}\,g_{01}\,\cdots g_{0(M-1)}|g_{10}\,g_{11}\,\cdots g_{1(M-1)}|\,\cdots\,|g_{(N-1)0}\,g_{(N-1)1}\,\cdots g_{(N-1)(M-1)}\right]^T,$$

with

$$g_{nm}=\left\langle\varphi_{\tau_b,nm}(\tau),g(\tau)\right\rangle_{w_{\tau_b,n}}=\int_0^{\tau_b}w_{\tau_b,n}(\tau)\varphi_{\tau_b,nm}(\tau)g(\tau)d\tau,$$

and

$$\Phi_{\tau_b,NM}(\tau)=\left[\varphi_{\tau_b,00}(\tau)\,\varphi_{\tau_b,01}(\tau)\ldots\varphi_{\tau_b,0(M-1)}(\tau)|\varphi_{\tau_b,10}(\tau)\,\varphi_{\tau_b,11}(\tau)\ldots\varphi_{\tau_b,1(M-1)}(\tau)|\ldots |\varphi_{\tau_b,(N-1)0}(\tau)\,\varphi_{\tau_b,(N-1)1}(\tau)\ldots\varphi_{\tau_b,(N-1)(M-1)}(\tau)\right]^T. \tag{27}$$

Error upper bound of the orthonormal piecewise Vieta-Lucas functions expansion

In the continuation, we obtain an error bound for the orthonormal piecewise VL functions expansion.

Theorem 4. Let M is a positive integer and $g\in C^M\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]$ for $n=0,1,\ldots,N-1$. If $g^*(\tau)=\mathbf{G}^T\Phi_{\tau_b,NM}(\tau)$ is the best approximation of $g(\tau)$ in the space

$$\Pi_{\tau_b,NM}=span\left\{\varphi_{\tau_b,nm}(\tau),\,n=0,1,\ldots,N-1,\,m=0,1,\ldots,M-1\right\}, \tag{28}$$

the upper bound of the error satisfies the following relation:

$$\|g(\tau)-g^*(\tau)\|_{L^2_{w_{\tau_b,n}}[0,\tau_b]}\leqslant\frac{\sqrt[4]{\pi}\overline{M}\tau_b^M}{M!N^M}\sqrt{\frac{\Gamma\left(2M+\frac{1}{2}\right)}{(2M)!}}, \tag{29}$$

where

$$\overline{M}=\max\left\{\sup_{\tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]}\left|g^{(M)}(\tau)\right|,\,n=0,1,\ldots,N-1\right\}.$$

Proof. From the L^2-norm, we have

$$\begin{aligned}\|g(\tau)-g^*(\tau)\|^2_{L^2_{w_{\tau_b,n}}[0,\tau_b]}&=\sum_{n=0}^{N-1}\|g(\tau)-g^*(\tau)\|^2_{L^2_{w_{\tau_b,n}}\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]}\\&=\sum_{n=0}^{N-1}\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}w_{\tau_b,n}(\tau)|g(\tau)-g^*(\tau)|^2d\tau.\end{aligned} \tag{30}$$

Since $g^*(\tau)$ is the best approximation of $g(\tau)$ in $\Pi_{\tau_b,NM}$, for any subinterval $\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]$, we have

$$|g(\tau)-g^*(\tau)|\leqslant|g(\tau)-\hat{g}(\tau)|,$$

where $\hat{g}(\tau)$ is the Taylor series expansion of $g(\tau)$. So, from the above relation and the Taylor series Theorem, we obtain

$$|g(\tau)-g^*(\tau)|\leqslant\frac{\left(\tau-\frac{n\tau_b}{N}\right)^M}{M!}\sup_{\tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]}\left|g^{(M)}(\tau)\right|, \tag{31}$$

for $n=0,1,\ldots,N-1$. Substituting (31) into (30), yields

$$\begin{aligned}\|g(\tau)-g^*(\tau)\|^2_{L^2_{w_{\tau_b,n}}[0,\tau_b]}&\leqslant\sum_{n=0}^{N-1}\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}w_{\tau_b,n}(\tau)\left(\frac{\left(\tau-\frac{n\tau_b}{N}\right)^M}{M!}\sup_{\tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]}\left|g^{(M)}(\tau)\right|\right)^2d\tau\\&\leqslant\frac{\overline{M}^2}{(M!)^2}\sum_{n=0}^{N-1}\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}w_{\tau_b,n}(\tau)\left(\tau-\frac{n\tau_b}{N}\right)^{2M}d\tau\\&=\frac{\overline{M}^2}{(M!)^2N^{2M+1}}\sum_{n=0}^{N-1}\int_0^{\tau_b}w_{\tau_b}(s)s^{2M}ds\\&=\frac{\overline{M}^2}{(M!)^2N^{2M+1}}\sum_{n=0}^{N-1}\frac{\sqrt{\pi}\tau_b^{2M}\Gamma\left(2M+\frac{1}{2}\right)}{(2M)!}\\&=\frac{\overline{M}^2}{(M!)^2N^{2M}}\frac{\sqrt{\pi}\tau_b^{2M}\Gamma\left(2M+\frac{1}{2}\right)}{(2M)!}.\end{aligned}$$

Subsequently, by putting the square root on the both sides of the above relation, we obtain

$$\|g(\tau)-g^*(\tau)\|_{L^2_{w_{\tau_b,n}}[0,\tau_b]}\leqslant\frac{\sqrt[4]{\pi}\overline{M}\tau_b^M}{M!N^M}\sqrt{\frac{\Gamma\left(2M+\frac{1}{2}\right)}{(2M)!}},$$

which completes the proof. □

Hybrid expansion

A function $v(\zeta,\tau)$ defined over $[0,\zeta_b]\times[0,\tau_b]$ may be expanded by the above the orthonormal one variable VL polynomials and orthonormal piecewise VL functions as follows:

$$v(\zeta,\tau)\simeq\sum_{i=0}^{\widehat{N}}\sum_{j=1}^{NM}v_{ij}\psi_{\zeta_b,i}(\zeta)\hat{\varphi}_{\tau_b,j}(\tau)\triangleq\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\mathbf{V}\Phi_{\tau_b,NM}(\tau), \tag{32}$$

where $\hat{\varphi}_{\tau_b,j}(\tau)=\varphi_{\tau_b,nm}(\tau)$ with $j=nM+m+1$ for $n=0,1,\ldots,N-1$ and $m=0,1,\ldots,M-1$, and $\mathbf{V}=[v_{ij}]$ is an $\left(\widehat{N}+1\right)\times NM$ matrix with entries

$$v_{ij}=\int_0^{\zeta_b}\int_0^{\tau_b}w_{\zeta_b}(\zeta)w_{\tau_b,n}(\tau)\psi_{\zeta_b,i}(\zeta)\hat{\varphi}_{\tau_b,j}(\tau)v(\zeta,\tau)d\tau d\zeta,\quad 0\leqslant i\leqslant\widehat{N},\,1\leqslant j\leqslant NM.$$

Error upper bound of the hybrid expansion

Here, we derive an upper bound for the error of the above hybrid expansion. In the sequel, we assume that $\mathscr{X}=span\left\{\psi_{\zeta_b,0}(\zeta),\psi_{\zeta_b,1}(\zeta),\ldots,\psi_{\zeta_b,\widehat{N}}(\zeta)\right\}$ and $\mathscr{Y}=span\{\hat{\varphi}_{\tau_b,1}(\tau),\hat{\varphi}_{\tau_b,2}(\tau),\ldots,\hat{\varphi}_{\tau_b,NM}(\tau)\}$.

Theorem 5. Suppose that $v(\zeta,\tau)\in C^{q+1}\left([0,\zeta_b]\times\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]\right)$ for $n=0,1,\ldots,N-1$. If $v^*(\zeta,\tau)=\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\mathbf{V}\Phi_{\tau_b,NM}(\tau)$ is the best approximation of $v(\zeta,\tau)$ in the space $\mathscr{X}\times\mathscr{Y}$. Then, we have the below error upper bound:

$$\|v(\zeta,\tau)-v^*(\zeta,\tau)\|^2_{L^2_{w_{\zeta_b},w_{\tau_b,n}}([0,\zeta_b]\times[0,\tau_b])}\leqslant\frac{\mathscr{M}}{(q+1)!}\frac{1}{N^{q+1}}\sqrt{\sum_{k=0}^{2q+2}N^k\binom{2q+2}{k}\frac{\pi\zeta_b^k\tau_b^{2q-k+2}\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(2q-k+\frac{5}{2}\right)}{k!(2q-k+2)!}}, \tag{33}$$

where $\mathscr{M}=\max\{\mathscr{M}_n,\ n=0,1,\ldots,N-1\}$ and

$$\mathscr{M}_n=\max\left\{\sup_{(\zeta,\tau)\in[0,\zeta_b]\times\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]}\left|v^{(k)}(\zeta,\tau)\right|,\ k=0,1,\ldots,q+1\right\}.$$

Proof. we have

$$\begin{aligned}&\|v(\zeta,\tau)-v^*(\zeta,\tau)\|^2_{L^2_{w_{\zeta_b},w_{\tau_b,n}}([0,\zeta_b]\times[0,\tau_b])}\\&=\sum_{n=0}^{N-1}\|v(\zeta,\tau)-v^*(\zeta,\tau)\|^2_{L^2_{w_{\zeta_b},w_{\tau_b,n}}\left([0,\zeta_b]\times\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]\right)}\\&=\sum_{n=0}^{N-1}\int_0^{\zeta_b}\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}w_{\zeta_b}(\zeta)w_{\tau_b,n}(\tau)|v(\zeta,\tau)-v^*(\zeta,\tau)|^2d\tau d\zeta.\end{aligned} \tag{34}$$

Since $v^*(\zeta,\tau)$ is the best approximation of $v(\zeta,\tau)$ in $\mathscr{X}\times\mathscr{Y}$, for any sub-interval $\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right]$, we have

$$|v(\zeta,\tau)-v^*(\zeta,\tau)|\leqslant|v(\zeta,\tau)-\hat{v}(\zeta,\tau)|, \tag{35}$$

where $\hat{v}(\zeta,\tau)$ is the Taylor series expansion of $v(\zeta,\tau)$. On the other hand, we have

$$v(\zeta,\tau)=\underbrace{\sum_{k=0}^{q}\frac{1}{k!}\left(\zeta\frac{\partial}{\partial\zeta}+\left(\tau-\frac{n\tau_b}{N}\right)\frac{\partial}{\partial\tau}\right)^k v(\zeta,\tau)\Big|_{(\zeta,\tau)=\left(0,\frac{n\tau_b}{N}\right)}}_{\hat{v}(\zeta,\tau)}+\frac{1}{(q+1)!}\left(\zeta\frac{\partial}{\partial\zeta}+\left(\tau-\frac{n\tau_b}{N}\right)\frac{\partial}{\partial\tau}\right)^{q+1}v(\zeta,\tau)\Big|_{(\zeta,\tau)=(\hat{\zeta},\hat{\tau})}.$$

Thus, from the above two relations, we obtain

$$|v(\zeta,\tau)-\hat{v}(\zeta,\tau)|\leqslant\frac{\mathscr{M}_n}{(q+1)!}\left(\zeta+\left(\tau-\frac{n\tau_b}{N}\right)\right)^{q+1}. \tag{36}$$

From ()()()(34)–(36), we have

$$\begin{aligned}&\|v(\zeta,\tau)-v^*(\zeta,\tau)\|^2_{L^2_{w_{\zeta_b},w_{\tau_b,n}}([0,\zeta_b]\times[0,\tau_b])}\\&\leqslant\sum_{n=0}^{N-1}\frac{\mathscr{M}_n^2}{((q+1)!)^2}\int_0^{\zeta_b}\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}w_{\zeta_b}(\zeta)w_{\tau_b,n}(\tau)\left(\zeta+\left(\tau-\frac{n\tau_b}{N}\right)\right)^{2q+2}d\tau d\zeta\\&\leqslant\frac{\mathscr{M}^2}{((q+1)!)^2}\sum_{n=0}^{N-1}\int_0^{\zeta_b}\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}w_{\zeta_b}(\zeta)w_{\tau_b,n}(\tau)\left(\zeta+\left(\tau-\frac{n\tau_b}{N}\right)\right)^{2q+2}d\tau d\zeta.\end{aligned}$$

Also, we have

$$\begin{aligned}&\int_0^{\zeta_b}\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}w_{\zeta_b}(\zeta)w_{\tau_b,n}(\tau)\left(\zeta+\left(\tau-\frac{n\tau_b}{N}\right)\right)^{2q+2}d\tau d\zeta\\&=\frac{1}{N}\int_0^{\zeta_b}\int_0^{\tau_b}w_{\zeta_b}(\zeta)w_{\tau_b}(s)\left(\zeta+\frac{s}{N}\right)^{2q+2}d\zeta ds\\&=\frac{1}{N^{2q+3}}\sum_{k=0}^{2q+2}N^k\binom{2q+2}{k}\int_0^{\zeta_b}\int_0^{\tau_b}w_{\zeta_b}(\zeta)w_{\tau_b}(s)\zeta^k s^{2q-k+2}d\zeta ds\\&=\frac{1}{N^{2q+3}}\sum_{k=0}^{2q+2}N^k\binom{2q+2}{k}\frac{\pi\zeta_b^k\tau_b^{2q-k+2}\,\Gamma\left(k+\frac12\right)\Gamma\left(2q-k+\frac52\right)}{k!(2q-k+2)!}.\end{aligned}$$

Hence, from the above two relations, we get

$$\begin{aligned}&\|v(\zeta,\tau)-v^*(\zeta,\tau)\|^2_{L^2_{w_{\zeta_b},w_{\tau_b,n}}([0,\zeta_b]\times[0,\tau_b])}\\&\leqslant\frac{\mathscr{M}^2}{((q+1)!)^2}\frac{1}{N^{2q+2}}\\&\sum_{k=0}^{2q+2}N^k\binom{2q+2}{k}\frac{\pi\zeta_b^k\tau_b^{2q-k+2}\,\Gamma\left(k+\frac12\right)\Gamma\left(2q-k+\frac52\right)}{k!(2q-k+2)!}.\end{aligned}$$

Eventually, by putting the square root on both sides of the above relation, the desired result is obtained.□

Fractional derivatives of the orthonormal piecewise Vieta-Lucas functions

Here, we introduce two formulae for computing fractional derivatives of the orthonormal piecewise VL functions in the Caputo and ABC senses.

Theorem 6. Suppose that $\varphi_{\tau_b,nm}(\tau)$ are the functions given in (24) and $0\leqslant\alpha\leqslant1$ is a real number. Then, we have

$${}_0^CD_\tau^\alpha\varphi_{\tau_b,nm}(\tau)\triangleq\tilde{\varphi}_{\tau_b,nm}(\tau,\alpha)=\begin{cases}\varphi_{\tau_b,nm}(\tau), & \alpha=0,\\ \varphi'_{\tau_b,nm}(\tau), & \alpha=1,\\ \varphi^{(\alpha)}_{\tau_b,nm}(\tau), & 0<\alpha<1,\end{cases} \tag{37}$$

where

$$\varphi'_{\tau_b,nm}(\tau)=\begin{cases}0, & m=0,\\ \begin{cases}N^{\frac32}\sum\limits_{k=1}^{m}ka_{mk}^{(\tau_b)}(N\tau-n\tau_b)^{k-1}, & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ 0, & otherwise,\end{cases} & m=1,2,\ldots,M-1,\end{cases} \tag{38}$$

and

$$\varphi^{(\alpha)}_{\tau_b,nm}(\tau)=\begin{cases}0, & m=0,\\ \begin{cases}\theta_{\tau_b,nm}(\tau,\alpha), & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ \vartheta_{\tau_b,nm}(\tau,\alpha), & \tau\in\left[\frac{(n+1)\tau_b}{N},\tau_b\right],\\ 0, & otherwise,\end{cases} & m=1,2,\ldots,M-1,\end{cases} \tag{39}$$

with

$$\theta_{\tau_b,nm}(\tau,\alpha)=N^{\alpha+\frac12}\sum_{k=1}^{m}\frac{k!a_{mk}^{(\tau_b)}}{\Gamma(k-\alpha+1)}(N\tau-n\tau_b)^{k-\alpha}, \tag{40}$$

and

$$\begin{aligned}\vartheta_{\tau_b,nm}(\tau,\alpha)=&\frac{N^{\alpha+\frac12}}{\Gamma(1-\alpha)}\sum_{k=1}^{m}ka_{mk}^{(\tau_b)}\Bigg\{\prod_{l=1}^{k}\frac{1}{l-\alpha}\prod_{l=1}^{k-1}(k-l)(N\tau-n\tau_b)^{k-\alpha}\\&-\sum_{r=1}^{k}\prod_{l=1}^{r}\frac{1}{l-\alpha}\prod_{l=1}^{r-1}(k-l)\left[N^{k-r}\left(\frac{\tau_b}{N}\right)^{k-r}(N\tau-(n+1)\tau_b)^{r-\alpha}\right]\Bigg\}.\end{aligned} \tag{41}$$

Proof. For $\alpha=0$, the proof is obvious. For $\alpha=1$, from (15) and (24), we have $\frac{d\varphi_{\tau_b,n0}(\tau)}{d\tau}=0$. Also, for $m=1,2,\ldots,M-1$, we have

$$\frac{d\varphi_{\tau_b,nm}(\tau)}{d\tau}=\begin{cases}N^{\frac32}\sum\limits_{k=1}^{m}ka_{mk}^{(\tau_b)}(N\tau-n\tau_b)^{k-1}, & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ 0, & otherwise.\end{cases}$$

For $0<\alpha<1$, from Property 2 and relations (15) and (24), we obtain

$${}_0^CD_\tau^\alpha\varphi_{\tau_b,n0}(\tau)=0,$$

and for $m=1,2,\ldots,M-1$, we get

$$\begin{aligned}{}_0^CD_\tau^\alpha\varphi_{\tau_b,nm}(\tau)&=\frac{1}{\Gamma(1-\alpha)}\int_0^\tau(\tau-s)^{-\alpha}\frac{d\varphi_{\tau_b,nm}(s)}{ds}ds\\&=\begin{cases}\frac{\sqrt{N}}{\Gamma(1-\alpha)}\int_{\frac{n\tau_b}{N}}^{\tau}(\tau-s)^{-\alpha}\frac{d}{ds}\Big(\psi_{\tau_b,m}(Ns-n\tau_b)\Big)ds, & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ \frac{\sqrt{N}}{\Gamma(1-\alpha)}\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}(\tau-s)^{-\alpha}\frac{d}{ds}\Big(\psi_{\tau_b,m}(Ns-n\tau_b)\Big)ds, & \tau\in\left[\frac{(n+1)\tau_b}{N},\tau_b\right],\\ 0, & otherwise.\end{cases}\end{aligned} \tag{42}$$

Using (15) and (42), we obtain

$${}_0^CD_\tau^\alpha\varphi_{\tau_b,nm}(\tau)=\begin{cases}\frac{N^{\frac12}}{\Gamma(1-\alpha)}\sum\limits_{k=1}^{m}ka_{mk}^{(\tau_b)}N^k\int_{\frac{n\tau_b}{N}}^{\tau}(\tau-s)^{-\alpha}\left(s-\frac{n\tau_b}{N}\right)^{k-1}ds, & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ \frac{N^{\frac12}}{\Gamma(1-\alpha)}\sum\limits_{k=1}^{m}ka_{mk}^{(\tau_b)}N^k\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}(\tau-s)^{-\alpha}\left(s-\frac{n\tau_b}{N}\right)^{k-1}ds, & \tau\in\left[\frac{(n+1)\tau_b}{N},\tau_b\right],\\ 0, & otherwise.\end{cases} \tag{43}$$

Furthermore, Property 2 yields

$$\int_{\frac{n\tau_b}{N}}^{\tau}(\tau-s)^{-\alpha}\left(s-\frac{n\tau_b}{N}\right)^{k-1}ds=\frac{(k-1)!\Gamma(1-\alpha)}{\Gamma(k-\alpha+1)}\left(\tau-\frac{n\tau_b}{N}\right)^{k-\alpha}, \tag{44}$$

and integration by parts gives

$$\begin{aligned}&\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}(\tau-s)^{-\alpha}\left(s-\frac{n\tau_b}{N}\right)^{k-1}ds\\&=\prod_{l=1}^{k}\frac{1}{l-\alpha}\prod_{l=1}^{k-1}(k-l)\left(\tau-\frac{n\tau_b}{N}\right)^{k-\alpha}\\&-\sum_{r=1}^{k}\prod_{l=1}^{r}\frac{1}{l-\alpha}\prod_{l=1}^{r-1}(k-l)\left[\left(\frac{\tau_b}{N}\right)^{k-r}\left(\tau-\frac{(n+1)\tau_b}{N}\right)^{r-\alpha}\right].\end{aligned} \tag{45}$$

Substituting (44) and (45) into (43) yields

$${}_0^CD_\tau^\alpha\varphi_{\tau_b,nm}(\tau)=\begin{cases}\theta_{\tau_b,nm}(\tau,\alpha), & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ \vartheta_{\tau_b,nm}(\tau,\alpha), & \tau\in\left[\frac{(n+1)\tau_b}{N},\tau_b\right],\\ 0, & otherwise,\end{cases}$$

where $\theta_{\tau_b,nm}(\tau,\alpha)$ and $\vartheta_{\tau_b,nm}(\tau,\alpha)$ are introduced respectively in (40) and (41). Hence, the proof is completed. □

Theorem 7. Let $\varphi_{\tau_b,nm}(\tau)$ are the functions given in (24) and $0<\beta\leqslant 1$ is a real number. Then, we have

$${}_0^{ABC}D_\tau^\beta\varphi_{\tau_b,nm}(\tau)\triangleq\bar{\varphi}_{\tau_b,nm}(\tau,\beta)=\begin{cases}\varphi'_{\tau_b,nm}(\tau), & \beta=1,\\ \bar{\varphi}^{(\beta)}_{\tau_b,nm}(\tau), & 0<\beta<1,\end{cases} \tag{46}$$

where $\varphi'_{\tau_b,nm}(\tau)$ are as in (38) and

$$\bar{\varphi}^{(\beta)}_{\tau_b,nm}(\tau)=\begin{cases}0, & m=0,\\ \begin{cases}\eta_{\tau_b,nm}(\tau,\beta), & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ \sigma_{\tau_b,nm}(\tau,\beta), & \tau\in\left[\frac{(n+1)\tau_b}{N},\tau_b\right],\\ 0, & otherwise,\end{cases} & m=1,2,\ldots,M-1,\end{cases} \tag{47}$$

with

$$\eta_{\tau_b,nm}(\tau,\beta)=\frac{N^{\frac{1}{2}}AB(\beta)}{1-\beta}\sum_{k=1}^{m}a_{mk}^{(\tau_b)}k!(N\tau-n\tau_b)^k\mathbf{E}_{\beta,k+1}\left(\frac{-\beta N^{-\beta}(N\tau-n\tau_b)^\beta}{1-\beta}\right), \tag{48}$$

and

$$\begin{aligned}\sigma_{\tau_b,nm}(\tau,\beta)=&\frac{N^{\frac{1}{2}}AB(\beta)}{1-\beta}\sum_{k=1}^{m}a_{mk}^{(\tau_b)}kN^k\sum_{j=0}^{\infty}\frac{1}{\Gamma(j\beta+1)}\left(-\frac{\beta}{1-\beta}\right)^j\left\{\prod_{l=1}^{k}\frac{1}{l+j\beta}\prod_{l=1}^{k-1}(k-l)\left(\tau-\frac{n\tau_b}{N}\right)^{k+j\beta}\right.\\&\left.-\sum_{r=1}^{k}\prod_{l=1}^{r}\frac{1}{l+j\beta}\prod_{l=1}^{r-1}(k-l)\left[\left(\frac{\tau_b}{N}\right)^{k-r}\left(\tau-\frac{(n+1)\tau_b}{N}\right)^{j\beta+r}\right]\right\}.\end{aligned} \tag{49}$$

Proof. For $\beta=1$, the details of the proof are given in Theorem 6. For $0<\beta<1$, using Property 3 and relations (15) and (24), we get

$${}_0^{ABC}D_\tau^\beta\varphi_{\tau_b,n0}(\tau)=0, \tag{50}$$

and for $m=1,2,\ldots,M-1$, we have

$${}_0^{ABC}D_\tau^\beta\varphi_{\tau_b,nm}(\tau)=\begin{cases}\frac{N^{\frac{3}{2}}AB(\beta)}{1-\beta}\sum_{k=1}^{m}a_{mk}^{(\tau_b)}k\int_{\frac{n\tau_b}{N}}^{\tau}\mathbf{E}_\beta\left(\frac{-\beta(\tau-s)^\beta}{1-\beta}\right)(Ns-n\tau_b)^{k-1}ds, & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ \frac{N^{\frac{3}{2}}AB(\beta)}{1-\beta}\sum_{k=1}^{m}a_{mk}^{(\tau_b)}k\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}\mathbf{E}_\beta\left(\frac{-\beta(\tau-s)^\beta}{1-\beta}\right)(Ns-n\tau_b)^{k-1}ds, & \tau\in\left[\frac{(n+1)\tau_b}{N},\tau_b\right],\\ 0, & otherwise.\end{cases} \tag{51}$$

Property 3 yields

$$\begin{aligned}&\frac{kAB(\beta)}{1-\beta}\int_{\frac{n\tau_b}{N}}^{\tau}\mathbf{E}_\beta\left(\frac{-\beta(\tau-s)^\beta}{1-\beta}\right)(Ns-n\tau_b)^{k-1}ds\\&=\frac{N^{k-1}AB(\beta)k!\left(\tau-\frac{n\tau_b}{N}\right)^k}{1-\beta}\mathbf{E}_{\beta,k+1}\left(\frac{-\beta\left(\tau-\frac{n\tau_b}{N}\right)^\beta}{1-\beta}\right).\end{aligned} \tag{52}$$

Moreover, from the definition of the Mittag–Leffler function, we have

$$\begin{aligned}&\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}\mathbf{E}_\beta\left(\frac{-\beta(\tau-s)^\beta}{1-\beta}\right)(Ns-n\tau_b)^{k-1}ds\\&=N^{k-1}\sum_{j=0}^{\infty}\frac{1}{\Gamma(j\beta+1)}\left(-\frac{\beta}{1-\beta}\right)^j\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}(\tau-s)^{j\beta}\left(s-\frac{n\tau_b}{N}\right)^{k-1}ds.\end{aligned} \tag{53}$$

Furthermore, integration by parts gives

$$\begin{aligned}&\int_{\frac{n\tau_b}{N}}^{\frac{(n+1)\tau_b}{N}}(\tau-s)^{j\beta}\left(s-\frac{n\tau_b}{N}\right)^{k-1}ds\\&=\prod_{l=1}^{k}\frac{1}{l+j\beta}\prod_{l=1}^{k-1}(k-l)\left(\tau-\frac{n\tau_b}{N}\right)^{k+j\beta}\\&-\sum_{r=1}^{k}\prod_{l=1}^{r}\frac{1}{l+j\beta}\prod_{l=1}^{r-1}(k-l)\left[\left(\frac{\tau_b}{N}\right)^{k-r}\left(\tau-\frac{(n+1)\tau_b}{N}\right)^{j\beta+r}\right].\end{aligned} \tag{54}$$

Hence, from Eqs. (51)–(54), we have

$${}_0^{ABC}D_\tau^\beta\varphi_{\tau_b,nm}(\tau)=\begin{cases}\eta_{\tau_b,nm}(\tau,\beta), & \tau\in\left[\frac{n\tau_b}{N},\frac{(n+1)\tau_b}{N}\right],\\ \sigma_{\tau_b,nm}(\tau,\beta), & \tau\in\left[\frac{(n+1)\tau_b}{N},\tau_b\right],\\ 0, & otherwise,\end{cases}$$

where $\eta_{\tau_b,nm}(\tau,\beta)$ and $\sigma_{\tau_b,nm}(\tau,\beta)$ are introduced respectively in (48) and (49). Thus, the expressed assertion is proved.

Corollary 8. From the above Theorem, we have

$${}_0^PD_{\tau;\tau_1}^{\rho(\alpha),\beta}\varphi_{\tau_b,nm}(\tau)=\begin{cases}\int_0^1\rho(\alpha)\tilde{\varphi}_{\tau_b,nm}(\tau,\alpha)d\alpha, & 0\leqslant\tau<\tau_1,\\ \bar{\varphi}_{\tau_b,nm}(\tau,\beta), & \tau_1\leqslant\tau\leqslant\tau_b.\end{cases} \tag{55}$$

Two variables orthonormal Vieta-Lucas polynomials and their properties

In this section, we introduce the two variables orthonormal VL polynomials and derive some new results for them.

Two variables Vieta-Lucas polynomials

For $\widehat{N},\widehat{M}\in\mathbb{Z}^+$, we can define the two variables VL polynomials on $[0,\zeta_b]\times[0,\xi_b]$ as

$$\psi_{\zeta_b\xi_b,ij}(\zeta,\xi)=\psi_{\zeta_b,i}(\zeta)\psi_{\xi_b,i}(\xi), i=0,1,\ldots,\widehat{N},\ j=0,1,\ldots,\widehat{M}, \tag{56}$$

where $\psi_{\zeta_b,i}(\zeta)$ and $\psi_{\xi_b,i}(\xi)$ are defined similar to (15). These functions are orthonormal with respect to the weight function $\omega_{\zeta_b\xi_b}(\zeta,\xi)=\frac{1}{\sqrt{(\zeta_b\zeta-\zeta^2)(\xi_b\xi-\xi^2)}}$. A function $h\in L^2_{\omega_{\zeta_b\xi_b}}([0,\zeta_b]\times[0,\xi_b])$ can be approximated by these polynomials as

$$h(\zeta,\xi)\simeq\sum_{i=0}^{\widehat{N}}\sum_{j=0}^{\widehat{M}}h_{ij}\psi_{\zeta_b\xi_b,ij}(\zeta,\xi)\triangleq\mathbf{H}^T\mathbf{\Psi}_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi), \tag{57}$$

where

$$\mathbf{H} = \left[h_{00}\, h_{01}\, \ldots\, h_{0\widehat{M}}\, h_{10}\, h_{11}\, \ldots\, h_{1\widehat{M}}\, \ldots\, h_{\widehat{N}0}\, h_{\widehat{N}1}\, \ldots\, h_{\widehat{N}\widehat{M}}\right]^{\mathsf{T}},$$

with

$$h_{ij} = \int_0^{\zeta_b}\int_0^{\xi_b} \omega_{\zeta_b\xi_b}(\zeta,\xi)h(\zeta,\xi)\psi_{\zeta_b\xi_b,ij}(\zeta,\xi)d\xi d\zeta,$$

and

$$\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi) = \left[\psi_{\zeta_b\xi_b,00}(\zeta,\xi)\,\psi_{\zeta_b\xi_b,01}(\zeta,\xi)\ldots\psi_{\zeta_b\xi_b,0\widehat{M}}(\zeta,\xi)\,\psi_{\zeta_b\xi_b,10}(\zeta,\xi)\,\psi_{\zeta_b\xi_b,11}(\zeta,\xi)\ldots \psi_{\zeta_b\xi_b,1\widehat{M}}(\zeta,\xi)\ldots\psi_{\zeta_b\xi_b,\widehat{N}0}(\zeta,\xi)\,\psi_{\zeta_b\xi_b,\widehat{N}1}(\zeta,\xi)\ldots\psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right]^T. \tag{58}$$

Note that for convenience, we can rewrite (57) as follows:

$$h(\zeta,\xi) \simeq \sum_{l=0}^{(\widehat{N}+1)(\widehat{M}+1)-1} \tilde{h}_l\tilde{\psi}_{\zeta_b\xi_b,l}(\zeta,\xi) \triangleq \mathbf{H}^T\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi), \tag{59}$$

where $\tilde{h}_l = h_{ij}$ and $\tilde{\psi}_{\zeta_b\xi_b,l}(\zeta,\xi) = \psi_{\zeta_b\xi_b,ij}(\zeta,\xi)$ with $l = (\widehat{M}+1)i+j$ for $i = 0,1,\ldots,\widehat{N}$ and $j = 0,1,\ldots,\widehat{M}$.

Hybrid approximation

A function $v(\zeta,\xi,\tau)$ defined over $[0,\zeta_b]\times[0,\xi_b]\times[0,\tau_b]$ may be expanded via the orthonormal two variables VL polynomials and orthonormal piecewise VL functions as follows:

$$v(\zeta,\xi,\tau) \simeq \sum_{l=0}^{(\widehat{N}+1)(\widehat{M}+1)-1}\sum_{j=1}^{NM}\bar{v}_{lj}\tilde{\psi}_{\zeta_b\xi_b,l}(\zeta,\xi)\hat{\varphi}_{\tau_b,j}(\tau) \triangleq \left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau), \tag{60}$$

where $\bar{\mathbf{V}} = [\bar{v}_{lj}]$ is an $(\widehat{N}+1)(\widehat{M}+1)\times NM$ matrix with entries

$$\bar{v}_{lj} = \int_0^{\zeta_b}\int_0^{\xi_b}\int_0^{\tau_b}\omega_{\zeta_b\xi_b}(\zeta,\xi)w_{\tau_b,n}(\tau)\tilde{\psi}_{\zeta_b\xi_b,l}(\zeta,\xi)\hat{\varphi}_{\tau_b,j}(\tau)v(\zeta,\xi,\tau)d\tau d\xi d\zeta,$$

for $0 \leqslant l \leqslant (\widehat{N}+1)(\widehat{M}+1)$ and $1 \leqslant j \leqslant NM$, and $w_{\tau_b,n}(\tau)$ and $\omega_{\zeta_b\xi_b}(\zeta,\xi)$ have already been introduced.

Two-dimensional operational matrices

Here, we obtain some relationships for the classical derivatives of the orthonormal two variables VL polynomials.

Theorem 9. The first- and second-order derivatives of the vector $\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)$ defined in (58) can be expressed as follows:

$$\begin{aligned}\frac{\partial\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)}{\partial\zeta} &= \mathbf{P}^{(1)}_{\widehat{N}\widehat{M}}\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi),\\ \frac{\partial\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)}{\partial\xi} &= \mathbf{Q}^{(1)}_{\widehat{N}\widehat{M}}\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi),\end{aligned} \tag{61}$$

and

$$\begin{aligned}\frac{\partial^2\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)}{\partial\zeta^2} &= \mathbf{P}^{(2)}_{\widehat{N}\widehat{M}}\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi),\\ \frac{\partial^2\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)}{\partial\xi^2} &= \mathbf{Q}^{(2)}_{\widehat{N}\widehat{M}}\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi),\end{aligned} \tag{62}$$

where

$$\mathbf{P}^{(1)}_{\widehat{N}\widehat{M}} = \mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\otimes\mathbf{I}_{\widehat{M}} = \begin{pmatrix} \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{11}\mathbf{I}_{\widehat{M}} & \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{12}\mathbf{I}_{\widehat{M}} & \cdots & \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{1(\widehat{N}+1)}\mathbf{I}_{\widehat{M}}\\ \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{21}\mathbf{I}_{\widehat{M}} & \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{22}\mathbf{I}_{\widehat{M}} & \cdots & \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{2(\widehat{N}+1)}\mathbf{I}_{\widehat{M}}\\ \vdots & \vdots & \cdots & \vdots\\ \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{(\widehat{N}+1)1}\mathbf{I}_{\widehat{M}} & \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{(\widehat{N}+1)2}\mathbf{I}_{\widehat{M}} & \cdots & \left[\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right]_{(\widehat{N}+1)(\widehat{N}+1)}\mathbf{I}_{\widehat{M}} \end{pmatrix},$$

$$\mathbf{Q}^{(1)}_{\widehat{N}\widehat{M}} = \begin{pmatrix} \mathbf{D}^{(1)}_{\xi_b,\widehat{M}} & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \ldots & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}}\\ \mathbf{O}_{\widehat{M}} & \mathbf{D}^{(1)}_{\xi_b,\widehat{M}} & \mathbf{O}_{\widehat{M}} & \ldots & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}}\\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots\\ \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \ldots & \mathbf{D}^{(1)}_{\xi_b,\widehat{M}} & \mathbf{O}_{\widehat{M}}\\ \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \ldots & \mathbf{O}_{\widehat{M}} & \mathbf{D}^{(1)}_{\xi_b,\widehat{M}} \end{pmatrix},$$

and

$$\mathbf{P}^{(2)}_{\widehat{N}\widehat{M}} = \mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\otimes\mathbf{I}_{\widehat{M}} = \begin{pmatrix} \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{11}\mathbf{I}_{\widehat{M}} & \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{12}\mathbf{I}_{\widehat{M}} & \cdots & \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{1(\widehat{N}+1)}\mathbf{I}_{\widehat{M}}\\ \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{21}\mathbf{I}_{\widehat{M}} & \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{22}\mathbf{I}_{\widehat{M}} & \cdots & \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{2(\widehat{N}+1)}\mathbf{I}_{\widehat{M}}\\ \vdots & \vdots & \cdots & \vdots\\ \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{(\widehat{N}+1)1}\mathbf{I}_{\widehat{M}} & \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{(\widehat{N}+1)2}\mathbf{I}_{\widehat{M}} & \cdots & \left[\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right]_{(\widehat{N}+1)(\widehat{N}+1)}\mathbf{I}_{\widehat{M}} \end{pmatrix},$$

$$\mathbf{Q}^{(2)}_{\widehat{N}\widehat{M}} = \begin{pmatrix} \mathbf{D}^{(2)}_{\xi_b,\widehat{M}} & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \ldots & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}}\\ \mathbf{O}_{\widehat{M}} & \mathbf{D}^{(2)}_{\xi_b,\widehat{M}} & \mathbf{O}_{\widehat{M}} & \ldots & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}}\\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots\\ \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \ldots & \mathbf{D}^{(2)}_{\xi_b,\widehat{M}} & \mathbf{O}_{\widehat{M}}\\ \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \mathbf{O}_{\widehat{M}} & \ldots & \mathbf{O}_{\widehat{M}} & \mathbf{D}^{(2)}_{\xi_b,\widehat{M}} \end{pmatrix},$$

in which $\mathbf{P}^{(l)}_{\widehat{N}\widehat{M}}$ and $\mathbf{Q}^{(l)}_{\widehat{N}\widehat{M}}$ for $l = 1,2$ are $(\widehat{N}+1)(\widehat{M}+1)$-order square matrices, $\mathbf{D}^{(l)}_{\zeta_b,\widehat{N}}$ and $\mathbf{D}^{(l)}_{\xi_b,\widehat{M}}$ for $l = 1,2$ are the matrices derived in Theorem 2 and Corollary 3, $\otimes$ denotes the Kronecker product, $\mathbf{O}_{\widehat{M}}$ is an $(\widehat{M}+1)$-order zero matrix and $\mathbf{I}_{\widehat{M}}$ is an $(\widehat{M}+1)$-order identity matrix.

Proof. The proof is straightforward. So, we leave it to the interested reader. □

The proposed method for the one-dimensional problem

In this section, we establish a hybrid method based on the orthonormal one variable VL polynomials and orthonormal piecewise VL functions to solve the following one-dimensional piecewise fractional Galilei invariant advection–diffusion equation:

$$\begin{aligned}&{}^P_0D^{\rho(\alpha),\beta}_{\tau;\tau_1}v(\zeta,\tau) + \kappa_1 v_\zeta(\zeta,\tau) = \kappa_2\,{}^{RL}_0D^{1-\gamma}_\tau(v_{\zeta\zeta}(\zeta,\tau)) + w(\zeta,\tau,v(\zeta,\tau)),\\ &(\zeta,\tau)\in[0,\zeta_b]\times[0,\tau_b],\end{aligned} \tag{63}$$

with the initial condition

$$v(\zeta,0) = f(\zeta), \tag{64}$$

and non-local boundary conditions

$$\begin{aligned}v(0,\tau) - \varrho_0 v_\zeta(0,\tau) &= \textstyle\int_0^{\zeta_b}K_0(\zeta)v(\zeta,\tau)d\zeta + g(\tau),\\ v(\zeta_b,\tau) + \varrho_1 v_\zeta(\zeta_b,\tau) &= \textstyle\int_0^{\zeta_b}K_1(\zeta)v(\zeta,\tau)d\zeta + h(\tau),\end{aligned} \tag{65}$$

where $\kappa_1,\kappa_2 > 0, 0 < \gamma < 1, \zeta_b,\tau_b > 0, 0 < \tau_1 < \tau, \varrho_0,\varrho_1 > 0$ and $0 < \beta \leqslant 1$ are real numbers, w, f, K_0, K_1, g and h are continuous functions in their domains, ${}_{0}^{RL}D_{\tau}^{1-\gamma}v(\zeta,\tau)$ is the Riemann–Liouville fractional derivative of order $1-\gamma$ with respect to τ of $v(\zeta,\tau)$, and ${}_{0}^{P}D_{\tau;\tau_1}^{\rho(\alpha),\beta}v(\zeta,\tau)$ is the piecewise fractional derivative with respect to τ of $v(\zeta,\tau)$. To solve the above problem, we approximate $v(\zeta,\tau)$ as

$$v(\zeta,\tau)\simeq\sum_{i=0}^{\widehat{N}}\sum_{j=1}^{NM}v_{ij}\psi_{\zeta_b,i}(\zeta)\hat{\varphi}_{\tau_b,j}(\tau)\triangleq\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\mathbf{V}\Phi_{\tau_b,NM}(\tau),\tag{66}$$

where $\mathbf{V}=\left[v_{ij}\right]$ is an $\left(\widehat{N}+1\right)\times NM$ undetermined matrix as

$$\mathbf{V}=\begin{pmatrix}v_{00} & v_{01} & \dots & v_{0NM}\\ v_{10} & v_{11} & \dots & v_{1NM}\\ \vdots & \vdots & \dots & \vdots\\ v_{\widehat{N}0} & v_{\widehat{N}1} & \dots & v_{\widehat{N}NM}\end{pmatrix}.$$

From (12), (37), (46) and (66), we get

$${}_{0}^{P}D_{\tau;\tau_1}^{\rho(\alpha),\beta}v(\zeta,\tau)\simeq\begin{cases}\sum_{i=0}^{\widehat{N}}\sum_{j=1}^{NM}v_{ij}\psi_{\zeta_b,i}(\zeta)\int_0^1\rho(\alpha)\tilde{\tilde{\varphi}}_{\tau_b,j}(\tau,\alpha)d\alpha, & 0<\tau<\tau_1,\\ \sum_{i=0}^{\widehat{N}}\sum_{j=1}^{NM}v_{ij}\psi_{\zeta_b,i}(\zeta)\bar{\bar{\varphi}}_{\tau_b,j}(\tau,\beta), & \tau_1\leqslant\tau<\tau_b,\end{cases}\tag{67}$$

where $\tilde{\tilde{\varphi}}_{\tau_b,j}(\tau,\alpha)=\tilde{\varphi}_{\tau_b,nm}(\tau,\alpha)$ and $\bar{\bar{\varphi}}_{\tau_b,j}(\tau,\beta)=\bar{\varphi}_{\tau_b,nm}(\tau,\beta)$ with $j=nM+m+1$ for $n=0,1,\dots,N-1$ and $m=0,1,\dots,M-1$. The integrals in (67) can be computed by an $\bar{p}$-point Gauss–Legendre integration method as

$$\int_0^1\rho(\alpha)\tilde{\tilde{\varphi}}_{\tau_b,j}(\tau,\alpha)d\alpha\simeq\frac{1}{2}\sum_{r=1}^{\bar{p}}\hat{w}_r\rho\left(\frac{1}{2}(\hat{\tau}_r+1)\right)\tilde{\tilde{\varphi}}_{\tau_b,j}\left(\tau,\frac{1}{2}(\hat{\tau}_r+1)\right),\tag{68}$$

where

$$\hat{w}_r=\frac{2}{(1-\hat{\tau}_r^2)\left(L'_{\bar{p}}(\hat{\tau}_r)\right)^2},$$

and $\{\hat{\tau}_r\}_{r=1}^{\bar{p}}$ are the Gauss–Legendre integration nodes in $[-1,1]$. For more details, see [40]. Substituting (68) into (67) yields

$${}_{0}^{P}D_{\tau;\tau_1}^{\rho(\alpha),\beta}v(\zeta,\tau)\triangleq\tilde{v}(\zeta,\tau,\beta)$$
$$\simeq\begin{cases}\frac{1}{2}\sum_{i=0}^{\widehat{N}}\sum_{j=1}^{NM}v_{ij}\psi_{\zeta_b,i}(\zeta)\sum_{r=1}^{\bar{p}}\hat{w}_r\rho(\frac{1}{2}(\hat{\tau}_r+1))\tilde{\tilde{\varphi}}_{\tau_b,j}(\tau,\frac{1}{2}(\hat{\tau}_r+1)), & 0<\tau<\tau_1,\\ \sum_{i=0}^{\widehat{N}}\sum_{j=1}^{NM}v_{ij}\psi_{\zeta_b,i}(\zeta)\bar{\bar{\varphi}}_{\tau_b,j}(\tau,\beta), & \tau_1\leqslant\tau<\tau_b.\end{cases}\tag{69}$$

Theorem 2 and Corollary 3, together with relation (66), result in

$$\begin{aligned}v_{\zeta}(\zeta,\tau)&\simeq\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\left(\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}\Phi_{\tau_b,NM}(\tau),\\ v_{\zeta\zeta}(\zeta,\tau)&\simeq\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\left(\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}\Phi_{\tau_b,NM}(\tau).\end{aligned}\tag{70}$$

Property 1, together with Theorem 6 and relation (70), give

$$\begin{aligned}{}_{0}^{RL}D_{\tau}^{1-\gamma}(v_{\zeta\zeta}(\zeta,\tau))\simeq&\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\left(\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}\tilde{\Phi}_{\tau_b,NM}(\tau,\gamma)\\ &+\frac{\tau^{\gamma-1}}{\Gamma(\gamma)}\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\left(\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}\Phi_{\tau_b,NM}(0),\end{aligned}\tag{71}$$

where

$$\begin{aligned}\tilde{\Phi}_{\tau_b,NM}(\tau,\gamma)=&\left[\tilde{\varphi}_{\tau_b,00}(\tau,1-\gamma)\ \tilde{\varphi}_{\tau_b,01}(\tau,1-\gamma)\dots\tilde{\varphi}_{\tau_b,0(M-1)}\right.\\ &(\tau,1-\gamma)|\tilde{\varphi}_{\tau_b,10}(\tau,1-\gamma)\ \tilde{\varphi}_{\tau_b,11}(\tau,1-\gamma)\dots\\ &\tilde{\varphi}_{\tau_b,1(M-1)}(\tau,1-\gamma)|\dots|\tilde{\varphi}_{\tau_b,(N-1)0}(\tau,1-\gamma)\\ &\left.\tilde{\varphi}_{\tau_b,(N-1)1}(\tau,1-\gamma)\dots\tilde{\varphi}_{\tau_b,(N-1)(M-1)}(\tau,1-\gamma)\right]^T.\end{aligned}\tag{72}$$

Meanwhile, for the functions given in (64) and (65), the following approximations can be considered:

$$f(\zeta)\simeq\sum_{i=0}^{\widehat{N}}f_i\psi_{\zeta_b,i}(\zeta)\triangleq\mathbf{F}^T\Psi_{\zeta_b,\widehat{N}}(\zeta)=\Psi_{\zeta_b,\widehat{N}}(\zeta)^T\mathbf{F},\tag{73}$$

and

$$\begin{aligned}g(\tau)&\simeq\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}g_{nm}\varphi_{\tau_b,nm}(\tau)\triangleq\mathbf{G}^T\Phi_{\tau_b,NM}(\tau),\\ h(\tau)&\simeq\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}g_{nm}\varphi_{\tau_b,nm}(\tau)\triangleq\mathbf{H}^T\Phi_{\tau_b,NM}(\tau).\end{aligned}\tag{74}$$

By considering (64), (65), (66), (70), (73) and (74), we derive

$$\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\left(\underbrace{\mathbf{V}\Phi_{\tau_b,NM}(0)-\mathbf{F}}_{\Lambda_1}\right)\simeq 0,\tag{75}$$

and

$$\begin{aligned}&\left(\underbrace{\left(\Psi_{\zeta_b,\widehat{N}}(0)\right)^T\mathbf{V}-\varrho_0\left(\Psi_{\zeta_b,\widehat{N}}(0)\right)^T\left(\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}-\bar{\mathbf{K}}_0^T\mathbf{V}-\mathbf{G}^T}_{\Lambda_2}\right)\Phi_{\tau_b,NM}(\tau)\simeq 0,\\ &\left(\underbrace{\left(\Psi_{\zeta_b,\widehat{N}}(\zeta_b)\right)^T\mathbf{V}+\varrho_1\left(\Psi_{\zeta_b,\widehat{N}}(\zeta_b)\right)^T\left(\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}-\bar{\mathbf{K}}_1^T\mathbf{V}-\mathbf{H}^T}_{\Lambda_3}\right)\Phi_{\tau_b,NM}(\tau)\simeq 0,\end{aligned}\tag{76}$$

where

$$\bar{\mathbf{K}}_0=\left[\bar{k}_{00}\ \bar{k}_{01}\ \dots\ \bar{k}_{0\widehat{N}}\right]^T,\quad\bar{\mathbf{K}}_1=\left[\bar{k}_{10}\ \bar{k}_{11}\ \dots\ \bar{k}_{1\widehat{N}}\right]^T,$$

and

$$\bar{k}_{0i}=\int_0^{\zeta_b}K_0(\zeta)\psi_{\zeta_b,i}(\zeta)d\zeta,\quad\bar{k}_{1i}=\int_0^{\zeta_b}K_1(\zeta)\psi_{\zeta_b,i}(\zeta)d\zeta,\quad i=0,1,\dots,\widehat{N}.$$

Substituting (66), (69), (70) and (71) into (63) yields

$$\begin{aligned}\tilde{v}(\zeta,\tau,\beta)\ &+\kappa_1\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\left(\mathbf{D}^{(1)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}\Phi_{\tau_b,NM}(\tau)-\kappa_2\left\{\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\left(\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}\tilde{\Phi}_{\tau_b,NM}(\tau,\gamma)\right.\\ &\left.+\frac{\tau^{\gamma-1}}{\Gamma(\gamma)}\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\left(\mathbf{D}^{(2)}_{\zeta_b,\widehat{N}}\right)^T\mathbf{V}\Phi_{\tau_b,NM}(0)\right\}-w\left(\zeta,\tau,\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\mathbf{V}\Phi_{\tau_b,NM}(\tau)\right)\\ &\triangleq\mathbf{R}(\zeta,\tau,\beta,\gamma)\simeq 0.\end{aligned}\tag{77}$$

From ()()()(75)–(77), we extract the below system:

$$\begin{cases}\mathbf{R}\left(\frac{(2i-1)\zeta_b}{2\widehat{N}},\frac{(2j-1)\tau_b}{2NM},\beta,\gamma\right)=0, & 2\leqslant i\leqslant\widehat{N},\ 2\leqslant j\leqslant NM,\\ [\Lambda_1]_i=0, & 2\leqslant i\leqslant\widehat{N},\\ [\Lambda_2]_j=0,[\Lambda_3]_j=0, & 1\leqslant j\leqslant NM.\end{cases}\tag{78}$$

Eventually, we achieve a solution for the problem by solving system (78) for specific values of β and γ, and determining $\mathbf{V}$ and substitute it into (66).

The proposed method for the two-dimensional problem

Here, we propose a hybrid method based on the orthonormal two variables VL polynomials and orthonormal piecewise VL functions to solve the following two-dimensional piecewise fractional Galilei invariant advection–diffusion equation:

$$\begin{aligned}{}^{P}_{0}D^{\rho(\alpha),\beta}_{\tau;\tau_1}v(\zeta,\xi,\tau)+\bar{\kappa}_1 v_\zeta(\zeta,\xi,\tau)+\bar{\kappa}_2 v_\xi(\zeta,\xi,\tau)\\ ={}^{RL}_{0}D^{1-\gamma}_{\tau}\left(\bar{\kappa}_3 v_{\zeta\zeta}(\zeta,\xi,\tau)+\bar{\kappa}_4 v_{\xi\xi}(\zeta,\xi,\tau)\right)+\bar{w}(\zeta,\xi,\tau,v(\zeta,\xi,\tau)),\end{aligned}\tag{79}$$

with $(\zeta,\xi,\tau)\in[0,\zeta_b]\times[0,\xi_b]\times[0,\tau_b]$, under the initial condition

$$v(\zeta,\xi,0)=\bar{f}(\zeta,\xi),\tag{80}$$

and non-local boundary conditions

$$\begin{aligned}v(0,\xi,\tau)-\bar{\varrho}_0 v_\zeta(0,\xi,\tau)&=\int_0^{\zeta_b}\overline{K}_0(\zeta,\xi)v(\zeta,\xi,\tau)d\zeta+\bar{g}_0(\xi,\tau),\\ v(\zeta_b,\xi,\tau)+\bar{\varrho}_1 v_\zeta(\zeta_b,\xi,\tau)&=\int_0^{\zeta_b}\overline{K}_1(\zeta,\xi)v(\zeta,\xi,\tau)d\zeta+\bar{g}_1(\xi,\tau),\\ v(\zeta,0,\tau)-\bar{\varrho}_2 v_\xi(\zeta,0,\tau)&=\int_0^{\xi_b}\overline{K}_2(\zeta,\xi)v(\zeta,\xi,\tau)d\xi+\bar{g}_2(\zeta,\tau),\\ v(\zeta,\xi_b,\tau)+\bar{\varrho}_3 v_\xi(\zeta,\xi_b,\tau)&=\int_0^{\xi_b}\overline{K}_3(\zeta,\xi)v(\zeta,\xi,\tau)d\xi+\bar{g}_3(\zeta,\tau),\end{aligned}\tag{81}$$

where $\bar{\kappa}_l>0$ for $l=1,2,3,4,0<\gamma<1,\zeta_b,\xi_b,\tau_b>0,0<\tau_1<\tau,\bar{\varrho}_l>0$ for $l=0,1,2,3$ and $0<\beta\leqslant 1$ are real numbers, $\bar{w},\bar{f},\overline{K}_l$ and $\bar{g}_l$ for $l=0,1,2,3$ are continuous functions in their domains.

To solve the above problem, we assume

$$v(\zeta,\xi,\tau)\simeq\sum_{l=0}^{(\widehat{N}+1)(\widehat{M}+1)-1}\sum_{j=1}^{NM}\bar{v}_{lj}\tilde{\psi}_{\zeta_b\xi_b,l}(\zeta,\xi)\hat{\varphi}_{\tau_b,j}(\tau)\triangleq\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau),\tag{82}$$

where $\bar{\mathbf{V}}=[v_{lj}]$ is an $(\widehat{N}+1)(\widehat{M}+1)\times NM$ undetermined matrix. From (12), (37), (46) and (82), as well as employing the Gauss–Legendre integration method, we get

$${}^{P}_{0}D^{\rho(\alpha),\beta}_{\tau;\tau_1}\ v(\zeta,\xi,\tau)\triangleq\hat{v}(\zeta,\xi,\tau,\beta)\simeq\begin{cases}\frac{1}{2}\sum_{l=0}^{(\widehat{N}+1)(\widehat{M}+1)-1}\sum_{j=1}^{NM}\bar{v}_{lj}\tilde{\psi}_{\zeta_b\xi_b,l}(\zeta,\xi)\sum_{r=1}^{\bar{p}}\hat{w}_r\rho\left(\frac{1}{2}(\hat{\tau}_r+1)\right)\tilde{\bar{\varphi}}_{\tau_b,j}\left(\tau,\frac{1}{2}(\hat{\tau}_r+1)\right), & 0<\tau<\tau_1,\\ \sum_{l=0}^{(\widehat{N}+1)(\widehat{M}+1)-1}\sum_{j=1}^{NM}\bar{v}_{lj}\tilde{\psi}_{\zeta_b\xi_b,l}(\zeta,\xi)\bar{\bar{\varphi}}_{\tau_b,j}(\tau,\beta), & \tau_1\leqslant\tau<\tau_b.\end{cases}\tag{83}$$

Based on Theorem 9 and relation (82), we obtain

$$\begin{aligned}v_\zeta(\zeta,\xi,\tau)&\simeq\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{P}^{(1)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau),\\ v_{\zeta\zeta}(\zeta,\xi,\tau)&\simeq\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{P}^{(2)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)\\ v_\xi(\zeta,\xi,\tau)&\simeq\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{Q}^{(1)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau),\\ v_{\xi\xi}(\zeta,\xi,\tau)&\simeq\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{Q}^{(2)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau).\end{aligned}\tag{84}$$

Property 1, together with Theorem 6 and relation (84), yield

$$\begin{aligned}{}^{RL}_{0}D^{1-\gamma}_{\tau}(v_{\zeta\zeta}(\zeta,\xi,\tau))\simeq&\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{P}^{(2)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\tilde{\Phi}_{\tau_b,NM}(\tau,\gamma)\\ &+\tfrac{\tau^{\gamma-1}}{\Gamma(\gamma)}\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{P}^{(2)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(0)\triangleq u_1(\zeta,\xi,\tau,\gamma),\end{aligned}\tag{85}$$

and

$$\begin{aligned}{}^{RL}_{0}D^{1-\gamma}_{\tau}(v_{\xi\xi}(\zeta,\xi,\tau))\simeq&\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{Q}^{(2)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\tilde{\Phi}_{\tau_b,NM}(\tau,\gamma)\\ &+\tfrac{\tau^{\gamma-1}}{\Gamma(\gamma)}\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{Q}^{(2)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(0)\triangleq u_2(\zeta,\xi,\tau,\gamma),\end{aligned}\tag{86}$$

where $\tilde{\Phi}_{\tau_b,NM}(\tau,\gamma)$ has already been defined in (72). From (80) and (82), we get

$$\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(0)-\bar{f}(\zeta,\xi)\triangleq\Pi_0(\zeta,\xi)\simeq 0.\tag{87}$$

Also, from (81), (82) and (84), we obtain

$$\begin{aligned}&\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(0,\xi)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)-\bar{\varrho}_0\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(0,\xi)\right)^T\left(\mathbf{P}^{(1)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)\\ &\quad-\left(\Phi^{(0)}_{\xi_b,\widehat{N}\widehat{M}}(\xi)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)-\bar{g}_0(\xi,\tau)\triangleq\Pi_1(\xi,\tau)=0,\\ &\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta_b,\xi)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)+\bar{\varrho}_1\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta_b,\xi)\right)^T\left(\mathbf{P}^{(1)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)\\ &\quad-\left(\Phi^{(1)}_{\xi_b,\widehat{N}\widehat{M}}(\xi)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)-\bar{g}_1(\xi,\tau)\triangleq\Pi_2(\xi,\tau),\\ &\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,0)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)-\bar{\varrho}_2\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,0)\right)^T\left(\mathbf{Q}^{(1)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)\\ &\quad-\left(\Phi^{(2)}_{\zeta_b,\widehat{N}\widehat{M}}(\zeta)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)-\bar{g}_2(\zeta,\tau)\triangleq\Pi_3(\zeta,\tau),\\ &\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi_b)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)+\bar{\varrho}_3\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi_b)\right)^T\left(\mathbf{Q}^{(1)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)\\ &\quad-\left(\Phi^{(3)}_{\zeta_b,\widehat{N}\widehat{M}}(\zeta)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)-\bar{g}_3(\zeta,\tau)\triangleq\Pi_4(\zeta,\tau),\end{aligned}\tag{88}$$

where

$$\begin{aligned}&\Phi^{(0)}_{\xi_b,\widehat{N}\widehat{M}}(\xi)=\int_0^{\zeta_b}\overline{K}_0(\zeta,\xi)\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)d\zeta,\quad \Phi^{(1)}_{\xi_b,\widehat{N}\widehat{M}}(\xi)=\int_0^{\zeta_b}\overline{K}_1(\zeta,\xi)\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)d\zeta\\ &\Phi^{(3)}_{\zeta_b,\widehat{N}\widehat{M}}(\zeta)=\int_0^{\xi_b}\overline{K}_2(\zeta,\xi)\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)d\xi,\quad \Phi^{(4)}_{\zeta_b,\widehat{N}\widehat{M}}(\zeta)=\int_0^{\xi_b}\overline{K}_3(\zeta,\xi)\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)d\xi.\end{aligned}$$

By substituting Eqs. (82)–(86) into (79), we get

$$\begin{aligned}&\hat{v}(\zeta,\xi,\tau,\beta)+\bar{\kappa}_1\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{P}^{(1)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)+\bar{\kappa}_2\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\left(\mathbf{Q}^{(1)}_{\widehat{N}\widehat{M}}\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)\\ &-\bar{\kappa}_3 u_1(\zeta,\xi,\tau,\gamma)-\bar{\kappa}_4 u_2(\zeta,\xi,\tau,\gamma)-\bar{w}\left(\zeta,\xi,\tau,\left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T\bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau)\right)\\ &\triangleq\bar{\mathbf{R}}(\zeta,\xi,\tau,\beta,\gamma)\simeq 0.\end{aligned}\tag{89}$$

From (87)–(89), we extract the following system:

$$\begin{cases}\bar{\mathbf{R}}(\zeta_i,\xi_j,\tau_l,\beta,\gamma)=0, & i=2,3,\ldots,\widehat{N},j=2,3,\ldots,\widehat{M},l=2,3,\ldots,NM,\\ \Pi_0(\zeta_i,\xi_j)=0, & i=1,2,\ldots,\widehat{N}+1,j=1,2,\ldots,\widehat{M}+1,\\ \Pi_r(\xi_j,\tau_l)=0, & r=1,2,\ j=1,2,\ldots,\widehat{M}+1,l=2,3,\ldots,NM,\\ \Pi_r(\zeta_i,\tau_l)=0, & r=3,4,\ i=2,3,\ldots,\widehat{N},l=2,3,\ldots,NM,\end{cases}\tag{90}$$

where

$$\zeta_i=\frac{(2i-1)\zeta_b}{2(\widehat{N}+1)},\quad \xi_j=\frac{(2j-1)\xi_b}{2(\widehat{M}+1)},\quad \tau_l=\frac{(2l-1)\tau_b}{2NM}.$$

Finally, we obtain a solution for the problem by solving system (90) for specific values of β and γ, and determining $\bar{\mathbf{V}}$ and substitute it into (82).

Numerical examples

In this section, we examine the accuracy of the proposed methods by solving several examples. The following formulae are used to compute the accuracy of the obtained results:

One-dimensional problem:

The maximum absolute error is computed as

$$e_\infty=\max_{(\zeta,\tau)\in[0,\zeta_b]\times[0,\tau_b]}\left|v(\zeta,\tau)-\left(\Psi_{\zeta_b,\widehat{N}}(\zeta)\right)^T\mathbf{V}\Phi_{\tau_b,NM}(\tau)\right|,$$

where v is the exact solution. Also, the convergence order (CO) of the method is calculated as

$$CO = \left| \frac{\ln\left(\frac{e_\infty(\bar{m}_2)}{e_\infty(\bar{m}_1)}\right)}{\ln\left(\frac{\bar{m}_2}{\bar{m}_1}\right)} \right|,$$

where $\bar{m}_1$ and $\bar{m}_2$are the number of basis functions used in the first and second implementations, respectively.

Two-dimensional problem:

The maximum absolute error at the terminal time is computed as

$$e_\infty = \max_{(\zeta,\xi)\in[0,\zeta_b]\times[0,\xi_b]} \left| v(\zeta,\xi,\tau_b) - \left(\Psi_{\zeta_b\xi_b,\widehat{N}\widehat{M}}(\zeta,\xi)\right)^T \bar{\mathbf{V}}\Phi_{\tau_b,NM}(\tau_b) \right|,$$

where v is the exact solution. Also, the CO of the method is computed similar to the one-dimensional problem.

Note that Maple 18 (with 25 digits) on a X64-based PC with Intel (R) Core (TM) i7-7500U CPU @ 2.90 GHz and 32.0 GB of RAM is applied for all simulations. Meanwhile, the series in the Mittag–Leffler functions are truncated after 35th term. Moreover, for numerical integration, we put $\bar{p} = 15$.

Example 1. Consider the problem

$$\begin{aligned} &{}^{P}_{0}D^{\rho(\alpha),\beta}_{\tau;\frac{2}{3}} v(\zeta,\tau) + v_\zeta(\zeta,\tau) + \sin(\zeta)e^{-v(\zeta,\tau)} \\ &\quad = {}^{RL}_{0}D^{\frac{1}{2}}_{\tau}(v_{\zeta\zeta}(\zeta,\tau)) + w(\zeta,\tau), (\zeta,\tau) \in [0,2]\times[0,2], \end{aligned}$$

where $\rho(\alpha) = \Gamma(4-\alpha)$ and

$$w(\zeta,\tau) = \tau^3\cos(\zeta) + \frac{16\tau^{\frac{5}{2}}}{15\sqrt{\pi}}\sin(\zeta) + \sin(\zeta)e^{-\tau^3\sin(\zeta)} + \sin(\zeta)\begin{cases} \frac{6\tau^2(\tau-1)}{\ln(\tau)}, & 0<\tau<\frac{2}{3}, \\ \begin{cases} \frac{6AB(\beta)}{1-\beta}\tau^3\mathbf{E}_{\beta,4}\left(\frac{-\beta\tau^\beta}{1-\beta}\right), & 0<\beta<1, \\ 3\tau^2, & \beta=1, \end{cases} & \frac{2}{3}\leqslant\tau\leqslant 2. \end{cases}$$

The below conditions are imposed for this problem

$$v(\zeta,0) = 0,$$

and

$$\begin{aligned} &v(0,\tau) - \pi v_\zeta(0,\tau) = \int_0^2 \sin(\zeta)v(\zeta,\tau)d\zeta + \tau^3\left(\tfrac{1}{4}(\sin(4)-4)-\pi\right), \\ &v(2,\tau) + v_\zeta(2,\tau) = \int_0^2 \cos(\zeta)v(\zeta,\tau)d\zeta + \tau^3\left(\sin(2)+\cos(2)+\tfrac{1}{2}(\cos^2(2)-1)\right). \end{aligned}$$

The function $v(\zeta,\tau) = \tau^3\sin(\zeta)$ is the problem exact solution. The results acquired of the established approach are shown in Table 1 for two values of β. These results reflect the high accuracy of the method presented in the previous section. This table also confirms that the results have a high degree of convergence. The columns regarding the CPU time (seconds) confirm the low computational works of the expressed method. Fig. 1 illustrates the behavior of the obtained results for the case $\beta = 0.4$.

Example 2. Consider the problem

$$\begin{aligned} &{}^{P}_{0}D^{\rho(\alpha),\beta}_{\tau;\frac{1}{2}} v(\zeta,\tau) + 2v_\zeta(\zeta,\tau) + v(\zeta,\tau)(v(\zeta,\tau)-2) \\ &\quad = 2\,{}^{RL}_{0}D^{\frac{2}{3}}_{\tau}(v_{\zeta\zeta}(\zeta,\tau)) + w(\zeta,\tau), (\zeta,\tau)\in[0,3]\times[0,1], \end{aligned}$$

where $\rho(\alpha) = \Gamma(5-\alpha)$ and

$$w(\zeta,\tau) = \tau^4e^{-\zeta}(\tau^4e^{-\zeta}-4) - \frac{243\sqrt{3}}{35\pi}\Gamma\left(\frac{2}{3}\right)\tau^{\frac{10}{3}}e^{-\zeta} + e^{-\zeta}\begin{cases} \frac{24\tau^3(\tau-1)}{\ln(\tau)}, & 0<\tau<\frac{1}{2}, \\ \begin{cases} \frac{24AB(\beta)}{1-\beta}\tau^4\mathbf{E}_{\beta,5}\left(\frac{-\beta\tau^\beta}{1-\beta}\right), & 0<\beta<1, \\ 4\tau^3, & \beta=1, \end{cases} & \frac{1}{2}\leqslant\tau\leqslant 1. \end{cases}$$

The below conditions are imposed for this problem

$$v(\zeta,0) = 0,$$

and

Table 1
The results obtained with two values of β and some choices of $\widehat{N}$ where $(N=3, M=4)$ in Example 1.

$\widehat{N}$	N	M	$\beta=0.4$ e_∞	CO	CPU time	$\beta=0.8$ e_∞	CO	CPU time
6	3	4	1.2672×10^{-02}	–	17.00	8.5641×10^{-03}	–	20.40
8			1.4996×10^{-04}	15.4225	27.31	1.0125×10^{-04}	15.4258	30.84
10			1.0445×10^{-06}	22.2584	43.18	7.1087×10^{-07}	22.2227	63.07
12			4.8273×10^{-09}	29.4918	68.21	3.3095×10^{-09}	29.4517	71.26

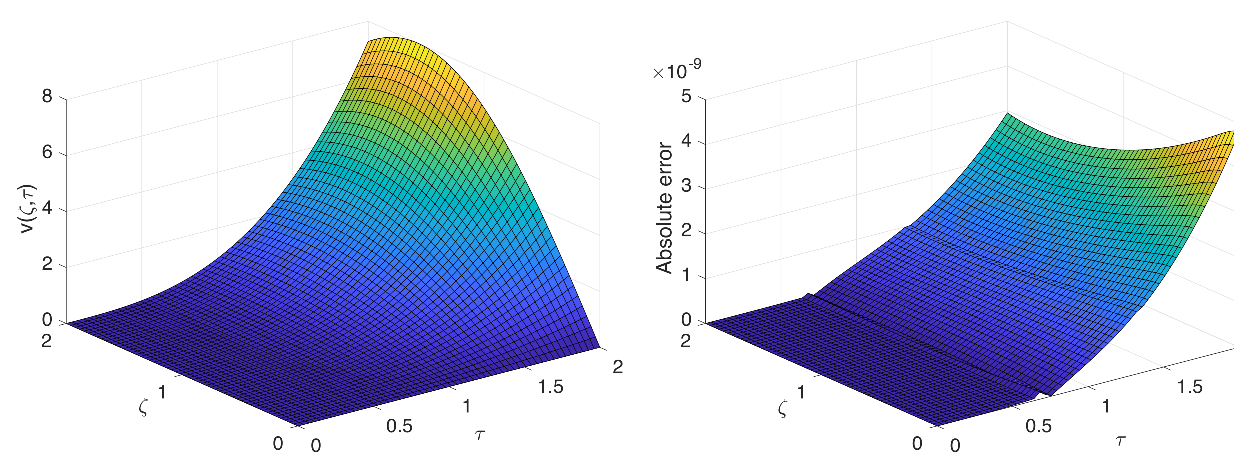

Fig. 1. Approximate solution (left) and associated absolute error function (right) with $\beta = 0.4$ where $(\widehat{N}=12, N=3, M=4)$ in Exa.mple 1.

$$v(0,\tau) - v_\zeta(0,\tau) = \int_0^3 e^{\zeta} v(\zeta,\tau)d\zeta - \tau^4,$$
$$v(3,\tau) + 2v_\zeta(3,\tau) = \int_0^3 e^{-\zeta} v(\zeta,\tau)d\zeta - \tfrac{1}{2}\tau^4\left(1+2e^{-3}-e^{-6}\right).$$

The problem exact solution is $v(\zeta,\tau)=\tau^4 e^{-\zeta}$. Table 2 is used to confirm the validity of the results obtained by the presented approach. For the case $\beta=0.2$ the extracted results are shown in Fig. 2.

EXAMPLE 3. Consider the problem

$$\begin{aligned}&{}_0^P D^{\rho(\alpha),\beta}_{\tau;\frac{3}{2}} v(\zeta,\tau) + 3v_\zeta(\zeta,\tau) + 2v(\zeta,\tau)\\ &= {}_0^{RL}D^{\frac{1}{4}}_\tau(v_{\zeta\zeta}(\zeta,\tau)) + w(\zeta,\tau), \quad (\zeta,\tau)\in[0,1]\times[0,2],\end{aligned}$$

where $\rho(\alpha)=\Gamma(3-\alpha)$ and

$$w(\zeta,\tau) = (g_1(\tau)+g_2(\tau))\cos(\zeta) + (2\cos(\zeta)-3\sin(\zeta))\begin{cases}\tau^2, & 0<\tau<\frac{3}{2},\\ \tau^3, & \frac{3}{2}\leqslant\tau\leqslant 2,\end{cases}$$

in which

$$g_1(\tau) = \begin{cases}\frac{2\tau(\tau-1)}{\ln(\tau)}, & 0<\tau<\frac{3}{2},\\ \begin{cases} g_3(\tau), & 0<\beta<1,\\ 3\tau^2, & \beta=1,\end{cases} & \frac{3}{2}\leqslant\tau\leqslant 2,\end{cases}$$

with

$$\begin{aligned}g_3(\tau) = &\tfrac{2AB(\beta)}{1-\beta}\Big\{\tfrac{3}{2}\left(\tfrac{3}{2}-\tau\right)\mathbf{E}_{\beta,2}\left(-\tfrac{\beta}{1-\beta}\left(\tau-\tfrac{3}{2}\right)^\beta\right) - \left(\tau-\tfrac{3}{2}\right)^2\mathbf{E}_{\beta,3}\left(-\tfrac{\beta}{1-\beta}\left(\tau-\tfrac{3}{2}\right)^\beta\right)\\ &+\tau^2\mathbf{E}_{\beta,3}\left(-\tfrac{\beta}{1-\beta}\tau^\beta\right)\Big\} + \tfrac{AB(\beta)}{1-\beta}\Big\{\tfrac{27}{4}\left(\tau-\tfrac{3}{2}\right)\mathbf{E}_{\beta,2}\left(-\tfrac{\beta}{1-\beta}\left(\tau-\tfrac{3}{2}\right)^\beta\right)\\ &+9\left(\tau-\tfrac{3}{2}\right)^2\mathbf{E}_{\beta,3}\left(-\tfrac{\beta}{1-\beta}\left(\tau-\tfrac{3}{2}\right)^\beta\right) + 6\left(\tau-\tfrac{3}{2}\right)^3\mathbf{E}_{\beta,4}\left(-\tfrac{\beta}{1-\beta}\left(\tau-\tfrac{3}{2}\right)^\beta\right)\Big\},\end{aligned}$$

and

$$g_2(\tau) = \frac{1}{\Gamma\left(\frac{3}{4}\right)}\begin{cases}\frac{32}{21}\tau^{\frac{7}{4}}, & 0<\tau<\frac{3}{2},\\ 5\left(\tau-\frac{3}{2}\right)^{\frac{3}{4}} + \frac{16}{3}\left(\tau-\frac{3}{2}\right)^{\frac{7}{4}} + \frac{32}{21}\tau^{\frac{7}{4}} + \frac{128}{77}\left(\tau-\frac{3}{2}\right)^{\frac{11}{4}}, & \frac{3}{2}\leqslant\tau\leqslant 2.\end{cases}$$

The following conditions are used for this problem:

$$v(\zeta,0)=0,$$

and

$$\begin{aligned}&v(0,\tau) - 2v_\zeta(0,\tau)\\ &= \int_0^1 \zeta\sin(\zeta)v(\zeta,\tau)d\zeta\\ &+\frac{1}{4}\left(3+2\cos^2(1)-\frac{1}{2}\sin(2)\right)\begin{cases}\tau^2, & 0\leqslant\tau<\frac{3}{2},\\ \tau^3, & \frac{3}{2}\leqslant\tau\leqslant 2,\end{cases}\\ &v(1,\tau)+3v_\zeta(1,\tau)\\ &= \int_0^1 \cos(\zeta)v(\zeta,\tau)d\zeta\\ &+\left(\cos(1)-3\sin(1)-\frac{1}{2}\left(\frac{1}{2}\sin(2)+1\right)\right)\begin{cases}\tau^2, & 0\leqslant\tau<\frac{3}{2},\\ \tau^3, & \frac{3}{2}\leqslant\tau\leqslant 2.\end{cases}\end{aligned}$$

The problem exact solution is

$$v(\zeta,\tau) = \cos(\zeta)\begin{cases}\tau^2, & 0\leqslant\tau<\frac{3}{2},\\ \tau^3, & \frac{3}{2}\leqslant\tau\leqslant 2.\end{cases}$$

We have reported the results achieved of applying the presented technique for two values of β in Table 3. For the case $\beta=0.6$, the extracted results are shown in Fig. 3. Taken together, these results confirm the high capability of the proposed method.

EXAMPLE 4. Consider the problem

$$\begin{aligned}&{}_0^P D^{\rho(\alpha),\beta}_{\tau;\frac{1}{2}} v(\zeta,\xi,\tau) + 2v_\zeta(\zeta,\xi,\tau) + 2v_\xi(\zeta,\xi,\tau)\\ &= {}_0^{RL}D^{\frac{3}{4}}_\tau(v_{\zeta\zeta}(\zeta,\xi,\tau) + v_{\xi\xi}(\zeta,\xi,\tau)) + v(\zeta,\xi,\tau) + \bar{w}(\zeta,\xi,\tau),\end{aligned}$$

with $(\zeta,\xi,\tau)\in[0,1]\times[0,1]\times[0,1]$, where $\rho(\alpha)=\Gamma(3-\alpha)$ and

Table 2
The results obtained with two values of β and some choices of $\widehat{N}$ where $(N=2, M=5)$ in Example 2.

$\widehat{N}$	N	M	$\beta=0.2$ e_∞	CO	CPU time	$\beta=0.6$ e_∞	CO	CPU time
5	2	5	2.1066×10^{-02}	–	09.40	2.0956×10^{-02}	–	11.78
7			3.6329×10^{-04}	12.0670	14.46	3.6511×10^{-04}	12.0365	16.98
9			5.6495×10^{-06}	16.5674	21.31	5.7010×10^{-06}	16.5512	24.04
11			6.1773×10^{-08}	22.5038	35.03	6.2487×10^{-08}	22.4917	38.54

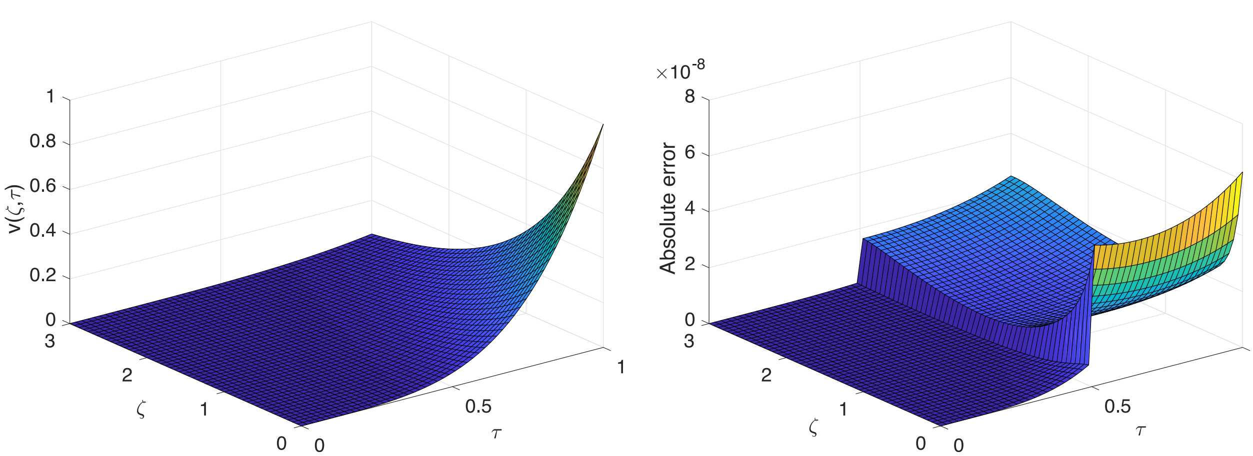

Fig. 2. Approximate solution (left) and associated absolute error function (right) with $\beta=0.2$ where $(\widehat{N}=11, N=2, M=5)$ in Exa.mple 2.

Table 3
The results obtained with two values of β and some choices of $\widehat{N}$ where $(N = M = 4)$ in Example 3.

$\widehat{N}$	N	M	$\beta = 0.6$ e_∞	CO	CPU time	$\beta = 0.8$ e_∞	CO	CPU time
5	4	4	1.7158×10^{-03}	–	17.85	1.5813×10^{-03}	–	20.62
7			7.1428×10^{-06}	16.2911	21.56	6.6171×10^{-06}	16.2757	27.17
9			1.6229×10^{-08}	24.2208	25.84	1.5081×10^{-08}	24.2086	28.62
11			2.3193×10^{-11}	32.6440	33.10	2.1586×10^{-11}	32.6362	36.45

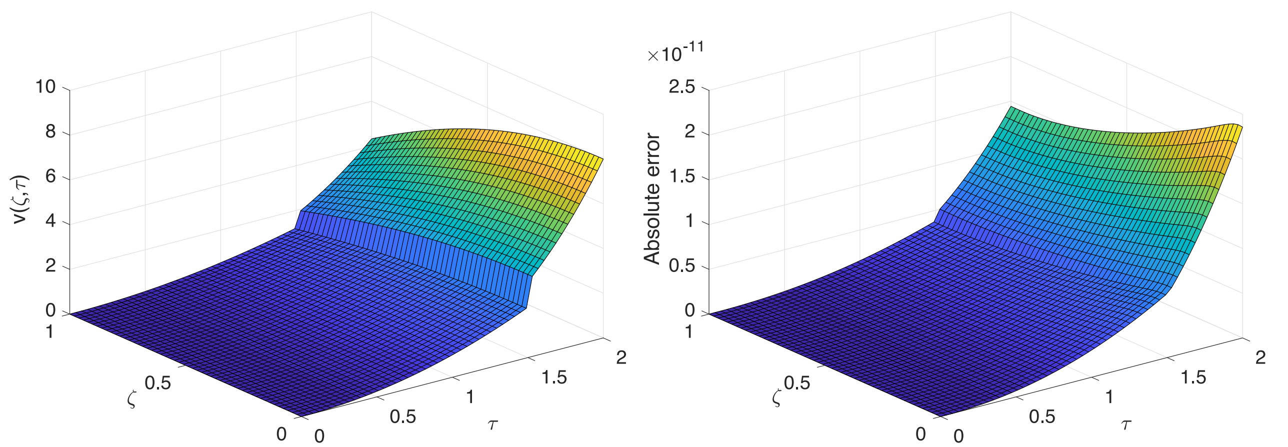

Fig. 3. Approximate solution (left) and associated absolute error function (right) with $\beta = 0.6$ where $(\widehat{N} = 11, N = M = 4)$ in Exa.mple 3.

$$\bar{w}(\zeta,\xi,\tau) = 2\tau^2\cos(\zeta)\sin(\xi) + 2\tau^2\sin(\zeta)\cos(\xi) + \frac{32\sqrt{2}\Gamma(\frac{3}{4})\tau^{\frac{5}{4}}\sin(\zeta)\sin(\xi)}{5\pi} - \tau^2\sin(\zeta)\sin(\xi) + \sin(\zeta)\sin(\xi)\begin{cases} \frac{2\tau(\tau-1)}{\ln(\tau)}, & 0<\tau<\frac{1}{2}, \\ \begin{cases} \frac{2AB(\beta)}{1-\beta}\tau^2\mathbf{E}_{\beta,3}\left(\frac{-\beta\tau^\beta}{1-\beta}\right), & 0<\beta<1, \\ 2\tau, & \beta=1, \end{cases} & \frac{1}{2}\leqslant\tau\leqslant1. \end{cases}$$

The following conditions are used for this problem:

$v(\zeta,\xi,0) = 0,$

and

$$v(0,\xi,\tau) - \frac{1}{10}v_\zeta(0,\xi,\tau) = \int_0^1 \zeta v(\zeta,\xi,\tau)d\zeta - \frac{1}{10}\tau^2\sin(\xi) - \tau^2\sin(\xi)(\sin(1)-\cos(1)),$$
$$v(1,\xi,\tau) + \frac{1}{10}v_\zeta(1,\xi,\tau) = \frac{1}{2}\int_0^1 \zeta^2 v(\zeta,\xi,\tau)d\zeta - \frac{1}{5}\tau^2\sin(\xi)(2\cos(1)-5),$$
$$v(\zeta,0,\tau) + \frac{1}{100}v_\xi(\zeta,0,\tau) = \int_0^1 \xi v(\zeta,\xi,\tau)d\xi - \frac{1}{100}\tau^2\sin(\xi)(1+100\sin(1)-100\cos(1)),$$
$$v(\zeta,1,\tau) + \frac{1}{100}v_\xi(\zeta,1,\tau) = \int_0^1 \zeta\xi^3 v(\zeta,\xi,\tau)d\xi + \frac{1}{100}\tau^2\sin(\xi)(300\xi\sin(1) - 500\xi\cos(1) + 100\sin(1) + \cos(1)).$$

The exact solution of the problem is $v(\zeta,\xi,\tau) = \tau^2\sin(\zeta)\sin(\xi)$. The results obtained by the proposed technique for two values of β at $\tau = 1$ are shown in Table 4. These results confirm the high accuracy of the proposed method in solving this example. In the case of $\beta = 0.3$, the obtained results are shown in Fig. 4.

EXAMPLE 5. Consider the problem

$$ {}^{P}_{0}D^{\rho(\alpha),\beta}_{\tau;\frac{3}{4}} v(\zeta,\xi,\tau) + v_\zeta(\zeta,\xi,\tau) + v_\xi(\zeta,\xi,\tau) = {}^{RL}_{0}D^{\frac{1}{2}}_{\tau}\left(\frac{1}{2}v_{\zeta\zeta}(\zeta,\xi,\tau) + \frac{1}{2}v_{\xi\xi}(\zeta,\xi,\tau)\right) + \sin(\tau)v^2(\zeta,\xi,\tau) + \bar{w}(\zeta,\xi,\tau),$$

with $(\zeta,\xi,\tau) \in [0,1]\times[0,1]\times\left[0,\frac{3}{2}\right]$, where $\rho(\alpha) = \Gamma(4-\alpha)$ and

$$\bar{w}(\zeta,\xi,\tau) = -\tau^3(\sin(\zeta)\sin(\xi) - \cos(\zeta)\cos(\xi)) + \frac{16}{5\sqrt{\pi}}\tau^{\frac{5}{2}}\cos(\zeta)\sin(\xi) - \tau^6\sin(\tau)\cos^2(\zeta)\sin^2(\xi) + \cos(\zeta)\sin(\xi)\begin{cases} \frac{6\tau^2(\tau-1)}{\ln(\tau)}, & 0<\tau<\frac{3}{4}, \\ \begin{cases} \frac{6AB(\beta)}{1-\beta}\tau^3\mathbf{E}_{\beta,4}\left(\frac{-\beta\tau^\beta}{1-\beta}\right), & 0<\beta<1, \\ 3\tau^2, & \beta=1, \end{cases} & \frac{3}{4}\leqslant\tau\leqslant\frac{3}{2}. \end{cases}$$

The initial condition for this problem is

$v(\zeta,\xi,0) = 0,$

and the boundary conditions are

$$v(0,\xi,\tau) - \frac{1}{2}v_\zeta(0,\xi,\tau) = \int_0^1 \sin(\zeta)\cos(\xi)v(\zeta,\xi,\tau)d\zeta + \frac{1}{2}\tau^3\left((\cos^2(1)-1)\cos(\xi)+2\right),$$
$$v(1,\xi,\tau) + v_\zeta(1,\xi,\tau) = \int_0^1 \sin(\zeta)\cos(\xi)v(\zeta,\xi,\tau)d\zeta + \frac{1}{2}\tau^3\sin(\xi)\left((\cos^2(1)-1)\cos(\xi) + 2(\cos(1)-\sin(1))\right),$$
$$v(\zeta,0,\tau) + v_\xi(\zeta,0,\tau) = \int_0^1 \cos(\zeta)\sin(\xi)v(\zeta,\xi,\tau)d\xi + \frac{1}{2}\tau^3\cos(\zeta)\left(\frac{1}{2}\sin(2)\cos(\zeta) - \cos(\zeta) - 2\right),$$
$$v(\zeta,1,\tau) + \frac{1}{2}v_\xi(\zeta,1,\tau) = \int_0^1 \cos(\zeta)\sin(\xi)v(\zeta,\xi,\tau)d\xi + \frac{1}{2}\tau^3\cos(\zeta)\left(\left(\frac{1}{2}\sin(2)-1\right)\cos(\zeta) + \cos(1) + 2\sin(1)\right).$$

Table 4
The results obtained with two values of β and some choices of $(\widehat{N}, \widehat{M}, N, M)$ in Example 4.

$\widehat{N}$	$\widehat{M}$	N	M	$\beta = 0.3$ e_∞	CO	CPU time	$\beta = 0.9$ e_∞	CO	CPU time
4	4	1	3	9.1325×10^{-03}	–	003.18	7.2820×10^{-03}	–	003.23
5	5	1	4	6.4626×10^{-06}	09.8826	009.98	6.4432×10^{-06}	09.5782	009.90
6	6	2	5	4.9878×10^{-07}	01.9998	191.60	4.7105×10^{-07}	02.0421	192.03
7	7	2	6	2.6051×10^{-08}	06.0170	442.68	2.5094×10^{-08}	05.9767	453.45

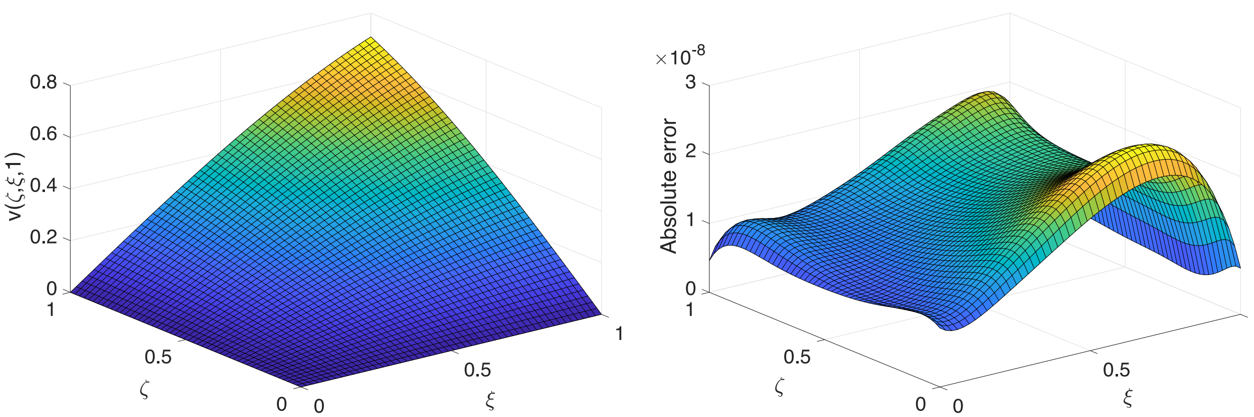

Fig. 4. Approximate solution (left) and associated absolute error function (right) at $\tau = 1$ with $\beta = 0.3$ where $(\widehat{N} = \widehat{M} = 7, N = 2, M = 6)$ in Exa.mple 4.

Table 5
The results obtained with two values of β and some choices of $(\widehat{N}, \widehat{M})$ at $\tau = \frac{3}{2}$ where $(N = 1, M = 4)$ in Example 5.

$\widehat{N}$	$\widehat{M}$	N	M	e_∞	$\beta = 0.45$ CO	CPU time	e_∞	$\beta = 0.95$ CO	CPU time
4	4	1	4	2.6351×10^{-03}	–	022.62	1.3443×10^{-03}	–	017.48
5	5			3.3100×10^{-04}	04.6484	098.76	1.2555×10^{-04}	05.3125	064.03
6	6			1.4468×10^{-05}	08.5842	184.29	7.8835×10^{-06}	07.5907	150.85
7	7			1.2541×10^{-06}	07.9322	398.68	5.0177×10^{-07}	08.9340	345.32
8	8			4.0907×10^{-08}	12.8167	898.23	2.3347×10^{-08}	11.4867	797.32

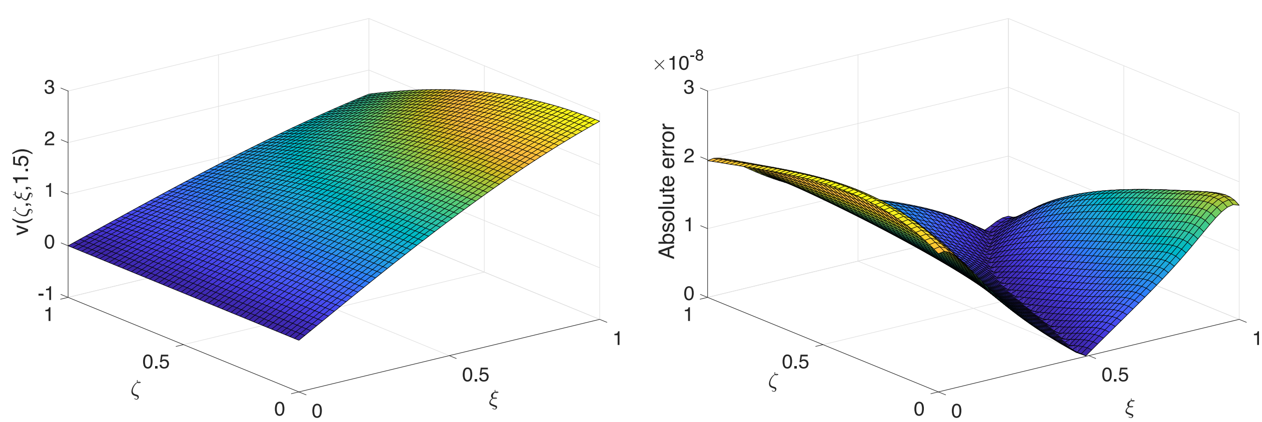

Fig. 5. Approximate solution (left) and associated absolute error function (right) at $\tau = 1.5$ with $\beta = 0.95$ where $(\widehat{N} = \widehat{M} = 8, N = 1, M = 4)$ in Exa.mple 5.

The analytic solution of this problem is $v(\zeta, \xi, \tau) = \tau^3 \cos(\zeta) \sin(\xi)$. We have used the hybrid method proposed for the two-dimensional problem to solve this example. The obtained results with two values of β at $\tau = \frac{3}{2}$ are reported in Table 5. These outcomes confirm the high accuracy of the method in solving this example. Note that the execution time of the program is relatively long due to the non-linear nature of the problem. For $\beta = 0.95$, the behaviors of obtained results are illustrated in Fig. 5.

Conclusion

In this paper, the distributed-order fractional derivative in the Caputo type together with the ABC fractional derivative were used to define a new kind of piecewise fractional derivative. This derivative was used to define piecewise fractional forms of the one- and two-dimensional Galilei invariant advection–diffusion equations. The orthonormal piecewise Vieta-Lucas (VL) functions (as a useful class of basis functions) were generated using the orthonormal VL polynomials to construct two hybrid methods for solving these problems. Analytical formulas regarding the Caputo and ABC fractional derivatives of these piecewise functions were obtained. The proposed methods transformed solving the problems under consideration into solving systems of algebraic equations. Several examples were studied to investigate the validity of the obtained results. These dervived results confirmed the high capability and accuracy of the proposed methods. As a future research direction, the methods introduced in this paper can be easily developed by generating fractional Vieta-Lucas functions to solve problems with fractional solutions.

CREDIT Author Statement

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Data Availability Statement

This manuscript has no associate data.

Compliance with Ethics Requirements

This paper does not contain any studies with human or animal subjects

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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