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http://www.aimspress.com/journal/era

ERA, 30(8): 3058–3070.
DOI: 10.3934/era.2022155
Received: 08 May 2022
Revised: 28 May 2022
Accepted: 01 June 2022
Published: 08 June 2022

Research article

Positivity analysis for the discrete delta fractional differences of the Riemann-Liouville and Liouville-Caputo types

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Abstract: In this article, we investigate some new positivity and negativity results for some families of discrete delta fractional difference operators. A basic result is an identity which will prove to be a useful tool for establishing the main results. Our first main result considers the positivity and negativity of the discrete delta fractional difference operator of the Riemann-Liouville type under two main conditions. Similar results are then obtained for the discrete delta fractional difference operator of the Liouville-Caputo type. Finally, we provide a specific example in which the chosen function becomes nonincreasing on a time set.

Keywords: discrete fractional calculus; discrete Riemann-Liouville operators; discrete Liouville-Caputo fractional operators; monotonicity and positivity

1. Introduction

Discrete fractional operators (DFOs) provide a rich source of interaction between continuous and nonfractional-order operators (see, for example, [1, 2]). In many scientific applications, DFOs are of key importance and they have achieved remarkable success in a number of domains including mathematical modeling [3, 4], stability analysis [5, 6], mathematical physics [7, 8], and uncertainty theory [9, 10]. In fact, there are many real-world applications of fractional difference equations and fractional discrete time systems, which are capable of addressing many problems that fractional differential equations cannot address. For example, we can mention such application areas as fractional chaotic maps for image encryption [11], variable-order recurrent neural networks [12], tempered fractional discrete systems [13], discrete fractional calculus for fuzzy and interval-valued functions [14], and so on. Many other domains of related studies, which are based upon discrete fractional calculus, can be found in [15–20] and also in the references cited therein.

There have been many recent works concerned with the DFOs of the standard kernel such as the discrete Riemann-Liouville fractional operators. These DFOs were extended and examined by many researchers [1, 21, 22]. Similar to the present work, these discrete Riemann-Liouville fractional operators have gained a lot of attention because of their connections to other types of DFOs, such as the discrete Liouville-Caputo fractional operators (see, for example, [23, 24]).

The discrete fractional operators are useful in studying monotonicity and positivity of the nabla and delta analyses described in terms of discrete fractional sum or difference operators (see, for details, [25–28]).

In an earlier influential work, Liu et al. [29] suggested that, if the discrete nabla fractional difference operator of the Riemann-Liouville type satisfies $\left({}^{RL}_{a_0}\nabla^{\alpha}_{h}f\right)(x) \leqq 0$ or $(\geqq 0)$ and the Liouville-Caputo type satisfies $\left({}^{C}_{a_0}\nabla^{\alpha}_{h}f\right)(x) \leqq 0$ or $(\geqq 0)$, $(\Delta_h f)(x)$ could be nonpositive or nonnegative by analysing the nabla fractional differences.

Based on the above-mentioned article by Liu et al. [29], the goal here is to analyse the discrete delta fractional difference operators of the Riemann-Liouville and Liouville-Caputo types for classes of discrete delta operators which induce $(\Delta_h f)(x)$. Our objective is twofold:

- Establish and analyse the positivity and negativity of $(\Delta_h f)(x)$ via the positivity and negativity of the corresponding discrete delta fractional differences in the sense of the Riemann-Liouville operators together with an initial condition.
- Establish and analyse the positivity and negativity of $(\Delta_h f)(x)$ via the positivity and negativity of the corresponding discrete delta fractional differences in the sense of the Liouville-Caputo operators combined with an initial condition.

The outline of this study is as follows. First, in Section 2, we give a brief summary of the discrete delta fractional Riemann-Liouville and Liouville-Caputo type operators. In Section 3, we demonstrate our main theorems and corollaries concerning the positivity and negativity of the discrete delta fractional difference operators by using some conditions. Next, in Section 4, we consider a test example to illustrate the theory which we have presented in this paper. Finally, in our last Section 5, we include our conclusions and comments on further study.

2. Discrete delta fractional operators

The related concepts regarding the discrete delta fractional sums and differences used in this study are recalled in this section.

Definition 2.1 (see [23,24])**.** Let us denote the set $\{a_0, a_0+h, a_0+2h, \ldots\}$ by $\mathbb{N}_{a_0,h}$ with a starting point $a_0 \in \mathcal{R}$. Assume that f is defined on $\mathbb{N}_{a_0,h}$. Then the Δ_h Riemann-Liouville fractional sum of order α (> 0) is expressed as follows:

$$\left({}_{a_0}\Delta_h^{-\alpha} f\right)(x) = \frac{h}{\Gamma(\alpha)} \sum_{r=\frac{a_0}{h}}^{\frac{x}{h}-\alpha} (x-(r+1)h)_h^{[\alpha-1]} f(rh) \quad \text{for } x \text{ in } \mathbb{N}_{a_0+\alpha h,h}, \tag{2.1}$$

where $x_h^{[\alpha]}$ is defined by

$$x_h^{[\alpha]} = h^\alpha \frac{\Gamma\left(\frac{x+h}{h}\right)}{\Gamma\left(\frac{x+h}{h}-\alpha\right)} \quad \text{for } x \text{ and } \alpha \text{ in } \mathcal{R}, \tag{2.2}$$

and we use the convention that $x_h^{[\alpha]} = 0$ for $\frac{x+h}{h} - \alpha$ to not be a nonpositive integer and $\frac{x+h}{h}$ to be a nonpositive integer.

Definition 2.2 (see [24])**.** Let f be defined on $\mathbb{N}_{a_0,h}$. Then the Δ_h difference operator is given by

$$(\Delta_h f)(x) = \frac{1}{h}\{f(x+h) - f(x)\} \quad \text{for } x \text{ in } \mathbb{N}_{a_0,h}.$$

Moreover, the Δ_h Riemann-Liouville fractional difference of order α ($0 \leqq \alpha < 1$) is defined by

$$\begin{aligned}\left({}_{a_0}^{RL}\Delta_h^{\alpha} f\right)(x) &= \left(\Delta_h \; {}_{a_0}\Delta_h^{-(1-\alpha)} f\right)(x) \\ &= \frac{h}{\Gamma(1-\alpha)} \Delta_h \left[\sum_{r=\frac{a_0}{h}}^{\frac{x}{h}+\alpha-1} (x-(r+1)h)_h^{[-\alpha]} f(rh)\right] \quad \text{for } x \text{ in } \mathbb{N}_{a_0+(1-\alpha)h,h}.\end{aligned}$$

An equivalent definition to Definition 2.2 is stated in the following theorem.

Theorem 2.1 (see [25])**.** *The Δ_h Riemann-Liouville fractional differences of order α ($0 < \alpha < 1$) can be expressed as follows:*

$$\left({}_{a_0}^{RL}\Delta_h^{\alpha} f\right)(x) = \frac{h}{\Gamma(-\alpha)} \sum_{r=\frac{a_0}{h}}^{\frac{x}{h}+\alpha} (x-(r+1)h)_h^{[-\alpha-1]} f(rh) \quad \text{for } x \text{ in } \mathbb{N}_{a_0-\alpha h,h}. \tag{2.3}$$

Definition 2.3 (see [24])**.** For a function f defined on $\mathbb{N}_{a_0,h}$, the Δ_h Liouville-Caputo type fractional difference of order α ($0 \leqq \alpha < 1$) is defined by

$$\begin{aligned}\left({}_{a_0}^{C}\Delta_h^{\alpha} f\right)(x) &= \left({}_{a_0}\Delta_h^{-(1-\alpha)} \, \Delta_h f\right)(x) \\ &= \frac{h}{\Gamma(1-\alpha)} \sum_{r=\frac{a_0}{h}}^{\frac{x}{h}+\alpha-1} (x-(r+1)h)_h^{[-\alpha]} (\Delta_h f)(rh) \quad \text{for } x \text{ in } \mathbb{N}_{a_0+(1-\alpha)h,h}.\end{aligned}$$

Lemma 2.1 (see [25, Lemma 1]). *For positive values of α and h, the following result holds true*:

$$\Delta_h\left(x_h^{[\alpha]}\right) = \alpha\, x_h^{[\alpha-1]},$$

for each x in $\mathbb{N}_{0,h}$.

The following proposition gives the relationship between the Δ_h Riemann-Liouville and Liouville-Caputo fractional differences.

Proposition 2.1 (see [25, Proposition 1]). *For $\alpha \in (0,1)$, we have*

$$\left({}_{a_0}^{C}\Delta_h^{\alpha} f\right)(x) = \left({}_{a_0}^{RL}\Delta_h^{\alpha} f\right)(x) - \frac{1}{\Gamma(1-\alpha)}(x-a_0)_h^{[-\alpha]} f(a), \tag{2.4}$$

for x in $\mathbb{N}_{a_0+(1-\alpha)h,h}$.

Lemma 2.2 (see [1, Theorem 2.40]). *Let $\alpha \in (0,1)$, $h > 0$ and $\mu > -1$. Then*

$${}_{a_0+\mu h}^{RL}\Delta_h^{\alpha}(x-a_0)_h^{[\mu]} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)}(x-a_0)_h^{[\mu-\alpha]}, \tag{2.5}$$

for $x \in \mathbb{N}_{a_0+(\mu+1-\alpha)h,h}$.

3. Positivity and main results

Let's start our main results on the Riemann-Liouville and Liouville-Caputo differences. Moreover, the following identity is the main lemma to start off our work here.

Lemma 3.1. *Let $\alpha \in (0,1)$ and $h > 0$. Then*

$$\frac{-1}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath}(\jmath h - \alpha h - \ell\, h)_h^{[-\alpha-1]}\frac{\Gamma(\alpha+\ell)}{\Gamma(\alpha)\Gamma(\ell+1)} = h^{-\alpha-1}\frac{\Gamma(\alpha+\jmath+1)}{\Gamma(\alpha)\Gamma(\jmath+2)},$$

for $\jmath \in \mathbb{N}_0$.

Proof. According to Lemma 2.2, we have

$$\begin{aligned}\left({}_{a_0+\alpha h}^{RL}\Delta_h^{\alpha}\frac{(x-h-a_0)_h^{[\alpha-1]}}{\Gamma(\alpha)}\right)(x) &= \left({}_{a_0+h+(\alpha-1)h}^{RL}\Delta_h^{\alpha}\frac{(x-(a_0+h))_h^{[\alpha-1]}}{\Gamma(\alpha)}\right)(x)\\ &= \frac{1}{\Gamma(0)}(x-h-a_0)_h^{[-1]} = 0,\end{aligned}$$

for $x \in \mathbb{N}_{a_0+h,h}$. Considering Theorem 2.1, it follows that

$$\begin{aligned}\left({}_{a_0+\alpha h}^{RL}\Delta_h^{\alpha}\frac{(x-h-a_0)_h^{[\alpha-1]}}{\Gamma(\alpha)}\right)(x) &= \frac{h}{\Gamma(-\alpha)}\sum_{r=\frac{a_0}{h}+\alpha}^{\frac{x}{h}+\alpha}(x-(r+1)h)_h^{[-\alpha-1]}\frac{(rh-h-a_0)_h^{[\alpha-1]}}{\Gamma(\alpha)}\\ &= \frac{h^{\alpha}}{\Gamma(-\alpha)}\sum_{r=\frac{a_0}{h}+\alpha}^{\frac{x}{h}+\alpha}(x-(r+1)h)_h^{[-\alpha-1]}\frac{\Gamma\left(r-\frac{a_0}{h}\right)}{\Gamma(\alpha)\Gamma\left(r-\frac{a_0}{h}-\alpha+1\right)} = 0.\end{aligned}$$

At $x = a_0 + (\jmath + 1)h$, we get

$$\begin{aligned}
0 &= \frac{h^\alpha}{\Gamma(-\alpha)} \sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+\jmath+1} (a_0 + \jmath h - rh)_h^{[-\alpha-1]} \frac{\Gamma\left(r - \frac{a_0}{h}\right)}{\Gamma(\alpha)\Gamma\left(r - \frac{a_0}{h} - \alpha + 1\right)} \\
&= \frac{h^\alpha}{\Gamma(-\alpha)} \sum_{\ell=0}^{\jmath+1} (\jmath h - \alpha h - \ell h)_h^{[-\alpha-1]} \frac{\Gamma(\alpha + \ell)}{\Gamma(\alpha)\Gamma(\ell + 1)} \\
&= h^{-1} \frac{\Gamma(\alpha + \jmath + 1)}{\Gamma(\alpha)\Gamma(\jmath + 2)} + \frac{h^\alpha}{\Gamma(-\alpha)} \sum_{\ell=0}^{\jmath} (\jmath h - \alpha h - \ell h)_h^{[-\alpha-1]} \frac{\Gamma(\alpha + \ell)}{\Gamma(\alpha)\Gamma(\ell + 1)},
\end{aligned}$$

which rearranges to the required result.

Theorem 3.1. *Suppose that $f : \mathbb{N}_{a_0+\alpha h,h} \longrightarrow \mathcal{R}$ satisfies the following conditions*:

$$\begin{aligned}
&\text{(i)} \quad \left({}^{RL}_{a_0+\alpha h}\Delta_h^\alpha f\right)(x) \leqq 0 \quad \text{for each } x \in \mathbb{N}_{a_0+h,h}, \\
&\text{(ii)} \quad f(a_0 + (\alpha + \ell)h) \geqq \frac{\Gamma(\alpha + \ell + 1)}{\Gamma(\alpha)\Gamma(\ell + 2)} f(a_0 + \alpha h) \quad \text{for } \ell \in \mathbb{N}_0,
\end{aligned}$$

for $\alpha \in (0, 1]$. Then $(\Delta_h f)(x) \leqq 0$ for $x \in \mathbb{N}_{a_0+\alpha h,h}$.

Proof. The case when $\alpha = 1$ it is straightforward. For $0 < \alpha < 1$, we firstly try to show that $f(a_0 + (\alpha + \ell)h) \leqq \frac{\Gamma(\alpha+\ell)}{\Gamma(\alpha)\Gamma(\ell+1)} f(a_0 + \alpha h)$ by strong induction process for $\ell \in \mathbb{N}_0$. From the assumption and Theorem 2.1 (Eq (2.3)) at $x = a_0 + h$, we have for $0 < \alpha < 1$:

$$\begin{aligned}
\left({}^{RL}_{a_0}\Delta_h^\alpha f\right)(a_0 + h) &= \frac{h}{\Gamma(-\alpha)} \sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+1} (a_0 + h - (r + 1)h)_h^{[-\alpha-1]} f(rh) \\
&= h^{-\alpha}\Big\{-\alpha f(a_0 + \alpha h) + f(a_0 + \alpha h + h)\Big\} \leqq 0,
\end{aligned}$$

which implies that

$$f(a_0 + \alpha h + h) \leqq \alpha f(a_0 + \alpha h).$$

On the other hand, by using condition (ii) at $\ell = 0$, we have

$$f(a_0 + \alpha h) \geqq \alpha\, f(a_0 + \alpha h).$$

Thus,

$$\begin{aligned}
(\Delta_h f)(x)\Big|_{x=a_0+\alpha h} &= \frac{f(a_0 + (\alpha + 1)h) - f(a_0 + \alpha h)}{h} \\
&\leqq \frac{1}{h}[\alpha f(a_0 + \alpha h) - \alpha f(a_0 + \alpha h)] = 0.
\end{aligned}$$

By using Eq (2.3) at $x = a_0 + 2h$ and the assumption, we have for $0 < \alpha < 1$:

$$\left({}^{RL}_{a_0}\Delta_h^\alpha f\right)(a_0 + 2h) = \frac{h}{\Gamma(-\alpha)} \sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+2} (a_0 + 2h - (r + 1)h)_h^{[-\alpha-1]} f(rh)$$

$$= h^{-\alpha}\left\{\frac{(1-\alpha)(-\alpha)}{2}f(a_0+\alpha h)+(-\alpha)f(a_0+\alpha h+h)+f(a_0+\alpha h+2h)\right\} \leqq 0,$$

which leads to

$$f(a_0+\alpha h+2h) \leqq \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(3)}f(a_0+\alpha h).$$

On the other hand, considering condition (ii) at $\ell = 1$ to get

$$f(a_0+(\alpha+1)h) \geqq \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(3)}f(a_0+\alpha h). \tag{3.1}$$

Therefore,

$$\begin{aligned}(\Delta_h f)(x)\Big|_{x=a_0+(\alpha+1)h} &= \frac{f(a_0+(\alpha+2)h)-f(a_0+(\alpha+1)h)}{h}\\ &\leqq \frac{1}{h}\left[\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\Gamma(3)}f(a_0+\alpha h)-f(a_0+(\alpha+1)h)\right] \underset{\text{Eq. (3.1)}}{\overset{by}{\leqq}} 0.\end{aligned}$$

Now, we suppose that

$$f(a_0+(\alpha+\ell)h) \leqq \frac{\Gamma(\alpha+\ell)}{\Gamma(\alpha)\Gamma(\ell+1)}f(a_0+\alpha h), \tag{3.2}$$

for $\ell = 0, 1, \ldots, \jmath$ and some $\jmath \in \mathbb{N}_0$. Then, we will try to show that the rule is true at $\ell = \jmath + 1$. By using Eq (2.3) at $x = a_0 + (\jmath + 1)h$ and the assumption, we have:

$$\begin{aligned}\left({}^{RL}_{a_0}\Delta^{\alpha}_{h}f\right)(a_0+2h) &= \frac{h}{\Gamma(-\alpha)}\sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+\jmath+1}(a_0+\jmath h-rh)_h^{[-\alpha-1]}f(rh)\\ &= h^{-\alpha}f(a_0+(\alpha+\jmath+1)h)+\frac{h}{\Gamma(-\alpha)}\sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+\jmath}(a_0+\jmath h-rh)_h^{[-\alpha-1]}f(rh)\\ &= h^{-\alpha}f(a_0+(\alpha+\jmath+1)h)+\frac{h}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath}(\jmath h-\alpha h-\ell\, h)_h^{[-\alpha-1]}f(a_0+(\alpha+\ell)h) \leqq 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}f(a_0+(\alpha+\jmath+1)h) &\leqq \frac{-h^{\alpha+1}}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath}(\jmath h-\alpha h-\ell\, h)_h^{[-\alpha-1]}f(a_0+(\alpha+\ell)h)\\ &\underset{(3.2)}{\overset{by}{\leqq}} -f(a_0+\alpha h)\frac{h^{\alpha+1}}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath}(\jmath h-\alpha h-\ell\, h)_h^{[-\alpha-1]}\frac{\Gamma(\alpha+\ell)}{\Gamma(\alpha)\Gamma(\ell+1)}\\ &\underset{\text{Lemma 3.1}}{\overset{by}{=}} \frac{\Gamma(\alpha+\jmath+1)}{\Gamma(\alpha)\Gamma(\jmath+2)}f(a_0+\alpha h),\end{aligned}$$

which gives that $f(a_0+(\alpha+\jmath+1)h) \leqq \frac{\Gamma(\alpha+\jmath+1)}{\Gamma(\alpha)\Gamma(\jmath+2)} f(a_0+\alpha h)$. Thus, by using this together with the Condition (ii), we have

$$\begin{aligned}(\Delta_h f)(x)\Big|_{x=a_0+(\alpha+\jmath)h} &= \frac{f(a_0+(\alpha+\jmath+1)h)-f(a_0+(\alpha+\jmath)h)}{h} \\ &\leqq \frac{1}{h}\left[\frac{\Gamma(\alpha+\jmath+1)}{\Gamma(\alpha)\Gamma(\jmath+2)} f(a_0+\alpha h)-f(a_0+(\alpha+\jmath)h)\right] \leqq 0,\end{aligned}$$

for $\jmath \in \mathbb{N}_0$. Consequently, we have $(\Delta_h f)(x) \leqq 0$ for all $x \in \mathbb{N}_{a_0+\alpha h,h}$.

Corollary 3.1. *If the function $f:\mathbb{N}_{a_0+\alpha h,h} \longrightarrow \mathcal{R}$ satisfies the following conditions*:

$$\begin{aligned}&\text{(i)} \quad \left({}^{RL}_{a_0+\alpha h}\Delta^{\alpha}_{h} f\right)(x) \geqq 0 \quad \text{for each } x \in \mathbb{N}_{a_0+h,h}, \\ &\text{(ii)} \quad f(a_0+(\alpha+\ell)h) \leqq \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha)\Gamma(\ell+2)} f(a_0+\alpha h) \quad \text{for } \ell \in \mathbb{N}_0,\end{aligned}$$

for $\alpha \in (0,1]$, then $(\Delta_h f)(x) \geqq 0$ for $x \in \mathbb{N}_{a_0+\alpha h,h}$.

Proof. Define $g := -f$. Thus, the proof follows immediately from Theorem 3.1 by applying it for the function g.

Theorem 3.2. *Suppose that $f:\mathbb{N}_{a_0+\alpha h,h} \longrightarrow \mathcal{R}$ satisfies the following conditions*:

$$\begin{aligned}&\text{(i)} \quad \left({}^{C}_{a_0+\alpha h}\Delta^{\alpha}_{h} f\right)(x) \leqq 0 \quad \text{for each } x \in \mathbb{N}_{a_0+2h,h}, \\ &\text{(ii)} \quad (1-\alpha) f(a_0+(\alpha+\ell)h) \geqq \frac{h^{\alpha}}{\Gamma(1-\alpha)}((\ell+1)h-\alpha h)_h^{[-\alpha]} f(a_0+\alpha h) \\ &\qquad\qquad - \frac{h^{\alpha+1}}{\Gamma(-\alpha)}\sum_{r=0}^{\ell-1}(\ell h-rh-\alpha h)_h^{[-\alpha-1]} f(a_0+(\alpha+r)h) \quad \text{for } \ell \in \mathbb{N}_0,\end{aligned}$$

for $\alpha \in (0,1]$. Then $(\Delta_h f)(x) \leqq 0$ for $x \in \mathbb{N}_{a_0+(\alpha+1)h,h}$.

Proof. The result is clear for $\alpha = 1$. Let $\alpha \in (0,1)$. Then, according to Proposition 2.1, Theorem 2.1 and the assumption, one can have

$$\begin{aligned}\left({}^{C}_{a_0+\alpha h}\Delta^{\alpha}_{h} f\right)(x) &= \left({}^{RL}_{a_0+\alpha h}\Delta^{\alpha}_{h} f\right)(x) - \frac{1}{\Gamma(1-\alpha)}(x-a-\alpha h)_h^{[-\alpha]} f(a_0+\alpha h) \\ &= \frac{h}{\Gamma(-\alpha)}\sum_{r=\frac{a_0}{h}+\alpha}^{\frac{x}{h}+\alpha}(x-(r+1)h)_h^{[-\alpha-1]} f(rh) - \frac{1}{\Gamma(1-\alpha)}(x-a-\alpha h)_h^{[-\alpha]} f(a_0+\alpha h) \leqq 0.\end{aligned} \tag{3.3}$$

For $x = a_0+2h$, it follows that

$$\begin{aligned}\left({}^{C}_{a_0+\alpha h}\Delta^{\alpha}_{h} f\right)(a_0+2h) &= \frac{h}{\Gamma(-\alpha)}\sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+2}(a_0+h-rh)_h^{[-\alpha-1]} f(rh) - \frac{1}{\Gamma(1-\alpha)}(2h-\alpha h)_h^{[-\alpha]} f(a_0+\alpha h) \\ &= h^{-\alpha}\left\{\frac{(1-\alpha)(-\alpha)}{2} f(a_0+\alpha h) + (-\alpha) f(a_0+\alpha h+h) + f(a_0+\alpha h+2h)\right\}\end{aligned}$$

$$- h^{-\alpha}\frac{(2-\alpha)(1-\alpha)}{2}f(a_0+\alpha h) \leqq 0,$$

which leads to

$$f(a_0+(\alpha+2)h) \leqq (1-\alpha)f(a_0+\alpha h)+\alpha\, f(a_0+(\alpha+1)h). \tag{3.4}$$

On the other hand, by considering condition (ii) at $\ell = 0$, we have

$$f(a_0+\alpha h+h) \leqq f(a_0+\alpha h). \tag{3.5}$$

Therefore, both inequalities (3.4) and (3.5) imply that

$$\begin{aligned}
(\Delta_h f)(x)\Big|_{x=a_0+(\alpha+1)h} &= \frac{f(a_0+(\alpha+2)h)-f(a_0+(\alpha+1)h)}{h}\\
&\leqq \frac{1}{h}[(1-\alpha)f(a_0+\alpha h)+\alpha\, f(a_0+\alpha h+h)-f(a_0+(\alpha+1)h)]\\
&\leqq \frac{(1-\alpha)}{h}[f(a_0+\alpha h)-f(a_0+\alpha h+h) \leqq 0.
\end{aligned}$$

By substituting $x = a_0+(\jmath+1)h$ int (3.3), we obtain

$$\begin{aligned}
\left({}_{a_0+\alpha h}^{\ \ C}\Delta_h^{\alpha} f\right)(a_0+(\jmath+1)h) &= \frac{h}{\Gamma(-\alpha)}\sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+\jmath+1}(a_0+\jmath h-rh)_h^{[-\alpha-1]}f(rh)\\
&\quad-\frac{1}{\Gamma(1-\alpha)}((\jmath+1)h-\alpha h)_h^{[-\alpha]}f(a_0+\alpha h)\\
&= \frac{h}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath+1}(\jmath h-\alpha h-\ell\, h)_h^{[-\alpha-1]}f(a_0+(\alpha+\jmath)h)\\
&\quad-\frac{1}{\Gamma(1-\alpha)}((\jmath+1)h-\alpha h)_h^{[-\alpha]}f(a_0+\alpha h)\\
&= h^{-\alpha}f(a_0+(\alpha+\jmath+1)h)+\frac{h}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath}(\jmath h-\alpha h-\ell\, h)_h^{[-\alpha-1]}f(a_0+(\alpha+\jmath)h)\\
&\quad-\frac{1}{\Gamma(1-\alpha)}((\jmath+1)h-\alpha h)_h^{[-\alpha]}f(a_0+\alpha h) \leqq 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
f(a_0+(\alpha+\jmath+1)h) \leqq{}& \frac{h^{\alpha}}{\Gamma(1-\alpha)}((\jmath+1)h-\alpha h)_h^{[-\alpha]}f(a_0+\alpha h)\\
&-\frac{h^{\alpha+1}}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath}(\jmath h-\alpha h-\ell\, h)_h^{[-\alpha-1]}f(a_0+(\alpha+\jmath)h)
\end{aligned}$$

$$= \frac{h^{\alpha}}{\Gamma(1-\alpha)}((\jmath+1)h-\alpha h)_h^{[-\alpha]}f(a_0+\alpha h)+\alpha\, f(a_0+(\alpha+\jmath)h)$$

$$-\frac{h^{\alpha+1}}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath-1}(\jmath h-\alpha h-\ell\, h)_h^{[-\alpha-1]}f(a_0+(\alpha+\jmath)h). \quad (3.6)$$

Hence, by using inequality (3.6) combined with the condition (ii), we have

$$\begin{aligned}(\Delta_h f)(x)\Big|_{x=a_0+(\alpha+\jmath)h} &= \frac{f(a_0+(\alpha+\jmath+1)h)-f(a_0+(\alpha+\jmath)h)}{h}\\ &\leqq \frac{1}{h}\Bigg[\frac{h^{\alpha}}{\Gamma(1-\alpha)}((\jmath+1)h-\alpha h)_h^{[-\alpha]}f(a_0+\alpha h)-(1-\alpha)f(a_0+(\alpha+\jmath)h)\\ &\quad -\frac{h^{\alpha+1}}{\Gamma(-\alpha)}\sum_{\ell=0}^{\jmath-1}(\jmath h-\alpha h-\ell\, h)_h^{[-\alpha-1]}f(a_0+(\alpha+\jmath)h)\Bigg] \underset{\text{condition (ii)}}{\overset{by}{\leqq}} 0,\end{aligned}$$

for $\jmath\in\mathbb{N}_0$. Thus, we get $(\Delta_h f)(x)\leqq 0$ for all $x\in\mathbb{N}_{a_0+(\alpha+1)h,h}$.

Corollary 3.2. *If the function* $f:\mathbb{N}_{a_0+\alpha h,h}\longrightarrow\mathcal{R}$ *satisfies the following conditions*:

(i) $\left({}^{C}_{a_0+\alpha h}\Delta_h^{\alpha}f\right)(x)\geqq 0$ for each $x\in\mathbb{N}_{a_0+2h,h}$,

(ii) $$(1-\alpha)f(a_0+(\alpha+\ell)h)\leqq\frac{h^{\alpha}}{\Gamma(1-\alpha)}((\ell+1)h-\alpha h)_h^{[-\alpha]}f(a_0+\alpha h) -\frac{h^{\alpha+1}}{\Gamma(-\alpha)}\sum_{r=0}^{\ell-1}(\ell h-rh-\alpha h)_h^{[-\alpha-1]}f(a_0+(\alpha+r)h)\quad\text{for } \ell\in\mathbb{N}_0,$$

for $\alpha\in(0,1]$, *then* $(\Delta_h f)(x)\geqq 0$ *for* $x\in\mathbb{N}_{a_0+(\alpha+1)h,h}$.

Proof. First, we define define $g:=-f$. Therefore, the proof follows immediately from Theorem 3.2 applying for the new defined function g.

4. Application: A specific example

This section provides a specific example to illustrate our previous theoretical results.

Consider the function

$$f(x)=\left(\frac{2}{5}\right)^{x}\quad\text{for } x\in\mathbb{N}_{a_0+\alpha h,h}.$$

At first, we will try to show that $\left({}^{RL}_{a_0+\alpha h}\Delta_h^{\alpha}f\right)(x)\leqq 0$ for $x\in\{a_0+h,a_0+2h\}$, $\alpha=\frac{1}{2}$, $a_0=0$ and $h=1$. From Definition (2.3) at $x=a_0+h$, we have

$$\begin{aligned}\left({}^{RL}_{a_0}\Delta_h^{\alpha}f\right)(a_0+h) &= \frac{h}{\Gamma(-\alpha)}\sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+1}(a_0+h-(r+1)h)_h^{[-\alpha-1]}f(rh)\\ &= h^{-\alpha}\Big\{-\alpha f(a_0+\alpha h)+f(a_0+\alpha h+h)\Big\}\\ &= -\frac{1}{2}\left(\frac{2}{5}\right)^{\frac{1}{2}}+\left(\frac{2}{5}\right)^{\frac{3}{2}}=-\frac{281}{4443}\leqq 0,\end{aligned}$$

which leads to

$$f(a_0 + (\alpha + 1)h) \leqq \alpha f(a_0 + \alpha h). \tag{4.1}$$

Moreover, Definition (2.3) at $x = a_0 + 2h$ gives

$$\begin{aligned}\left({}^{RL}_{a_0}\Delta^{\alpha}_{h} f\right)(a_0 + 2h) &= \frac{h}{\Gamma(-\alpha)} \sum_{r=\frac{a_0}{h}+\alpha}^{\frac{a_0}{h}+\alpha+2} (a_0 + 2h - (r+1)h)_h^{[-\alpha-1]} f(rh) \\ &= h^{-\alpha}\left\{\frac{(1-\alpha)(-\alpha)}{2} f(a_0 + \alpha h) + (-\alpha) f(a_0 + \alpha h + h) + f(a_0 + \alpha h + 2h)\right\} \\ &= -\frac{1}{8}\left(\frac{2}{5}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{2}{5}\right)^{\frac{3}{2}} + \left(\frac{2}{5}\right)^{\frac{5}{2}} = -\frac{496}{4753} \leqq 0,\end{aligned}$$

which implies that

$$f(a_0 + (\alpha + 2)h) \leqq \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha)\Gamma(3)} f(a_0 + \alpha h) = \frac{\alpha(\alpha + 1)}{2} f(a_0 + \alpha h). \tag{4.2}$$

On the other hand, we will test the condition:

$$f(a_0 + (\alpha + \ell)h) \geqq \frac{\Gamma(\alpha + \ell + 1)}{\Gamma(\alpha)\Gamma(\ell + 2)} f(a_0 + \alpha h),$$

at $\ell = 0, 1$. At $\ell = 0$, it follows that

$$\left(\frac{2}{5}\right)^{\frac{1}{2}} = f(a_0 + \alpha h) \geqq \frac{1}{2}\left(\frac{2}{5}\right)^{\frac{1}{2}} = \alpha\, f(a_0 + \alpha h),$$

which means that

$$f(a_0 + \alpha h) \geqq \alpha\, f(a_0 + \alpha h). \tag{4.3}$$

Moreover, at $\ell = 1$, it follows that

$$\begin{aligned}\frac{509}{2012} = \left(\frac{2}{5}\right)^{\frac{3}{2}} &= f(a_0 + (\alpha + 1)h) \\ &\geqq \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha)\Gamma(3)} f(a_0 + \alpha h) = \frac{3}{8}\left(\frac{2}{5}\right)^{\frac{1}{2}} = \frac{171}{721},\end{aligned}$$

which is equivalent to

$$f(a_0 + (\alpha + 1)h) \geqq \frac{\alpha(\alpha + 1)}{2} f(a_0 + \alpha h). \tag{4.4}$$

Thus, inequalities (4.1)–(4.4) conclude that

$$\begin{aligned}f(a_0 + (\alpha + 2)h) &\leqq \frac{\alpha(\alpha + 1)}{2} f(a_0 + \alpha h) \\ &\leqq f(a_0 + (\alpha + 1)h) \leqq \alpha f(a_0 + \alpha h) \leqq f(a_0 + \alpha h).\end{aligned}$$

These inequalities imply that f is nonincreasing on the time set $\{a_0 + \alpha h, a_0 + (\alpha + 1)h\}$.

5. Concluding remarks

In this article, new positivity and negativity results for the discrete delta fractional difference operators of the Riemann-Liouville and Liouville-Caputo types have been established on $\mathbb{N}_{a_0+\alpha h,h}$. These results can be summarized as follows:

- An identity has been obtained in Lemma 3.1, which has been used in establishing the main results.
- $(\Delta_h f)(x) \leqq 0$ (or $\geqq 0$) for $x \in \mathbb{N}_{a_0+\alpha h,h}$ under the conditions given by $\left({}_{a_0+\alpha h}^{RL}\Delta_h^{\alpha} f\right)(x) \leqq 0$ (or $\geqq 0$) for each $x \in \mathbb{N}_{a_0+h,h}$ and $f(a_0 + (\alpha + \ell)h) \geqq 0$ (or $\leqq 0$)$\frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha)\Gamma(\ell+2)} f(a_0 + \alpha h)$ for $\ell \in \mathbb{N}_0$ and $\alpha \in (0, 1]$ in Theorem 3.1 and Corollary 3.1.
- $(\Delta_h f)(x) \leqq 0$ (or $\geqq 0$) for $x \in \mathbb{N}_{a_0+(\alpha+1)h,h}$ under the following conditions: $\left({}_{a_0+\alpha h}^{C}\Delta_h^{\alpha} f\right)(x) \leqq 0$ (or $\geqq 0$) for each $x \in \mathbb{N}_{a_0+2h,h}$ and $(1-\alpha) f(a_0 + (\alpha + \ell)h) \geqq 0$ (or $\leqq 0$)$\frac{h^{\alpha}}{\Gamma(1-\alpha)}((\ell + 1)h - \alpha h)_h^{[-\alpha]} f(a_0 + \alpha h) - \frac{h^{\alpha+1}}{\Gamma(-\alpha)} \sum_{r=0}^{\ell-1} (\ell h - rh - \alpha h)_h^{[-\alpha-1]} f(a_0 + (\alpha + r)h)$ for $\ell \in \mathbb{N}_0$ and $\alpha \in (0, 1]$ in Theorem 3.2 and Corollary 3.2.

Finally, we have dedicated the last section to show that a function is nonincreasing under the above conditions on the time set $\{a_0 + \alpha h, a_0 + (\alpha + 1)h\}$.

Acknowledgments

This work was supported by the Taif University Researchers Supporting Project Number (TURSP-2020/86), Taif University, Taif, Saudi Arabia.

Conflict of interest

The authors declare there is no conflicts of interest.

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