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Research Article

Shape Preserving Piecewise KNR Fractional Order Biquadratic C^2 Spline

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In a recent article, a piecewise cubic fractional spline function is developed which produces C^1 continuity to given data points. In the present paper, an interpolant continuity class C^2 is preserved which gives visually pleasing piecewise curves. The behavior of the resulting representations is analyzed intrinsically with respect to variation of the shape control parameters t and s. The data points are restricted to be strictly monotonic along real line.

1. Introduction

Among the various methods in computer aided geometric designing, piecewise spline-based techniques are the conventional methods. In many applications, one inclines interpolate or approximate univariate data by spline functions possessing certain geometric properties or shapes such as monotonicity, convexity, or nonnegativity. Due to the verity of spline algorithm, designers do not find any strain to adopt these techniques. Ample work has been done in this regard and researchers are still working on varied techniques by refining them to make it more and more diverse. The aim of spline interpolation is to get an interpolation formula that is continuous and smooth in both within the intervals and at the interpolating points. In recent past, a hatful of work have been done in the field of piecewise polynomial spline curve [1–4], rational spline [5], trigonometric spline [6], exponential spline [7], and spline-based surfaces which are used to preserve the C^2 continuity. This paper is a continuation of a previous paper [8] in which piecewise C^1 continuity is preserved. The fractional biquadratic spline is represented in terms of first and second order derivative values at the knots and provides an alternative to the ordinary spline. This paper is an attempt to embrace a novel technique on piecewise biquadratic polynomial.

Fractional calculus has been an Annex of ordinary calculus that encapsulated integrals and derivatives that are defined for arbitrary real orders. The journey of fractional calculus commenced in seventeenth century and underscored different derivatives [1] with significant pros and cons ranging from Riemann–Liouville, Hadamard, and Grünwald–Letnikov to Caputo, and so forth. Selecting apt fractional derivatives is pertinent to its considered systems; therefore, fractional operators were also a prevalent focus of various research works. Concurrently, studying generalized fractional operators is also indispensable in the field of computer graphics [9–11].

Fractional order derivatives are rapid emerging concept in different fields of mathematics, physics, and engineering in recent years [12–15]. Due to application of new approach of fractional order derivative, the computational cost is reduced. In this paper, an efficient and intuitive technique which is able to produce piecewise smooth curves in each given subinterval, $[x_i, x_{i+1}]$, $i = 0, 1, 2, 3, \ldots n$, $\forall x_i \in \mathbb{R}$, is adopted by combining both concepts of spline and Caputo–Fabrizio fractional order derivatives. With biquadratic piecewise polynomial assistance, higher accuracy is ensured.

The paper is organized in the following way. In Section 2, the formula using continuity condition is established. In Section 3, all the results are included, and in Section 4, discussion related to the novel technique is highlighted.

2. Preliminaries

There are heaps of definitions of fractional integral and derivatives; among them, few are Riemann–Liouville, Riesz, Caputo [8], Riesz–Caputo, Hadamard, Weyl, Grünwald–Letnikov, Chen, etc. Here, we are discussing Riemann–Liouville and Caputo. The proofs of results may be found in [16, 17].

Let g: $[a, b] \longrightarrow \mathscr{R}$ be a function, α a positive real number, n the integer satisfying $n-1 \leq \alpha < n$, and Γ the Euler gamma function [11]. Then, the left and right Riemann–Liouville fractional integrals of order α are defined,

$$
\begin{aligned}
{}_aI_y^{\alpha} g(y) &= \frac{1}{\Gamma(\alpha)} \int_a^y (y-\tau)^{\alpha-1} g(\tau)\mathrm{d}\tau, \\
{}_yI_b^{\alpha} g(y) &= \frac{1}{\Gamma(\alpha)} \int_y^b (\tau-y)^{\alpha-1} g(\tau)\mathrm{d}\tau,
\end{aligned}
\tag{1}
$$

respectively.

The left and right Riemann–Liouville fractional derivatives of order α are defined by

$$
\begin{aligned}
{}_aD_y^{\alpha} g(y) &= \frac{d^m}{\mathrm{d}y^m} {}_aI_y^{m-\alpha} g(y) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{\mathrm{d}y^m} \int_a^y (y-\tau)^{m-\alpha-1} g(\tau) d\tau, \\
{}_yD_b^{\alpha} g(y) &= \frac{d^m}{\mathrm{d}y^m} {}_yI_b^{m-\alpha} g(y) = \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{\mathrm{d}y^m} \int_a^y (y-\tau)^{m-\alpha-1} g(\tau) d\tau.
\end{aligned}
\tag{2}
$$

Therefore, the right and left Caputo fractional derivatives of order α are defined by

$$
\begin{aligned}
{}_a^CD_y^{\alpha} g(y) &= {}_aI_y^{m-\alpha} \frac{\mathrm{d}^m}{\mathrm{d}y^m} g(y) \\
&= \frac{1}{\Gamma(m-\alpha)} \int_a^y (y-\tau)^{m-\alpha-1} g^{(m)}(\tau)\mathrm{d}\tau, \\
{}_y^CD_b^{\alpha} g(y) &= (-1)^m {}_yI_b^{m-\alpha} \frac{d^m}{\mathrm{d}y^m} g(y) \\
&= \frac{1}{\Gamma(m-\alpha)} \int_y^b (-1)^m (\tau-y)^{m-\alpha-1} g^{(m)}(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{3}
$$

Intrinsically, there exists a relation between Caputo fractional and Riemann–Liouville derivatives, and as a consequence, we have the following relations:

If $g(a) = g'(a) = \ldots = g^{(m-1)}(a) = 0$, then ${}_a^CD_y^{\alpha} g(y) = {}_aI_y^{\alpha} g(y)$;

If $g(b) = g'(b) = \ldots = g^{(m-1)}(b) = 0$, then ${}_y^CD_b^{\alpha} g(y) = {}_yD_b^{\alpha} g(y)$.

If $g \in C^m[a, b]$, then the right and left Caputo derivatives are continuous on $[a, b]$. There are some properties which are valid for integer integration and integer differentiation which are also reflected in fractional integration and differentiation [18].

3. Piecewise KNR Fractional Order Biquadratic C^2 Spline

Let $P_i(x)$, $i = 1, 2, 3, \ldots, n$, be a piecewise polynomial in a subinterval $[x_i, x_{i+1}]$ for $x \in [x_i, x_{i+1}]$:

$$
\begin{aligned}
P_i(x) &= a_i(x-x_i)^4 + b_i(x-x_i)^3 + c_i(x-x)_i^{\,2} + d_i(x-x_i) \\
&\quad + e_i, \quad i = 0, 1, 2, 3, \ldots, n, \ x \in [x_i, x_{i+1}],
\end{aligned}
\tag{4}
$$

where a_i, b_i, c_i, d_i, and e_i are unknown constants which need to be calculated by means of the given continuity and differentiability conditions:

$$
\begin{aligned}
P_i(x_{i+1}) &= P_{i+1}(x_{i+1}), \\
P_i'(x_{i+1}) &= P_{i+1}'(x_{i+1}), \\
P_i''(x_{i+1}) &= P_{i+1}''(x_{i+1}), \\
P_i^{\alpha}(x_{i+1}) &= -P_{i+1}^{\alpha}(x_{i+1}), \ 1 < \alpha < 2.
\end{aligned}
\tag{5}
$$

The parameter α that appears in the above conditions is known as fractional order derivative. It is quite evident from the given conditions that the resulting piecewise curves will be smooth in each segment and will possess C^2 continuity. The fractional order derivative of a function $f(x) \in \mathrm{AC}^n[a, b]$ such that f is absolutely continuous of order α with $n-1 < \alpha \leq n$, where n denotes the order of derivative, which is

$$
\left({}_CD_a^{\alpha} f\right)(\xi) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\xi)}{(x-\xi)^{\alpha-n+1}} \mathrm{d}\xi, \quad x > a, \tag{6}
$$

where

$$\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} \mathrm{d}u. \quad (7)$$

Let $P_i(x)$ and $P_{i+1}(x)$ be two piecewise spline polynomials with common point at $x = x_{i+1}$. The application of the above continuity and differentiability conditions will result in ten unknown constants which need to be evaluated for practical applications. Since the spline curve passes through the given data points, it will result in $e_i = y_i$ and $e_{i+1} = y_{i+1}$. The remaining eight unknowns can be calculated by applying Caputo fractional and derivative conditions.

$$\frac{1}{\Gamma(2-\alpha)} \int_t^{x_{i+1}} \frac{p_i''(\tau)}{(x_{i+1}-\tau)^{\alpha-1}} \mathrm{d}\tau = -\frac{1}{\Gamma(2-\alpha)} \int_{x_{i+1}}^{s} \frac{p_{i+1}''(\tau)}{(\tau - x_{i+1})^{\alpha-1}} \mathrm{d}\tau, \quad 1 < \alpha \leq 2. \quad (8)$$

The given system of linear equations is of the form

$$\begin{aligned} a_i A_{\alpha_j} + b_i B_{\alpha_j} + c_i C_{\alpha_j} &= -\left(a_{i+1} E_{\alpha_j} + b_{i+1} F_{\alpha_j} + c_{i+1} G_{\alpha_j}\right), \\ A_{\alpha_j} &= -\frac{12(-t + x[i+1])^{2-\alpha_j}(K + L - M + N)}{(-4+\alpha_j)(-3+\alpha_j)(-2+\alpha_j)}, \end{aligned} \quad (9)$$

where

$$\begin{aligned} &K = 2x[i+1]^2 + 2x[i]x[i+1](-4+\alpha_j),\ L = x[i]^2(-4+\alpha_j)(-3+\alpha_j), \\ &M = 2t\left(x[i+1] + x[i](-4+\alpha_j)\right)(-2+\alpha_j),\ N = t^2(-3+\alpha_j)(-2+\alpha_j), \\ &B_{\alpha_j} = -\frac{6(-t + x[i+1])^{2-\alpha_j}\left(-x[i+1] - x[i](-3+\alpha_j) + t(-2+\alpha_j)\right)}{(-3+\alpha_j)(-2+\alpha_j)}, \\ &C_{\alpha_j} = -\frac{2(-t + x[i+1])^{2-\alpha_j}}{-2+\alpha_j},\ E_{\alpha_j} = \frac{12(s - x[i+1])^{4-\alpha_j}}{4-\alpha_j},\ F_{\alpha_j} = \frac{6(s - x[i+1])^{3-\alpha_j}}{3-\alpha_j}, \\ &G_{\alpha_j} = \frac{2(s - x[i+1])^{2-\alpha_j}}{2-\alpha_j}, \quad j = 1, 2, 3, \text{ and } 4. \end{aligned} \quad (10)$$

We will have four linear equations.

The other four linear equations can be derived from continuity and differentiability conditions as follows:

$$\begin{aligned} a_i h_i^4 + b_i h_i^3 + c_i h_i^2 + d_i h_i &= y_{i+1} - y_i, \\ a_{i+1} h_{i+1}^4 + b_{i+1} h_{i+1}^3 + c_{i+1} h_{i+1}^2 + d_{i+1} h_{i+1} &= y_{i+2} - y_{i+1}, \\ 4a_i h_i^3 + 3b_i h_i^2 + 2c_i h_i + d_i &= d_{i+1}, \\ 12a_i h_i^2 + 6b_i h_i + 2c_i &= 2c_{i+1}, \end{aligned} \quad (11)$$

where $h_i = x_{i+1} - x_i$ and $h_{i+1} = x_{i+2} - x_{i+1}$.

The above system of linear equations will give rise to a unique solution of unknowns $a_i, b_i, c_i, d_i, a_{i+1}, b_{i+1}, c_{i+1}$, and d_{i+1}.

As an example, for a given set of data points, we have a piecewise biquadratic fractional spline curve. In Figures 1 and 2, we have two kinds of curves: one is concave while the other one is convex. The fractional order derivatives used in both curves are given by Table 1. These figures also indicate the potency of the technique at the bending points. We also have a liberty to control the bending due to the introduction of two parameters denoted by t and s.

$$t \in (x_i, x_{i+1}),\ s \in (x_{i+1}, x_{i+2}). \quad (12)$$

They both will serve as shape control parameters. Different choices of these parameters will cause changes in the final shapes. The piecewise curve (Figure 3) shows a C^2 KNR biquadratic fractional spline curve, whereas Figure 4 indicates the exact location of the points and Figure 5 indicates the concentration of the points.

In this method, we have the liberty to modify the path of the curve. Figures 6–9 are good examples of different

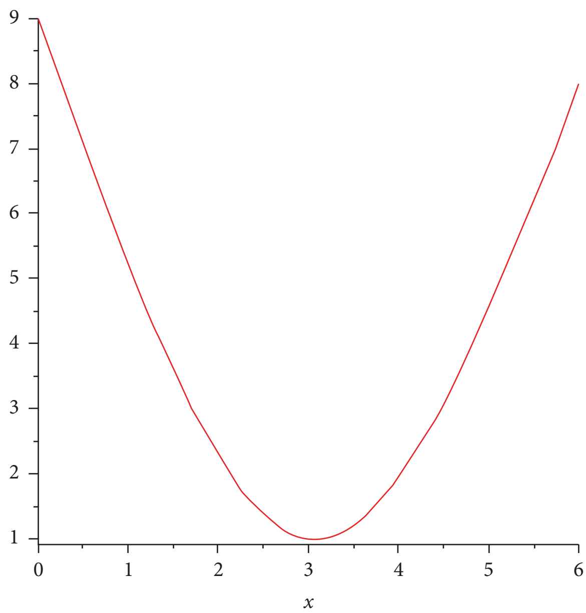

Figure 1: Convex function.

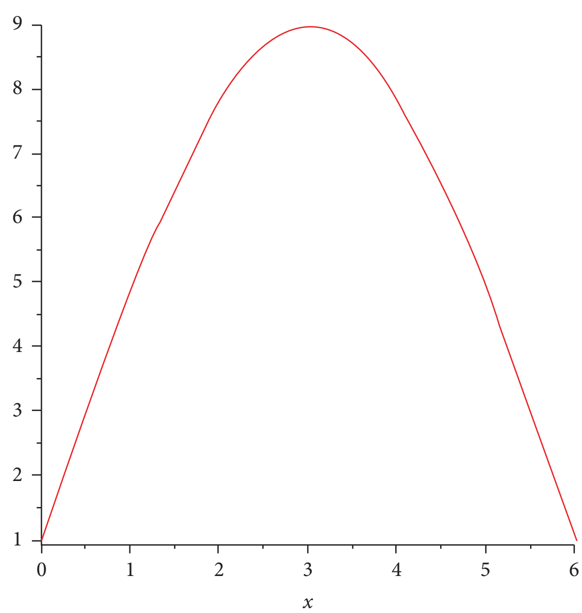

Figure 2: Concave function.

Table 1: Order of fractional derivatives used for both curves.

α_1	α_2	α_3	α_4
1.87	1.9	1.92	1.95

values of shape parameters t and s. As these parameters move away from the connecting point x_{i+1}, the curve starts to flatten at the point and will have effect on the final shape of the curve.

Figures 10–12 indicate the evidence for the effectiveness of the novel technique. The data equally reflect back after application of the newly adopted technique. The straight lines can also be graphed accordingly. Constant

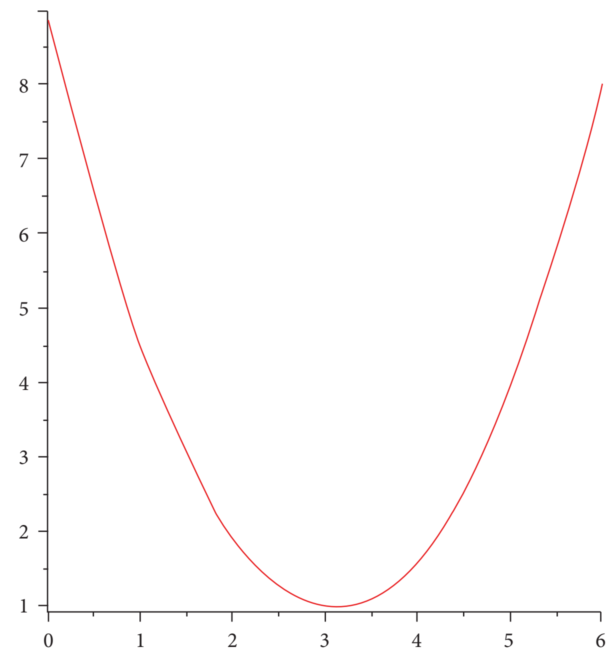

Figure 3: C^2 KNR biquadratic fractional spline curve.

function (in y-values) as shown in Figure 11 and monotone increasing data as shown in Figure 12 can also be preserved, which indicates the accuracy of the technique. In all these shapes, Table 1 is used. Effect on final shape can also be observed if the fractional order derivatives are changed.

4. Comparison of KNR Biquadratic Fractional Spline with Ordinary Cubic Spline

Since ordinary cubic spline is a conventional tool for curve generation, the given comparison indicates that the newly adopted technique coincides with the ordinary one.

For different choices of shape parameters t and s, Figures 13––15 show that the given piecewise curves can be manipulated by the choice of shape parameters. The slight adjustment of the shape parameters can give rise to different shapes. It also indicates that a small change can be made in final shape by altering these parameters.

Geometrically, we have $t \in (x_i, x_{i+1})$ and $s \in (x_{i+1}, x_{i+2})$, which gives us better control on curve's path. Different values of these parameters can change the whole geometry/pattern of the curves. Although the given fractional spline curve will pass through the given data points, but still we can have improved control on the curve.

5. Application of Fractional Spline to n Data Points

Let (x_i, y_i), $i = 0, 1, 2, \ldots, n$, be a set of n data points. Using first three data points, we can find two patches of curves as defined in this paper above. Since all the unknown constants of these two patches are already known, they can be used to find three or more patches of the curves.

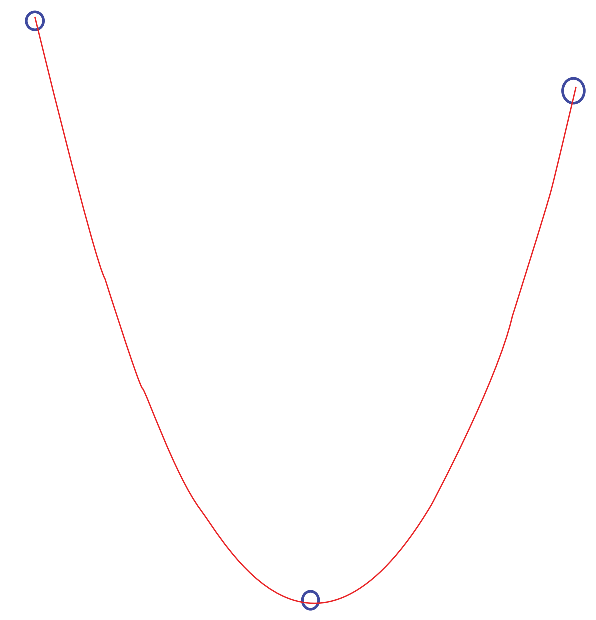

Figure 4: Location of the points.

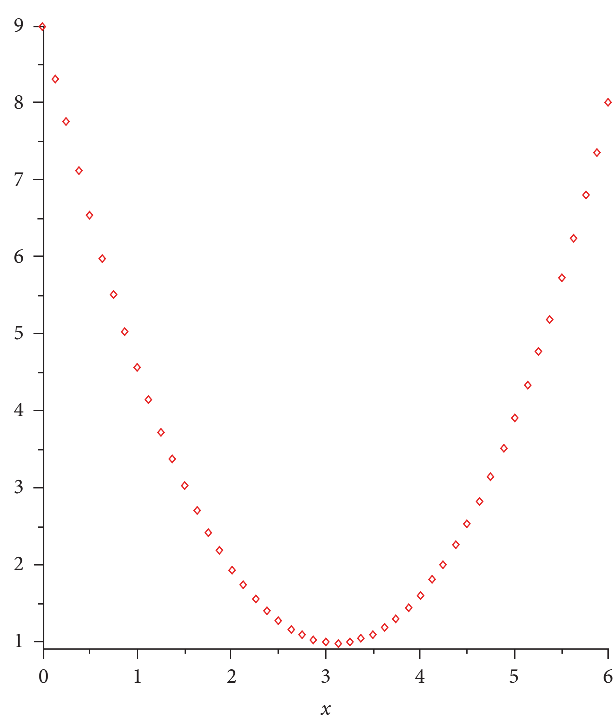

Figure 5: Concentration of the points.

By applying continuity and differentiability conditions, we have the following system of linear equations in three unknowns, namely, a_{i+1}, b_{i+1}, and c_{i+1}.

$$\begin{aligned} &a_{i+1}h_{i+1}^4 + b_{i+1}h_{i+1}^3 + c_{i+1}h_{i+1}^2 = y_{i+2} - y_{i+1} - d_{i+1}h_{i+1}, \\ &a_{i+1}E_{\alpha_j} + b_{i+1}F_{\alpha_j} + c_{i+1}G_{\alpha_j} = -\left(a_iA_{\alpha_j} + b_iB_{\alpha_j} + c_iC_{\alpha_j}\right), \quad j = 1, 2, \end{aligned} \tag{13}$$

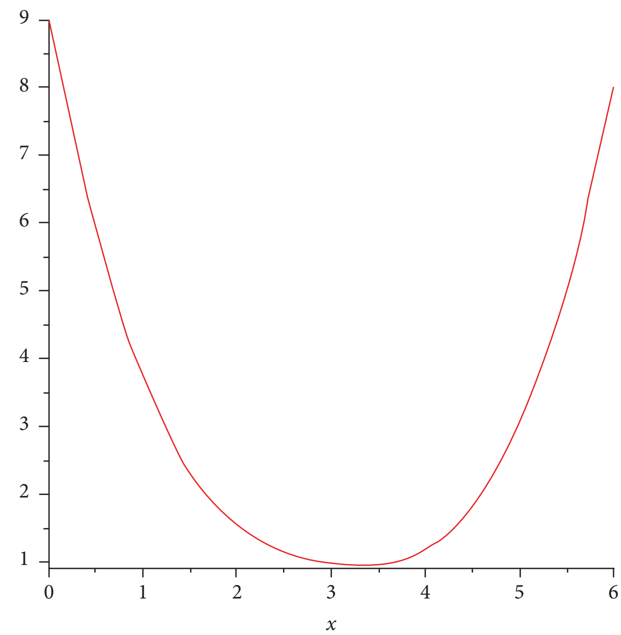

Figure 6: An example of different values of shape parameters t and s.

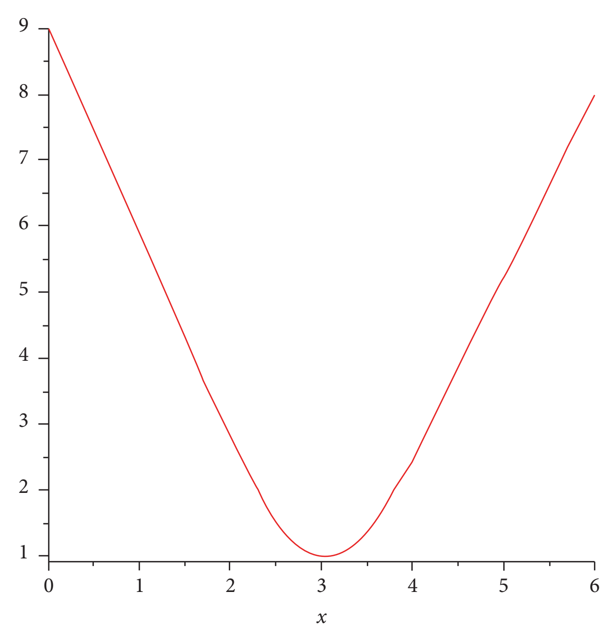

Figure 7: Impact of shape parameters t and as it moves away from connecting point.

where $h_{i+1} = x_{i+2} - x_{i+1}$, $d_{i+1} = 4a_ih_i^3 + 3b_ih_i^2 + 2c_ih_i - d_i$, A_{α_j}, B_{α_j}, C_{α_j}, E_{α_j}, F_{α_j}, and G_{α_j} are already calculated in the previous section.

The above system involves three linear equations for two values of j. In each subsequent segment of curves, we will repeatedly solve the above system for $n-1$ segments of curve. Hence, the above system is true for $i = 1, 2, \ldots, n-1$.

In Figure 16, curve segments in $[x_0, x_1]$ and $[x_1, x_2]$ intervals can easily be calculated by the algorithm as defined prior, whereas the curve segment in interval $[x_2, x_3]$, in

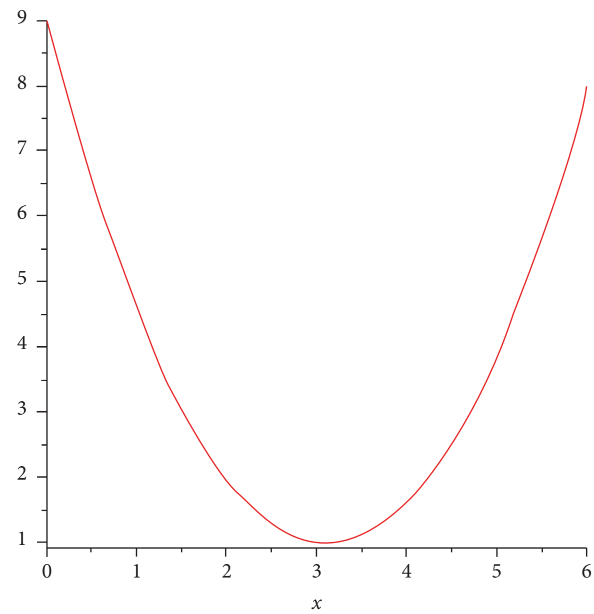

FIGURE 8: Impact of shape parameters t and as it moves away from connecting point.

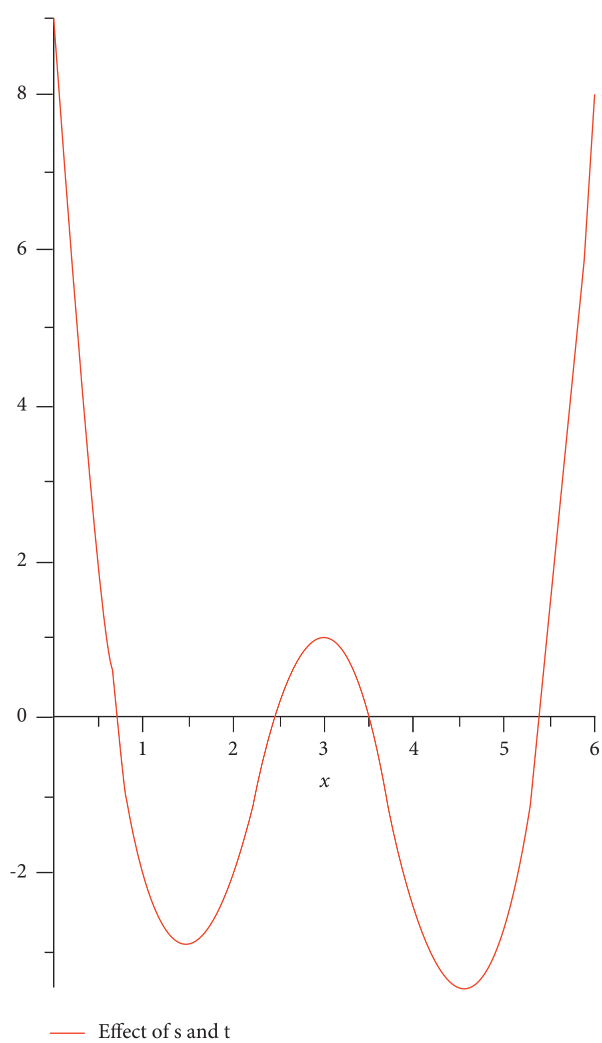

FIGURE 9: Impact of shape parameters t and as it moves away from connecting point.

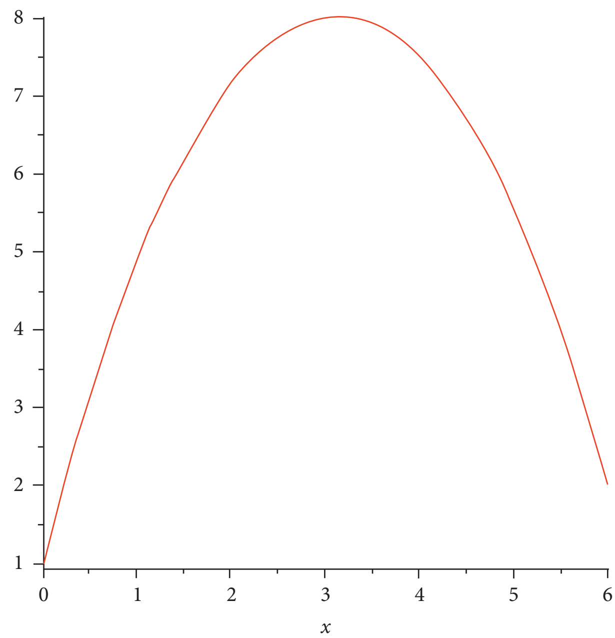

FIGURE 10: After application of the newly adopted technique.

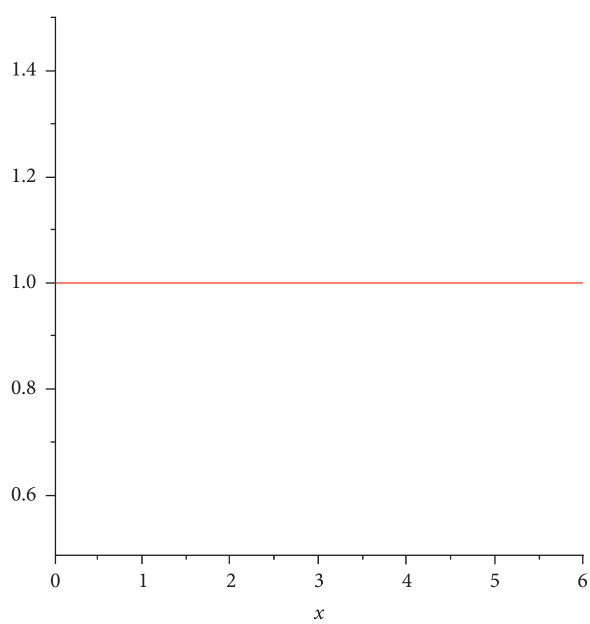

FIGURE 11: Constant functions are preserved.

which x_2 is the connecting point, can be evaluated by the following way:

$$\begin{aligned} P_2(x_2) &= P_3(x_2), \\ P_2'(x_2) &= P_3'(x_2), \\ P_2''(x_2) &= P_3''(x_2), \\ P_3(x_3) &= y_3, \\ P_2^{\alpha}(x_2) &= -P_3^{\alpha}(x_2). \end{aligned} \tag{14}$$

Here, in polynomial $P_3(x)$, we have five unknowns which can easily be calculated by the abovementioned conditions. Similarly, in Figure 17, one more curve segment is included by aforesaid way.

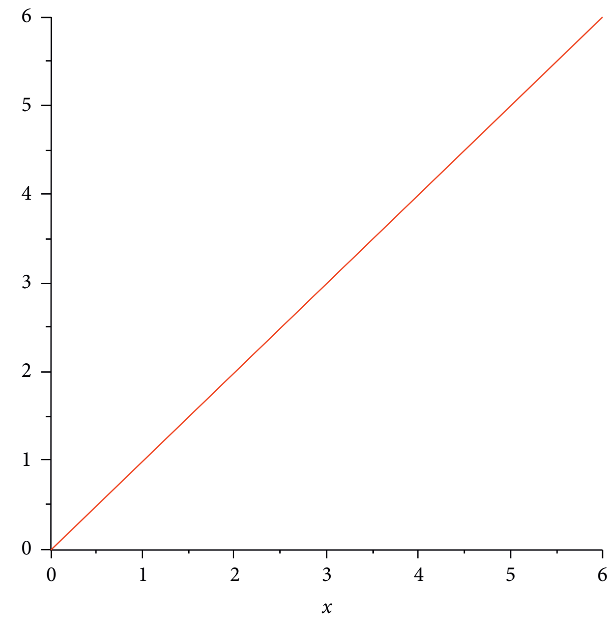

FIGURE 12: Monotone increasing data are preserved.

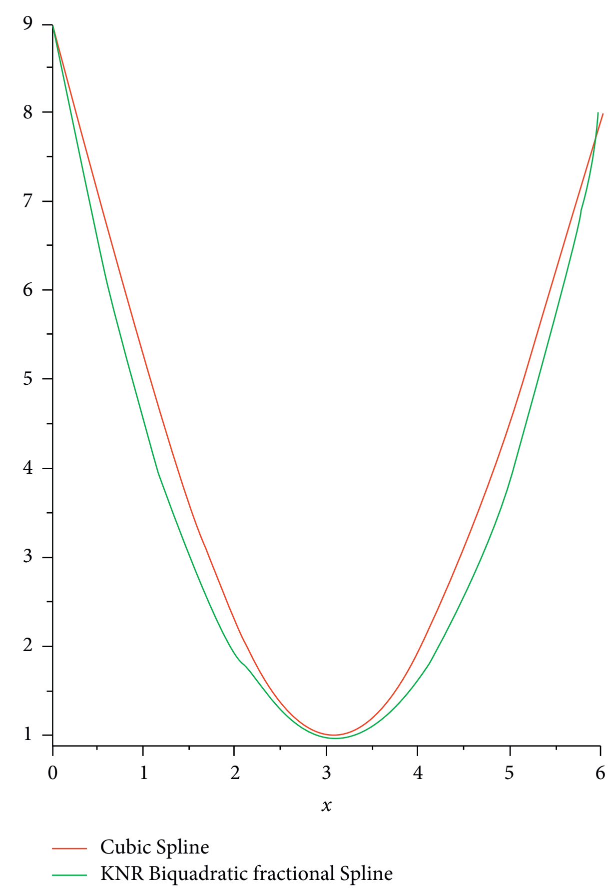

FIGURE 13: Piecewise curves can be manipulated by the choice of shape parameters 1.

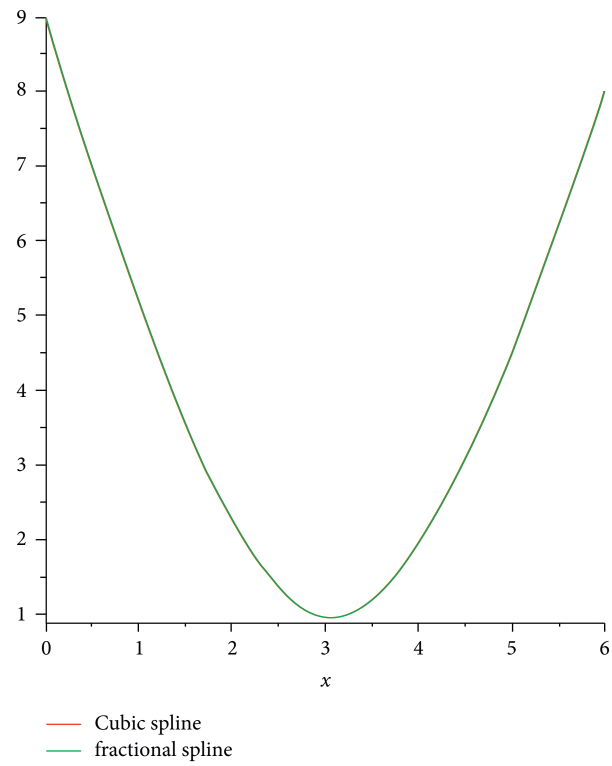

Figure 14: Piecewise curves can be manipulated by the choice of shape parameters 2.

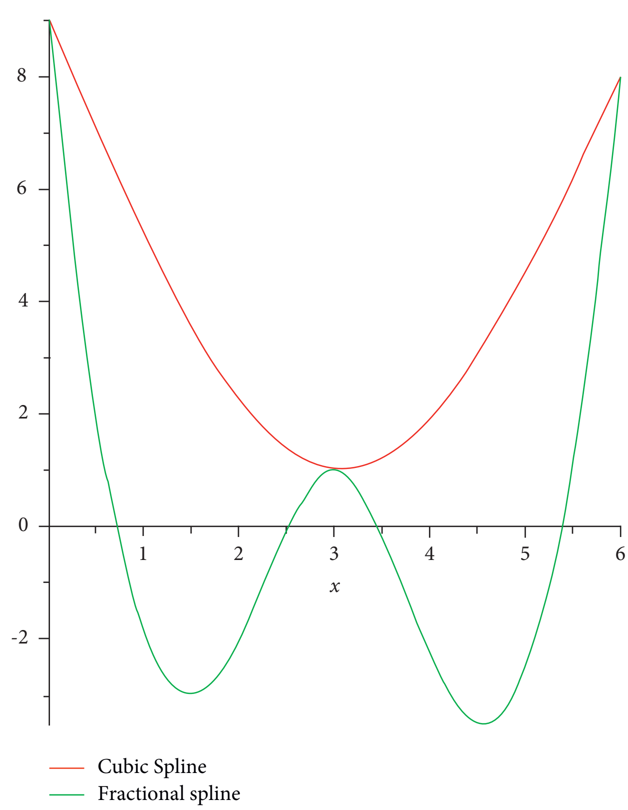

Figure 15: Piecewise curves can be manipulated by the choice of shape parameters 3.

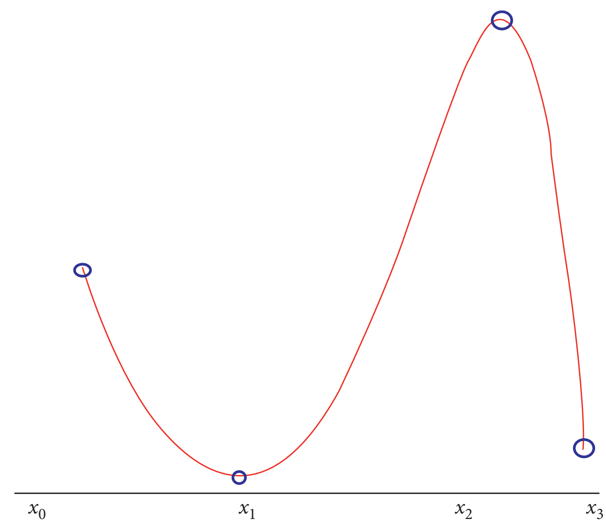

Figure 16: Curve segments.

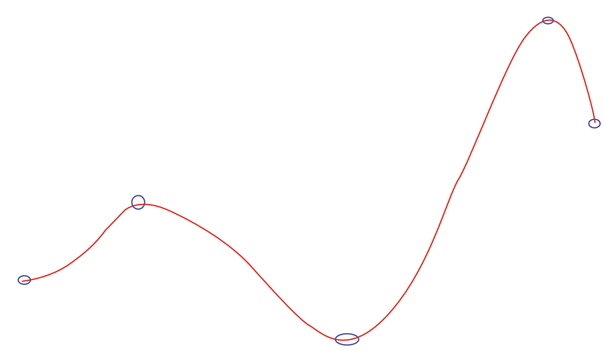

Figure 17: Another curve segment is included.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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