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Search for adequate closed form wave solutions to space–time fractional nonlinear equations

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ABSTRACT

The nonlinear space–time fractional Phi-4 equation and density dependent fractional reaction–diffusion equation (FRDE) are important models to interpret the fusion and fission phenomena ensued in solid state physics, plasma physics, chemical kinematics, astrophysical fusion plasma, electromagnetic interactions etc. In this study, we search advanced and wide-ranging wave solutions to the formerly reported nonlinear fractional evolution equations in diverse family through the new generalized (G'/G)-expansion technique. The solutions are developed with trigonometric, hyperbolic, exponential and rational functions including parameters. The technique is a compatible, functional and effective scientific scheme to examine diverse space–time fractional models in physics and engineering concerned with the real life problems.

1. Introduction

Most of the tangible incidents are modeled and interpreted by the nonlinear fractional or classical partial differential equations. Nonlinear fractional differential equations response speedily and efficiently in numerous branches of scientific and engineering arena, for instance, in fusion plasma, astrophysical dynamics, signal processing, optical fibers, system identification, finance, continuum mechanics, biology, solid state physics, geochemistry etc. Dispersion, reaction, diffusion, dissipation and convection concerning meaningful terms are intimately related to the aforesaid anomaly and can be examined successfully through fractional partial differential equations. Therefore, the study of the exact wave solutions to fractional nonlinear differential equations (FNDEs) as part of the investigation of nonlinear physical incidents is particularly significant. Over the years, many researches have been carried out through developing different techniques, and numerical, analytical and asymptotic solutions are established to the FNDEs. On account of this, several effective methods, as for example, the mixed monotone operator[1] method, the fractional sub-equation[2–6] method, the Adomain polynomial[7] approximation, the homotopy analysis transform,[8] the modified auxiliary equation[9] method, the first integral[10–14] method, the (G'/G)-expansion[15–19] method, the modified simple equation[20] method, the variational iteration[21,22] procedure, the Lie symmetry group[23] analysis, the modified Kudryashov[24] method, the F-expansion[25] method, the complex transform,[26] the new extended direct algebraic[27,28] method, the Cole–Hopf transformation,[29] the transformed rational function method,[30] the Hirota bilinear and tri-linear formation,[31–34] the functional variable[35] method, the Exp-function[36] method, the Liu's extended trial function[37] method, the extended sine–Gordon equation expansion[38] method, the unified method[39] and others[40–44] have been established and extended by diverse group of researchers.

Tariq and Akram[45] investigated wave solutions by utilizing tanh-method to the time fractional Phi-4 equation in a new approach. With the help of extended direct algebraic method, Rezazadeh et al.[46] determined new exact solutions to the nonlinear conformable time-fractional Phi-4 equation. Later, Akram et al.[47] extracted exact solutions of the nonlinear fractional Phi-4 equation analytically by using two reliable techniques, namely, the $\exp(-\phi(\xi))$ and the modified Kudryashov techniques in the sense of conformable time-fractional derivative. Sirisubtawee et al.[48] also investigated exact wave solutions to the space–time fractional Phi-4 equation using generalized Kudryashov method. Very recently, Abdelrahman and Alkhidhr[49] extracted closed-form solutions to the time fractional Phi-4 equation employing unified technique. Das et al.[50] examined an approximate solution of nonlinear FRDE by applying homotopy perturbation method. Merdan[51] used fractional variational iteration method for finding solutions to the time-fractional reaction–diffusion equation in the sense of modified Riemann–Liouville derivative. Guner and Bekir[52] applied exp-function method and established exact solutions to nonlinear FDRE arising in mathematical biology. Agarwal et al.[53] studied the analytic solution to the generalized space–time FRDE. In addition, Tripathi et al.[54] discussed about the

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solutions of higher order nonlinear time-fractional reaction–diffusion equation. Pandey et al.[55] utilized an efficient technique, named the homotopy perturbation for solving the space–time FRDE in porous media. In the recent times, Rui and Zhang[56] employed separation variable method combined with integral bifurcation method for solving time-fractional reaction–diffusion models.

It is noticed from the realistic and statistical evaluation that the space–time fractional Phi-4 equation and the density-dependent space–time FRDE have not yet been searched by the use of the generalized (G'/G)-expansion technique to formulate useful wave solutions in closed form. The generalized (G'/G)-expansion procedure is an efficient and compatible approach that instigates detailed solutions to FNDEs in a straightforward way. Thus, the objective of this article is to investigate further general and wide-ranging solutions comprised with free parameters to the earlier stated models. Definite values of these parameters reveal some existing solutions and establish a number of typical wave solutions established by using aforesaid method. The solutions are achieved in the combination of trigonometric, hyperbolic and rational functions. Merdan[51] studied the aforesaid models and accomplished only hyperbolic and trigonometric function solutions. It is interpreted herein the physical explanation along with the graphical representation of the solutions extensively.

The rest parts of this paper are scheduled as follows: Section 2 explains briefly the definition of conformable fractional derivatives. We outline the solving procedure of the stated method in Section 3. In Section 4, we determine advanced structured solution to the space–time fractional Phi-4 equation and the density-dependent space–time FRDE. In Section 5, the figures are presented graphically and explained physically and we conclude in Section 6.

2. The conformable fraction derivative

Different definitions of fractional derivatives were initiated in the history of fractional calculus, as for instance, Riemann–Lioville, Grunwald–Letnikov, Caputo, Riesz and Weyl, etc. Most of the researchers defined fractional derivatives in terms of fractional integrals having nonlocal properties of integrals. These definitions have huge application areas but they are different from classical Newton–Leibniz calculus. Besides, these derivatives do not follow the chain rule, product rule, quotient rule in course of derivative operation. Some inconsistencies emerge while we compare these derivatives with Newton's derivative. To overcome these challenges, Khalil et al.[44] proposed the concept of local fractional derivative named "conformable fractional derivative (CFD)" in 2014. Most properties of CFD introduced by Khalil match with Newton derivative and also calculate fractional models more effortlessly.

Definition 1. Currently, a new definition of fractional derivative termed as conformable fractional derivative (CFD) of order $\alpha > 0$ developed by Khalil et al.[44] is defined as follows:

Suppose $f : (0, \infty) \to \mathbb{R}$, the conformable fractional derivative of f of α-order is then defined as

$$ {}_tT_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0, \alpha \in (0, 1]$.

The following theorem refers to the features satisfied by the new definition:

Theorem 1. *Let us consider $\alpha \in (0, 1]$, and f, g be α-differentiable at a point t, then the following properties hold[44]:*

(i) ${}_tT_\alpha(af + bg) = a\,{}_tT_\alpha(f) + b\,{}_tT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.
(ii) ${}_tT_\alpha(t^\mu) = \mu t^{\mu-\alpha}$, *for all* $\mu \in \mathbb{R}$.
(iii) ${}_tT_\alpha(fg) = f\,{}_tT_\alpha(f) + g\,{}_tT_\alpha(f)$.
(iv) ${}_tT_\alpha\left(\frac{f}{g}\right) = \frac{g\,{}_tT_\alpha(f)+f\,{}_tT_\alpha(g)}{g^2}$.

Accordingly, if f is differentiable, then ${}_tT_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}$, wherein $t^{1-\alpha}$ denotes a fractional conformable function.

Theorem 2. *Let $f : (0, \infty) \to \mathbb{R}$ be a function wherein f is α-differentiable.[5,13] Let g be a function defined in the range of f and also differentiable, then one can found:*

$$ {}_tT_\alpha(f o g)(t) = t^{1-\alpha} g'(t) f'(g(t)).$$

We propose $\frac{\partial^\alpha}{\partial t^\alpha}(f)$ for ${}_tT_\alpha(f)$ to represent the conformable fractional derivatives of f with regard to the variable t of order α.

Let us consider that a nonlinear conformable fractional partial differential equations bearing (x, t) as independent variables and u as a dependent variable:

$$\mathcal{F}\left(u, \frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \dots\right) = 0, 0 < \alpha \le 1, \tag{2.2}$$

where $u(x,t)$ is an unknown function, $\mathcal{F}$ is a polynomial in u involving nonlinear terms and fractional derivatives of higher order.

3. Method descriptions

The sequential steps of building solutions using the new generalized (G'/G)-expansion technique will be analyzed in this section. We have drawn up an interrelation between the accommodated approach and the transformed rational function method at the end of this section. Consider a general nonlinear space–time fractional equation structured as:

$$\mathcal{H}\left(u, \frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \dots\right) = 0, 0 < \alpha \le 1, \tag{3.1}$$

where $\mathcal{H}$ is a polynomial of the unrevealed wave function $u(x, t)$; the nonlinear terms and the fractional order derivatives of u are associated. The indices denote the fractional derivatives.

Step 1: We bring together the spatial variable x and temporal variable t by a compound variable ξ as:

$$u(x,t) = u(\xi), \xi = \frac{\gamma x^\alpha}{\alpha} + \frac{V t^\alpha}{\alpha}, \tag{3.2}$$

wherein γ is the coefficient of spatial variable, V be the speed of the traveling wave and α signifies the fractional order derivative. The traveling wave variable (3.2) restructures the Eq. (3.1) into an ordinary differential equation (ODE) for $u = u(\xi)$:

$$\mathcal{R}\left(u, u', u'', u''', u^{iv}, u^{v}, \dots..\right) = 0, \tag{3.3}$$

where superscripts designate the derivative with regard to ξ, $\mathcal{R}$ is the polynomial of u and its derivatives.

Step 2: We integrate Eq. (3.3) one or many times as per possibility, and set the integral constant(s) to be zero, as soliton solutions are searched.

Step 3: In conformity with the new (G'/G)-expansion method the solution of (3.3) can be formulated as:

$$u(\xi) = \sum_{i=0}^{N} a_i (d + \mathcal{F})^i + \sum_{i=1}^{N} b_i (d + \mathcal{F})^{-i}, \tag{3.4}$$

where a_N or b_N could be zero, but at the same time they cannot be zero, $a_i, b_i (i = 0, 1, 2, 3, \dots, N)$ and d are constants to be calculate afterward and $\mathcal{F}(\xi)$ is given as follows:

$$\mathcal{F}(\xi) = (G'/G), \tag{3.5}$$

where $G = G(\xi)$ meets the subsequent nonlinear equation:

$$PGG'' - QGG' - SG^2 - R(G')^2 = 0, \tag{3.6}$$

where P, Q, R and S are the indeterminate and the dashes indicate the derivatives with respect to ξ.

Step 4: By balancing the highest order exponent and the derivative, the score of the definite number N appearing in (3.3) can be calculated.
Step 5: Inserting (3.4) and (3.6) along with (3.5) into (3.3) and in conjunction with the value of N found in Step 4 yields a polynomial of $(d+\mathcal{F})^N$ and $(d+\mathcal{F})^{-N}$, $(N = 0,1,2,3,\dots)$. It can be collected each coefficient of the reported polynomial to zero affords a class of algebraic equations for $a_i (i=0,1,2,3,\dots,N)$, $b_i (i=1,2,3,\dots,N)$, d, γ and V.
Step 6: We presume the constants a_i, b_i, $(i=0,1,2,3,\dots,N)$, d and γ might be determined by unraveling the algebraic equations achieved in Step 5. Since the solutions of (3.6) are known, setting the values of $a_i (i=0,1,2,3,\dots,N)$, $b_i (i=1,2,3,\dots,N)$, d, γ and V into (3.4), we found wide-ranging, further comprehensive and fresh soliton solutions to the nonlinear space–time fractional differential equation (3.1).

It is noteworthy to notice that the series expansion (3.4) is a special case of the transformed rational function method investigated by Ma and Lee. Also Eq. (3.6), by the use of (3.5) can be transformed to the subsequent Riccati equation

$$\mathcal{F}' = (-1 + R/P)\mathcal{F}^2 + (Q/P)\mathcal{F} + S/P, \tag{3.7}$$

The Eq. (3.7) ensures that another general Riccati equation with different coefficients is satisfied by $\mathcal{F}+d$. The general solutions to the general Riccati equation with constant coefficients were presented by Ma and Fuchssteiner in Ref. 29 by (40)–(42). The solutions of (3.6) are subject to the relations of the associated parameters. Therefore, with the aid of Refs. 29, 30 and in conjunction with (3.7), the following are the solutions of (3.6):
Family 1: For $Q \neq 0, \psi = P - R$ and $\rho = Q^2 + 4S(P-R) > 0$,

$$\mathcal{F}(\xi) = \left(G'/G\right) = \frac{Q}{2\psi} + \frac{\sqrt{\rho}}{2\psi}\frac{C_{11}\sinh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right) + C_{22}\cosh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right)}{C_{11}\cosh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right) + C_{22}\sinh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right)}. \tag{3.8}$$

Family 2: When $Q \neq 0, \psi = P - R$ and $\rho = Q^2 + 4S(P-R) < 0$,

$$\mathcal{F}(\xi) = (G'/G) = \frac{Q}{2\psi} + \frac{\sqrt{-\rho}}{2\psi}\frac{-C_{11}\sin\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right) + C_{22}\cos\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right)}{C_{11}\cos\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right) + C_{22}\sin\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right)}. \tag{3.9}$$

Family 3: When $Q \neq 0, \psi = P - R$ and $\rho = Q^2 + 4S(P-R) = 0$,

$$\mathcal{F}(\xi) = \left(G'/G\right) = \frac{Q}{2\psi} + \frac{C_{22}}{C_{11} + C_{22}\xi}. \tag{3.10}$$

Family 4: When $Q \neq 0, \psi = P - R$ and $\Delta = \psi S > 0$,

$$\mathcal{F}(\xi) = \left(G'/G\right) = \frac{\sqrt{\Delta}}{\psi} + \frac{\sqrt{\Delta}}{2\psi}\frac{C_{11}\sinh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right) + C_{22}\cosh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)}{C_{11}\cosh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right) + C_{22}\sinh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)}. \tag{3.11}$$

Family 5: When $Q \neq 0, \psi = P - R$ and $\Delta = \psi S < 0$,

$$\mathcal{F}(\xi) = (G'/G) = \frac{\sqrt{-\Delta}}{\psi} + \frac{\sqrt{\Delta}}{2\psi}\frac{-C_{11}\sin\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right) + C_{22}\cos\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)}{C_{11}\cos\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right) + C_{22}\sin\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)}. \tag{3.12}$$

Although the transformed rational function method is the general case, we have put in use the new generalized (G'/G)-expansion method in this article, inasmuch as the adopted technique is straightforward, easy to compute, compatible and user friendly mathematical tool to extract exact wave solutions. By using maple-like computation software, the ascending algebraic equations generated by this method can be easily calculated by this technique. For this reason, using the new generalized (G'/G)-expansion approach, we have explored adequate closed-form wave solutions for space–time fractional models.

4. Formulation of the solutions

In this portion, we formulate some advance and wide-ranging soliton solutions to the nonlinear space–time fractional Phi-4 equation and the density-dependent space time FRDE through the new generalized (G'/G)-expansion approach explained in Section 3.

4.1. The nonlinear space–time fractional Phi-4 equation

Soliton interactions are elastic or particle-like phenomena. On account of this, there is no physical change in the characteristics of speed and energy after the clashes among the solitons but only the phase is shifted. Due to developing the symbolic computation software, nowadays, soliton solutions to fusion and fission phenomena are being investigated both theoretically and experimentally. In this sub-section, we establish scores of soliton solutions to the space–time fractional Phi-4 equation through executing the above reported method.

Consider the space–time fractional Phi-4 equation[45–49]:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} - \frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} + u(x,t) - u^3(x,t) = 0; \quad t > 0, 0 < \alpha \le 1, \tag{4.1.1}$$

herein α is the fractional order derivative. By means of the wave variable $\xi = \frac{\omega x^\alpha}{\alpha} + \frac{V t^\alpha}{\alpha}$, the fractional Phi-4 Eq. (4.1.1) turns out to be the ODE as:

$$\left(V^2 - \omega^2\right)u'' + u - u^3 = 0. \tag{4.1.2}$$

By means of the homogeneous balance theory u'', the highest order derivative and u^3, the highest order exponent in Eq. (4.1.2), we find out $N = 1$. Thus, the solution structure of Eq. (4.1.2) becomes:

$$u(\xi) = a_0 + a_1(d+F) + b_1(d+F)^{-1}, \tag{4.1.3}$$

where a_0, a_1, b_1, d are constants that is to be evaluated.

After substitution (4.1.3) along with (3.5) and (3.6) into (4.1.2), the left side is transmuted to the polynomial in $(d+\mathcal{F})^N$ and $(d+\mathcal{F})^{-N}$, $(N = 0,1,2,\dots)$. Picking up all coefficients of this developed polynomial and putting them to zero leads to a set of algebraic equations for a_0, a_1, b_1, d, ω and V. The algebraic equations are not expressed here for avoiding complexity. Addressing these algebraic equations via computation software Maple, we establish the sets of solutions as follows:

$$\text{Set 1: } V = \pm\frac{\sqrt{\left(Q^2+4S\psi\right)\left(\omega^2Q^2+2P^2+4\omega^2S\psi\right)}}{Q^2+4S\psi}, \quad a_0 = \mp\frac{Q+2d\psi}{\sqrt{Q^2+4S\psi}}, a_1 = \pm\frac{2\psi}{\sqrt{Q^2+4S\psi}}, b_1 = 0, \tag{4.1.4}$$

where $\psi = P - R$, d, P, Q, R, S are free parameters, ω is a wave constant.

$$\text{Set 2: } V = \pm\frac{\sqrt{\left(Q^2+4S\psi\right)\left(\omega^2Q^2+2P^2+4\omega^2S\psi\right)}}{Q^2+4S\psi}, \quad a_0 = \mp\frac{Q+2d\psi}{\sqrt{Q^2+4S\psi}}, a_1 = 0, b_1 = \pm\frac{2\left(d^2\psi+Qd-S\right)}{\sqrt{Q^2+4S\psi}}, \tag{4.1.5}$$

$$\text{Set 3: } V = \pm\frac{\sqrt{2}\sqrt{\left(Q^2+4S\psi\right)\left(2\omega^2Q^2+P^2+8\omega^2S\psi\right)}}{2\left(Q^2+4S\psi\right)}, \quad d = -\frac{Q}{2\psi}, a_0 = 0, a_1 = \pm\frac{2\psi}{\sqrt{Q^2+4S\psi}}, \quad b_1 = \pm\frac{\sqrt{Q^2+4S\psi}}{4\psi}, \tag{4.1.6}$$

$$\text{Set 4: } V = \pm\frac{\sqrt{-\left(Q^2+4S\psi\right)\left(P^2-\omega^2Q^2+4\omega^2S\psi\right)}}{Q^2+4S\psi},$$

$$d=-\frac{Q}{2\psi},a_0=0,a_1=\pm\frac{2\psi}{\sqrt{\left(-2Q^2-8S\psi\right)}},$$
$$b_1=\pm\frac{\sqrt{\left(-2Q^2-8S\psi\right)}}{4\psi},\tag{4.1.7}$$

When $Q\neq 0$, $\psi=P-R$ and $\rho=Q^2+4S(P-R)>0$, inserting the values of the constraints assorted in (4.1.4) into the solution (4.1.3) and after simplifying, we attain the soliton solutions as (for $C_{11}\neq 0$; $C_{22}=0$ and $C_{22}\neq 0$; $C_{11}=0$):

$$u_1(\xi)=\pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\tanh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right),$$
$$u_2(\xi)=\pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\coth\left(\frac{\sqrt{\rho}}{2\psi}\xi\right),$$

where $\xi=\frac{\omega x^\alpha}{\alpha}\pm\frac{\sqrt{(Q^2+4S\psi)(\omega^2Q^2+2P^2+4\omega^2S\psi)}}{Q^2+4S\psi}\frac{t^\alpha}{\alpha}$.

In terms of the temporal and spatial variable, the formerly established solutions become

$$u_{1_{11}}(x,t)=\pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\tanh\left(\frac{\sqrt{\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right),$$
$$u_{2_{11}}(x,t)=\pm\sqrt{\frac{\rho}{Q^2+4S\psi}}\coth\left(\frac{\sqrt{\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right),$$

where $V=\pm\frac{\sqrt{(B^2+4E\psi)(\omega^2B^2+2A^2+4\omega^2E\psi)}}{B^2+4E\psi}$ and ω is a free parameter.

When $Q\neq 0,\psi=P-R$ and $\rho=Q^2+4S(P-R)<0$, for the values assembled in (4.1.4), from solution (4.1.3), we ascertain the subsequent wave solutions (for $C_{11}\neq 0$; $C_{22}=0$ and $C_{22}\neq 0$; $C_{11}=0$):

$$u_3(\xi)=\mp i\sqrt{\frac{\rho}{Q^2+4S\psi}}\tan\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right),$$
$$u_4(\xi)=\pm i\sqrt{\frac{\rho}{Q^2+4S\psi}}\cot\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right).$$

Subject to the variable x and t, the above solutions develop into

$$u_{3_{11}}(x,t)=\mp i\sqrt{\frac{\rho}{Q^2+4S\psi}}\tan\left(\frac{\sqrt{-\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right),$$
$$u_{4_{11}}(x,t)=\pm i\sqrt{\frac{\rho}{Q^2+4S\psi}}\cot\left(\frac{\sqrt{-\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right).$$

On the other hand, for $Q\neq 0$, $\psi=P-R$ and $\rho=Q^2+4S(P-R)=0$, for the values gathered in (4.1.4), from solution (4.1.3) we carry out the under mentioned solution:

$$u_5(\xi)=\pm\frac{2\psi}{\sqrt{Q^2+4S\psi}}\left(\frac{C_{22}}{C_{11}+C_{22}\xi}\right).$$

Making use of wave variable ξ, the solution u_5 turns into:

$$u_{5_{11}}(x,t)=\pm\frac{2\psi}{\sqrt{Q^2+4S\psi}}\frac{C_{22}}{C_{11}+C_{22}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)}.$$

Moreover, for $Q\neq 0,\psi=P-R$ and $\Delta=\psi S>0$ and for the values of the constants laid out in (4.1.4), from (4.1.3) we achieve the ensuing solutions (for $C_{11}\neq 0$; $C_{22}=0$ and $C_{22}\neq 0$; $C_{11}=0$):

$$u_6(\xi)=\pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-\mathrm{Q}+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right\},$$
$$u_7(\xi)=\pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-\mathrm{Q}+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right\}.$$

In relation to the (x,t) variable the former solution varies as follows

$$u_{6_{11}}(x,t)=\pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-\mathrm{Q}+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right)\right\},$$
$$u_{7_{11}}(x,t)=\pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-\mathrm{Q}+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right)\right\}.$$

On the other side, when $Q\neq 0,\psi=P-R$ and $\Delta=\psi S<0$, for the values of the constraints sorted out in (4.1.4), from (4.1.3) we found the following wave solutions (for $C_{11}\neq 0$; $C_{22}=0$ and $C_{22}\neq 0$; $C_{11}=0$):

$$u_8(\xi)=\pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-\mathrm{Q}-2\mathrm{i}\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right\},$$
$$u_9(\xi)=\pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-\mathrm{Q}+2\mathrm{i}\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right\}.$$

Regarding to the variable (x,t), the preceding solutions varies as follows

$$u_{8_{11}}(x,t)=\pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-\mathrm{Q}-2\mathrm{i}\sqrt{\Delta}\tan\left\{\frac{\sqrt{-\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right\}\right\},$$
$$u_{9_{11}}(x,t)=\pm\frac{1}{\sqrt{Q^2+4\Delta}}\left\{-\mathrm{Q}+2\mathrm{i}\sqrt{\Delta}\cot\left\{\frac{\sqrt{-\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right\}\right\}.$$

If we use the other values of parameters accumulated in (4.1.5)–(4.1.7), in the similar fashion, it is possible to construct more wide-ranging general solutions to the fractional Phi-4 equation but for conciseness, the rest of the solutions available for those values are not displayed here. Comparing the formerly established results with those found in Refs. 45–49, it might be emphasized that the wave solutions $u_{1_{11}}(x,t)-u_{9_{11}}(x,t)$ to the space–time fractional Phi-4 equation are useful and compatible and were not established in the earlier research. The above solutions will be expedient to interpret the fusion and fission phenomena, quantum relativistic one-particle theory etc.

4.2. The density-dependent space time fractional reaction–diffusion equation

In this sub-section, we will make use of the new generalized (G'/G)-expansion method to ascertain the solitary wave solutions to the density-dependent space–time FRDE connected with nonlinear wave profiles. The density-dependent space–time FRDE is given by[52]:

$$D\frac{\partial^{2\alpha}w}{\partial x^{2\alpha}}-\frac{\partial^\alpha w}{\partial t^\alpha}-cw\frac{\partial^\alpha w}{\partial x^\alpha}+aw-bw^2=0;\quad t>0,0<\alpha\leq 1,\tag{4.2.1}$$

wherein the parameter α be a fractional order derivative and a,b,c and also the density D be the real constants. Moreover, the fractional constant $\alpha=0.5$ is especially well-known. This is why large amount of models have been developed using this fractional order derivative in fractional calculus. Fractional order derivatives are fruitful to study the anomalous behavior of dynamical systems in viscoelasticity, diffusivity, electrochemistry, chaotic theory etc. For wave translation $\xi=\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}$, Eq. (4.2.1) transmutes to the ODE in terms of the variable $w(\xi)$:

$$D\omega^2w''-Vw'-c\omega ww'+aw-bw^2=0.\tag{4.2.2}$$

The equation specifies a cluster of nonlinear traveling waves. The waves are effortlessly recognized when they do not reform their profiles during propagation. This article reflects kink type traveling waves, periodic waves, solitary waves etc. The solitary waves are stable waves show asymptotic results that tend to zero for extensive path. The nature of kink waves is moving up-and-down from one asymptotic frame to another.

Keeping balance the term w'', highest order linear term and w^2, the nonlinear term of highest order, from Eq. (4.2.2), we obtain the value $N=1$. Thus, the solution of Eq. (4.2.2) appears as follows:

$$w(\xi)=a_0+a_1(d+\mathcal{F})+b_1(d+\mathcal{F})^{-1},\tag{4.2.3}$$

where a_0, a_1, b_1, d are constants whose values will be estimated.

Introducing the solution (4.2.3) along with (3.5) and (3.6) into (4.2.2), we get the polynomial subject to the indeterminate $(d+\mathcal{F})^N$ and $(d+\mathcal{F})^{-N}$, $N=0,1,2,\ldots$. We assemble all coefficients of the generated polynomial to zero gives us a set of algebraic equations due to a_0, a_1, b_1, d, V and ω. Here the equations are not advisable for avoiding complexity:

Set 1: $\omega = \pm\frac{caP}{2bD\sqrt{Q^2+4S\psi}}, V = \mp\frac{aA(4b^2D+ac^2)}{4b^2D\sqrt{Q^2+4S\psi}}, d = -\frac{Q}{2\psi}, a_0 = \frac{a}{2b},$

$$a_1 = \mp\frac{a\psi}{b\sqrt{Q^2+4S\psi}}, b_1 = \mp\frac{a\sqrt{Q^2+4S\psi}}{4b\psi}, \tag{4.2.4}$$

where $\psi = P - R, P, Q, R, S, a, b, c, D$ are free parameters.

Set 2: $\omega = \pm\frac{caA}{2bD\sqrt{Q^2+4S\psi}}, V = \mp\frac{aA(4b^2D+ac^2)}{4b^2D\sqrt{Q^2+4S\psi}}, a_0 = \pm\frac{a\left(Q+2d\psi+\sqrt{Q^2+4S\psi}\right)}{2b\sqrt{Q^2+4S\psi}},$

$$a_1 = \mp\frac{a\psi}{b\sqrt{Q^2+4S\psi}}, b_1 = 0, \tag{4.2.5}$$

Set 3: $\omega = \pm\frac{caP}{2bD\sqrt{Q^2+4S\psi}}, V = \mp\frac{aP(4b^2D+ac^2)}{4b^2D\sqrt{Q^2+4S\psi}}, d = -\frac{Q}{2\psi}, a_0 = \frac{a}{2b},$

$$a_1 = 0, b_1 = \mp\frac{a\sqrt{Q^2+4S\psi}}{4b\psi}, \tag{4.2.6}$$

When $Q \neq 0, \psi = P - R$ and $\rho = Q^2 + 4S(P - R) > 0$, for the values of the constants determined in (4.2.4), from (4.2.3) we obtain the following solutions to the fractional reaction–diffusion equation (for $C_{11} \neq 0$; $C_{22} = 0$ and $C_{22} \neq 0$; $C_{11} = 0$):

$$w_1(\xi) = \frac{a}{2b} \mp \frac{a\sqrt{\rho}}{2b\sqrt{Q^2+4S\psi}} \tanh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right) \mp \frac{a\sqrt{Q^2+4S\psi}}{2b\sqrt{\rho}}\left\{\tanh\left(\frac{\sqrt{\rho}}{2\psi}\xi\right)\right\}^{-1},$$

$$w_2(\xi) = \frac{a}{2b} \mp \frac{a\sqrt{\rho}}{2b\sqrt{Q^2+4S\psi}} \coth\left(\frac{\sqrt{\rho}}{2\psi}\xi\right) \mp \frac{a\sqrt{Q^2+4S\psi}}{2b\sqrt{\rho}}\left\{\coth\left(\frac{\sqrt{\rho}}{2\psi}\xi\right)\right\}^{-1},$$

where, $\xi = \pm\frac{caP}{2bD\sqrt{Q^2+4S\psi}}\frac{x^\alpha}{\alpha} \mp \frac{aP(4b^2D+ac^2)}{4b^2D\sqrt{Q^2+4S\psi}}\frac{t^\alpha}{\alpha}$.

Therefore, regarding to the spatial and temporal variables the erstwhile solutions transform to:

$$w_{1_{11}}(x,t) = \frac{a}{2b} \mp \frac{a\sqrt{\rho}}{2b\sqrt{Q^2+4S\psi}} \tanh\left(\frac{\sqrt{\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right) \mp \frac{a\sqrt{Q^2+4S\psi}}{2b\sqrt{\rho}}\left\{\tanh\left(\frac{\sqrt{\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right)\right\}^{-1},$$

$$w_{2_{11}}(x,t) = \frac{a}{2b} \mp \frac{a\sqrt{\rho}}{2b\sqrt{Q^2+4S\psi}} \coth\left(\frac{\sqrt{\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right) \mp \frac{a\sqrt{Q^2+4S\psi}}{2b\sqrt{\rho}}\left\{\coth\left(\frac{\sqrt{\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right)\right\}^{-1},$$

where $\omega = \pm\frac{caP}{2bD\sqrt{Q^2+4S\psi}}, V = \mp\frac{aP(4b^2D+ac^2)}{4b^2D\sqrt{Q^2+4S\psi}}$.

On the contrary, when $Q \neq 0$, $\psi = P - R$ and $\rho = Q^2 + 4S(P - R) < 0$, for the scores scheduled in (4.2.4), from solution (4.2.3), we achieve the under mentioned wave solution (for $C_{11} \neq 0$; $C_{22} = 0$ and $C_{22} \neq 0$; $C_{11} = 0$):

$$w_3(\xi) = \frac{a}{2b} \pm i\frac{a\sqrt{\rho}}{2b\sqrt{Q^2+4S\psi}} \tan\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right) \mp i\frac{a\sqrt{Q^2+4S\psi}}{2b\sqrt{\rho}}\tan^{-1}\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right),$$

$$w_4(\xi) = \frac{a}{2b} \mp i\frac{a\sqrt{\rho}}{2b\sqrt{Q^2+4S\psi}} \cot\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right) \pm i\frac{a\sqrt{Q^2+4S\psi}}{2b\sqrt{\rho}}\cot^{-1}\left(\frac{\sqrt{-\rho}}{2\psi}\xi\right).$$

Consequently, subject to elementary variables the exact wave solutions become:

$$w_{3_{11}}(x,t) = \frac{a}{2b} \pm i\frac{a\sqrt{\rho}}{2b\sqrt{Q^2+4S\psi}} \tan\left(\frac{\sqrt{-\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right) \mp i\frac{a\sqrt{Q^2+4S\psi}}{2b\sqrt{\rho}}\left\{\tan\left(\frac{\sqrt{-\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right)\right\}^{-1},$$

$$w_{4_{11}}(x,t) = \frac{a}{2b} \mp i\frac{a\sqrt{\rho}}{2b\sqrt{Q^2+4S\psi}} \cot\left(\frac{\sqrt{-\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right) \pm i\frac{a\sqrt{Q^2+4S\psi}}{2b\sqrt{\rho}}\left\{\cot\left(\frac{\sqrt{-\rho}}{2\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right)\right\}^{-1}.$$

Moreover, when $Q \neq 0, \psi = P - R$ and $\rho = Q^2 + 4S(P - R) = 0$, placing the values of the constants accumulated in (4.2.4) into solution (4.2.3), we obtain the following solitary solutions:

$$w_5(\xi) = \frac{a}{2b} \mp \frac{a\psi}{b\sqrt{Q^2+4S\psi}}\left(\frac{C_{22}}{C_{11}+C_{22}\xi}\right) \mp \frac{a\sqrt{Q^2+4S\psi}}{4b\psi}\left(\frac{C_{22}}{C_{11}+C_{22}\xi}\right)^{-1}.$$

In terms of the basic variable the former solution converted into:

$$w_{5_{11}}(x,t) = \frac{a}{2b} \mp \frac{a\psi}{b\sqrt{Q^2+4S\psi}}\frac{C_{22}}{C_{11}+C_{22}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)} \mp \frac{a\sqrt{Q^2+4S\psi}}{4b\psi}\left\{\frac{C_{22}}{C_{11}+C_{22}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)}\right\}^{-1}.$$

Now, for $Q \neq 0, \psi = P - R$ and $\Delta = \psi S > 0$ and for the values of the parameters set out in (4.2.4), from solution (4.2.3) we find out the resulting wave solutions (for $C_{11} \neq 0$; $C_{22} = 0$ and $C_{22} \neq 0$; $C_{11} = 0$):

$$w_6(\xi) = \frac{a}{2b} \mp \frac{a}{2b\sqrt{Q^2+4\Delta}}\left(-Q+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right) \mp \frac{a\sqrt{Q^2+4\Delta}}{2b}\left(-Q+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right)^{-1},$$

$$w_7(\xi) = \frac{a}{2b} \mp \frac{a}{2b\sqrt{Q^2+4\Delta}}\left(-Q+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right) \mp \frac{a\sqrt{Q^2+4\Delta}}{2b}\left(-Q+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right)^{-1}.$$

Therefore, concerning the primary variables the former solutions become

$$w_{6_{11}}(x,t) = \frac{a}{2b} \mp \frac{a}{2b\sqrt{Q^2+4\Delta}}\left\{-Q+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right)\right\} \mp \frac{a\sqrt{Q^2+4\Delta}}{2b}\left\{-Q+2\sqrt{\Delta}\tanh\left(\frac{\sqrt{\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right)\right\}^{-1},$$

$$w_{7_{11}}(x,t) = \frac{a}{2b} \mp \frac{a}{2b\sqrt{Q^2+4\Delta}}\left\{-Q+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right)\right\} \mp \frac{a\sqrt{Q^2+4\Delta}}{2b}\left\{-Q+2\sqrt{\Delta}\coth\left(\frac{\sqrt{\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha}+\frac{Vt^\alpha}{\alpha}\right)\right)\right\}^{-1}.$$

When $Q \neq 0$, $\psi = P - R$ and $\Delta = \psi S < 0$, in similar manner, substituting (4.2.4) into (4.2.3) and reducing the traveling wave solutions appear in the next form (for $C_{11} \neq 0$; $C_{22} = 0$ and $C_{22} \neq 0$; $C_{11} = 0$):

$$w_8(\xi) = \frac{a}{2b} \pm \frac{a}{2b\sqrt{Q^2+4\Delta}}\left(Q+2i\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right) \pm \frac{a\sqrt{Q^2+4\Delta}}{2b}\left(Q+2i\sqrt{\Delta}\tan\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right)^{-1},$$

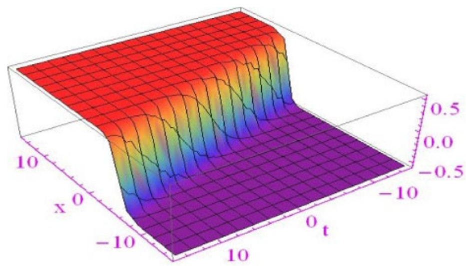

Fig. 1. Kink shape wave structure of solution $u_{1_{11}}(x,t)$.

$$w_9(\xi) = \frac{a}{2b} \mp \frac{a}{2b\sqrt{Q^2+4\Delta}}\left(-\mathrm{Q} + 2\mathrm{i}\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\psi}\xi\right)\right)$$
$$\mp \frac{a\sqrt{Q^2+4\Delta}}{2b}\left(-\mathrm{Q} + 2\mathrm{i}\sqrt{\Delta}\cot\left(\frac{\sqrt{\Delta}}{\psi}\xi\right)\right)^{-1}.$$

Therefore in relation to the original variables the above solutions become

$$w_{8_{11}}(x,t) = \frac{a}{2b} \pm \frac{a}{2b\sqrt{Q^2+4S\psi}}\left\{\mathrm{Q} + 2\mathrm{i}\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right)\right\}$$
$$\pm \frac{a\sqrt{Q^2+4\Delta}}{2b}\left\{\mathrm{Q} + 2\mathrm{i}\sqrt{\Delta}\tan\left(\frac{\sqrt{-\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right)\right\}^{-1},$$
$$w_{9_{11}}(x,t) = \frac{a}{2b} \mp \frac{a}{2b\sqrt{Q^2+4S\psi}}\left\{-\mathrm{Q} + 2\mathrm{i}\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right)\right\}$$
$$\frac{a\sqrt{Q^2+4\Delta}}{2b}\left\{-\mathrm{Q} + 2\mathrm{i}\sqrt{\Delta}\cot\left(\frac{\sqrt{-\Delta}}{\psi}\left(\frac{\omega x^\alpha}{\alpha} + \frac{Vt^\alpha}{\alpha}\right)\right)\right\}^{-1}.$$

By using the values organized in set 2 and set 3, we obtain further wide-ranging exact wave solutions to the reaction–diffusion equation, but for avoiding the repetition, the rest solutions for set 2 and set 3 are not written here. This equation demonstrates more accuracy if the population growth is not widespread although it is modulated by density dependent mortality. The reaction–diffusion system is a potential model to describe spatial dispersive processes. The determined solutions are reliable, compatible and functional to analyze the modulated events and have not been established by the authors.[50–57]

5. Graphical representation and physical explanations

In this section, we explain the established solutions of the models, namely, the density-dependent space–time FRDE and the nonlinear space–time fractional Phi-4 equation physically and graphically. The figures are sketched via the symbolic computation software named Mathematica to understand and disclose the nonlinear fractional phenomena.

The ascertained wave solutions are found in the form of rational, hyperbolic and trigonometric functions on account of $\rho > 0$ and $\rho < 0$ respectively originated from family 1 and family 2. For the values $P = 5$, $Q = 1$, $R = 2$, $S = 1$, $\omega = 2$, $\rho = 4$ of the constraints and the constant order $\alpha = 1.0$, solutions $u_{1_{11}}(x,t)$ and $w_{1_{11}}(x,t)$ are the kink-shape soliton, depicted within the interval $-15 \le x,t \le 15$ for family 1 and presented in Fig. 1.

It is observed that the profiles of the solutions $u_{2_{11}}(x,t)$, $u_{7_{11}}(x,t)$, $w_{2_{11}}(x,t)$ and $w_{7_{11}}(x,t)$ are singular kink shape wave for the similar values of parameters and displayed in Fig. 2.

For the values $P = 5$, $Q = 1$, $R = 2$, $S = 1$, $\omega = 2$, $\rho = -4$ and constant order $\alpha = 1.0$, the solutions $u_{3_{11}}(x,t)$, $u_{8_{11}}(x,t)$ and $w_{8_{11}}(x,t)$ are the periodic wave in the range $-15 \le x,t \le 15$ and portrayed in Fig. 3.

On the other hand, for the score of the constants $P = 5$, $Q = 1$, $R = 2$, $S = 1$, $D = 9$, $a = 1$, $b = 1$, $c = 1$, $\rho = -4$ and for the constant order $\alpha = 1.0$, the shape of the solution $w_{3_{11}}(x,t)$ within the range $-15 \le x,t \le 15$ is exact periodic wave solution and traced in Fig. 4.

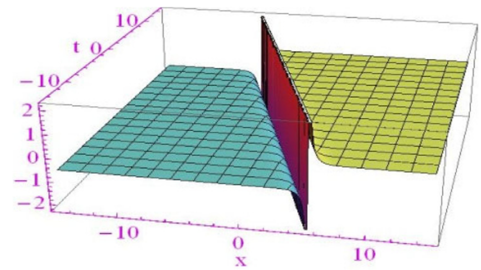

Fig. 2. Singular kink shape wave structure of solution $u_{2_{11}}(x,t)$.

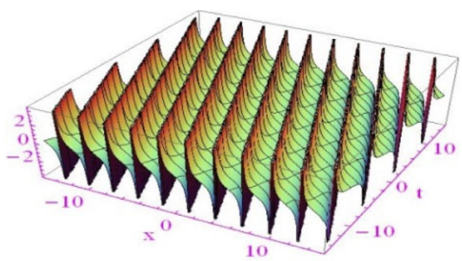

Fig. 3. Periodic wave structure originated from solution $u_{3_{11}}(x,t)$.

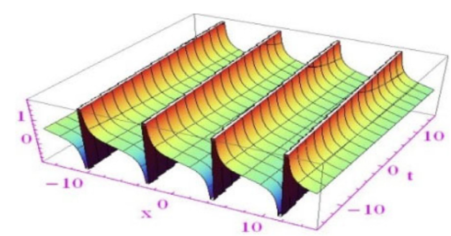

Fig. 4. Exact periodic wave structure portrayed from solution $w_{3_{11}}(x,t)$.

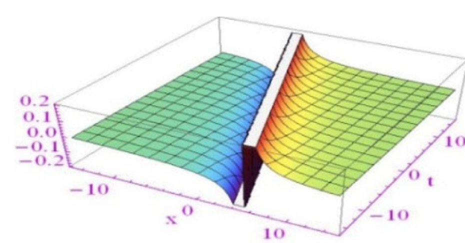

Fig. 5. Singular soliton sketched from solution $u_{5_{11}}(x,t)$.

For family 3, when $\rho = 0$, $P = 5$, $Q = 1$, $R = 2$, $S = 1$, $\omega = 2$, $C_{11} = 2$, $C_{22} = 2$ and for the constant value $\alpha = 1.0$, the solution $u_{5_{11}}(x,t)$ is a singular soliton within the range $-15 \le x,t \le 15$ and shown in Fig. 5.

When $\Delta > 0$, for family 4, the solutions $u_{6_{11}}(x,t)$ and $w_{6_{11}}(x,t)$ are exact periodic wave for $P = 6$, $Q = 1$, $R = 4$, $S = 1$, $\omega = 2$ and for the constant order $\alpha = 1.0$ within the limit $-15 \le x,t \le 15$ and specified in Fig. 6.

For $\Delta < 0$, solutions $u_{9_{11}}(x,t)$, $u_{4_{11}}(x,t)$, $w_{9_{11}}(x,t)$ and $w_{4_{11}}(x,t)$ characterize the singular periodic wave concerning the values $P = 3$, $Q = 1$, $R = 5$, $S = 1$, $\omega = 2$ to the parameters and constant order $\alpha = 1.0$ and within the range $-3 \le x,t \le 3$ and advisable in Fig. 7.

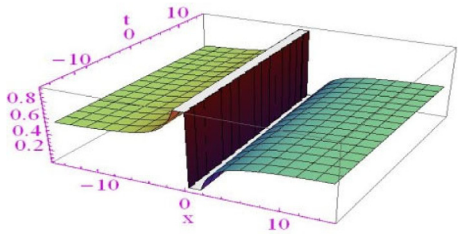

Fig. 6. Exact periodic wave structure originated from $u_{6_{11}}(x,t)$.

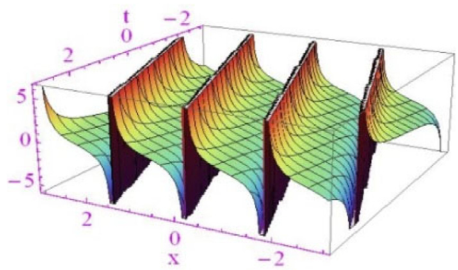

Fig. 7. Multiple soliton wave structure generated by $u_{9_{11}}(x,t)$.

6. Conclusion

In this study, we have established wide-ranging, further general and advanced soliton solutions, in particular, periodic wave, kink wave, single and multiple solitary waves of a couple of space–time fractional nonlinear models explicitly the density-dependent FRDE and the fractional Phi-4 equation. The new generalized (G'/G)-expansion method is effectively applicable to the FNDEs. The fractional complex transformation is highly significant for complex wave variable ξ that confirms that, a fractional order partial differential equation might be transformed into an integer order ODE. The established solutions are compatible to explicate the nonlinear fusion and fission phenomena noticed in many physical incidents, as for instance, plasma physics, electrodynamics, organic membrane etc. In cosmology, kink shape waves are accustomed to model domain walls. The solutions are derived relating to hyperbolic, trigonometric, and rational functions. The effectiveness of the reported approach is further feasible and simpler than the other techniques. The technique might be functional for subsequent assessment to other nonlinear fractional equations in applied mathematics, mathematical physics, engineering and other related fields. It is noteworthy that, the implemented method is the special case of the transformed rational function approach. The Frobenius decomposition approach[58] is one of the efficient techniques in establishing exact solutions to the FNDEs and complexity solutions are available in more integrated ways.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Intellectual property

We confirm that we have given due consideration to the protection of intellectual property associated with this work and that there are no impediments to publication, including the timing of publication, with respect to intellectual property. In so doing we confirm that we have followed the regulations of our institutions concerning intellectual property.

References

1. Wang T, Hao Z. Existence and uniqueness of positive solutions for singular nonlinear fractional differential equation via mixed monotone operator method. *J Function Spaces*. 2020. Article ID 2354927.
2. Jiang J, Feng Y, Li S. Improved fractional sub equation method and exact solution to fractional partial differential equations. *J Function Spaces*. 2020. Article ID 5840920.
3. Islam ME, Akbar MA. Stable wave solutions to the Landau–Ginzburg–Higgs equation and the modified equal width have equation using the IBSEF method. *Arab J Basic Appl Sci*. 2020;27(1):270–278.
4. Tariq KU, Younis M, Rezazadeh H, Rizvi STR, Osman MS. Optical solitons with quadratic-cubic nonlinearity and fractional temporal evolution. *Modern Phys Lett B*. 2018;32(26):1850317.
5. Aminikhah H, Sheikhani AR, Rezazadeh H. Sub-equation method for the fractional regularized long-wave equations with conformable fractional derivatives. *Sci Iran Trans b Mech Engg*. 2016;23(3):1048.
6. Khodadad FS, Nazari F, Eslami M, Rezazadeh H. Soliton solutions of the conformable fractional Zakharov–Kuznetsov equation with dual-power law nonlinearity. *Opt Quantum Electron*. 2017;49(11):384.
7. Al-Zhour Z, Al-Mutairi N, Alrawajeh F, Alkhasawneh R. Series solutions for the Laguerre and Lane-Emden fractional differential equations in the sense of conformable fractional derivative. *Alexandria Engg J*. 2019;58:1413–1420.
8. Maitama S, Zhao W. New homotopy analysis transforms method for solving multidimensional fractional diffusion equations. *Arab J Basic Appl Sci*. 2020;27(1):27–44.
9. Akbar MA, Ali NHM, Tanjim T. Adequate soliton solutions to the perturbed Boussinesq equation and the KdV-Caudrey–Dodd–Gibbon equation. *J King Saud Univ Sci*. 2020;32(6):2777–2785.
10. Rezazadeh H, Manafian J, Khodadad FS, Nazari F. Traveling wave solutions for density-dependent conformable fractional diffusion-reaction equation by the first integral method and the improved $\tan(\varphi(\xi)/2)$-expansion. *Opt Quantum Electron*. 2018;51:121.
11. Lu B. The first integral method for some time fractional differential equations. *J Math Anal Appl*. 2012;395(2):684–693.
12. Rezazadeh H, Manafian J, Khodadad FS, Nazari F. Traveling wave solutions for density-dependent conformable fractional diffusion-reaction equation by the first integral method and the improved $\tan(\varphi(\xi)/2)$-expansion. *Opt Quantum Electron*. 2018;51:121.
13. Eslami M, Rezazadeh H. The first integral method for Wu–Zhang system with conformable time-fractional derivative. *Calcolo*. 2016;53(3):475–485.
14. Eslami M, Khodadad FS, Bazari F, Rezazadeh H. The first integral method applied to the Bogoyavlenskii equations by means of conformable fractional derivative. *Opt Quantum Electron*. 2017;49(12):391.
15. Roy R, Akbar MA, Wazwaz AM. Exact wave solutions for the time fractional Sharma–Tasso–Olver equation and the fractional Klein–Gordon equation in mathematical physics. *Opt Quantum Electron*. 2017;50(1):25.
16. Akbar MA, Ali NHM, Roy R. Closed form solutions of two nonlinear time fractional wave equations. *Results Phys*. 2018;9:1031–1039.
17. Hossain MD, Alam MK, Akbar MA. Abundant wave solutions of the Boussinesq equation and the (2+1)-dimensional extended shallow water wave equation. *Ocean Engg*. 2018;165:69–76.
18. Bekir A, Guner O. The (G'/G)-expansion method using modified Riemann–Liouville derivative for some space–time fractional differential equations. *Ain Shams Engg J*. 2014;5:959–965.
19. Islam MT, Akbar MA, Azad MAK. Multiple closed form wave solutions to the KdV and modified KdV equations through the rational (G'/G)-expansion method. *J Assoc Arab Univ Basic Appl Sci*. 2017;24:160–168.
20. Roy R, Roy S, Hossain MN, Alam MZ. Study on nonlinear partial differential equation by implementing MSE method. *Global Sci J*. 2020;8(1):2320–9186.
21. El-Ajou A, Arqub OA, Momani S, Baleanu D, Alsaedi A. A novel expansion iterative method for solving linear partial differential equations of fractional order. *Appl Math Comput*. 2015;257:119–133.
22. Sing J, Kumar D, Baleanu D. On the analysis of chemical kinetics system pertaining to a fractional derivative with Mittag–Leffler type kernel. *Chaos*. 2017;27:103113.
23. Inc M, Yusuf A, Aliyu AI, Baleanu D. Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers–Huxley equation. *Opt Quantum Electron*. 2018;50:94.

24. Hosseini K, Mayeli P, Kumar D. New exact solutions of the coupled sine-Gordon equations in nonlinear optics using the modified Kudryashov method. *J Modern Optics*. 2018;65(3):361–364.
25. Islam MS, Akbar MA, Khan K. Analytical solutions of nonlinear Klein–Gordon equation using the improved F-expansion method. *Opt Quantum Electron*. 2018;50:224.
26. Herzallah MAE. Comments on different methods for (3+1)-dimensional space–time fractional modified KdV-Zakharov–Kuznetsov equation. *Comput Math Appl*. 2016;71(6):1259–1269.
27. Rezazadeh H, Mirhosseini-Alizamini SM, Eslami M, et al. New optical solitons of nonlinear conformable fractional Schrodinger–Hirota equation. *Optik*. 2018;172:545–553.
28. Rezazadeh H, Tariq H, Eslami M, Mirzazadeh M, Zhou Q. New exact solutions of nonlinear conformable time-fractional Phi-4 equation. *Chin J Phys*. 2018;56(6):2805–2816.
29. Ma WX, Fuchssteiner B. Explicit and exact solutions to a Kolmogorov–Petrovskii–Piskunov equation. *Int J Nonlinear Mech*. 1996;31(3):329–338.
30. Ma WX, Lee JH. A transformed rational function method and exact solutions to the 3+1 dimensional Jimbo–Miwa equation. *Chaos Solitons Fractals*. 2009;42(3):1356–1363.
31. Chen S-J, Yin Y-H, Ma WX, Lu X. Abundant exact solutions and interaction phenomena of the (2+1)-dimensional YTSF equation. *Anal Math Phys*. 2019;9:2329–2344.
32. Ma WX, Zhang Y, Tang Y. Symbolic computation of lump solutions to a combined equation involving three types of nonlinear terms. *East Asian J Appl Math*. 2020;10(4):732–745.
33. Yang J-Y, Ma WX, Khalique CM. Determining lump solutions for a combined soliton equation in (2+1)-dimensions. *Eur Phys J Plus*. 2020;135(6):494.
34. Ma WX, Zhang L. Lump solutions with higher-order rational dispersion relations. *Pramana J Phys*. 2020;43:94.
35. Liu W, Chen K. The functional variable method for finding exact solutions of some nonlinear time-fractional differential equations. *Pramana*. 2013;81(3):377–384.
36. Zheng B. Exp-function method for solving fractional partial differential equations. *Sci World J*. 2013;2013:465723.
37. Rezazadeh H, Osman MS, Eslami M, et al. Mitigating internet bottleneck with fractional temporal evolution of optical solitons having quadratic-cubic nonlinearity. *Optik*. 2018;164:84–92.
38. Bulut H, Sulaiman TA, Baskonus HM, Rezazadeh H, Eslami M, Mirzazadeh M. Optical solitons and other solutions to the conformable space–time fractional Fokas–Lenells equation. *Optik*. 2018;172:20–27.
39. Osman MS, Korkmaz A, Rezazadeh H, Mirzazadeh M, Eslami M, Zhou Q. The unified method for conformable time fractional schrodinger equation with perturbation terms. *Chin J Phys*. 2018;56(5):2500–2506.
40. Liu CS. Counterexamples on Jumarie's two basic fractional calculus formulae. *Commun Nonlinear Sci Numer Simul*. 2015;22(1):92–94.
41. Herzallah MAE. Notes on some fractional calculus operators and their properties. *J Fract Calc Appl*. 2014;5(19):1–10.
42. Liu CS. Counterexamples on Jumarie's three basic fractional calculus formulae for nondifferentiable continuous functions. *Chaos Solitons Fractals*. 2018;109:219–222.
43. Biswas A, Al-Amr MO, Rezazadeh H, et al. Resonant optical solitons with dual-power law nonlinearity and fractional temporal evolution. *Optik*. 2018;165:233–239.
44. Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. *J Comput Appl Math*. 2014;264:65–70.
45. Tariq H, Akram G. New approach for exact solutions of time fractional Cahn–Allen equation and time fractional Phi-4 equation. *Physica A*. 2017;473(1):352–362.
46. Rezazadeh H, Tariq H, Eslami M, Mirzazadeh M, Zhou Q. New exact solutions of nonlinear conformable time-fractional Phi-4 equation. *Chin J Phys*. 2018;56(6):2805–2816.
47. Akram G, Batool F, Raiz A. Two reliable techniques for the analytical study of conformable time-fractional Phi-4 equation. *Opt Quantum Electron*. 2018;50:22.
48. Sirisubtawee S, Koonprasert S, Sungnul S, Leekparn T. Exact traveling wave solutions of the space–time fractional complex Ginzburg–Landau equation and the space time fractional Phi-4 equation using reliable methods. *Adv Difference Equ*. 2019;2019:219.
49. Abdelrahman MAE, Alkhidhr HA. Closed-form solutions to the conformable space–time fractional simplified MCH equation and time fractional Phi-4 equation. *Results Phys*. 2020;18:103294.
50. Das S, Gupta PK, Ghosh P. An approximate solution of nonlinear fractional reaction–diffusion equation. *Appl Math Model*. 2011;35:4071–4076.
51. Merdan M. Solutions of time-fractional reaction–diffusion equation with modified Riemann–Liouville derivative. *Int J Phy Sci*. 2012;7(15):2317–2326.
52. Guner O, Bekir A. Exact solutions of some fractional differential equations arising in mathematical biology. *Int J Biomath*. 2015;8(1):1550003.
53. Agarwal R, Jain Sn, Agarwal RP. Analytic solution of generalized space time fractional reaction diffusion equation. *Fract Differ Calc*. 2017;7(1):169–184.
54. Tripathi NK, Das S, Ong SH, Jafari H, Quarshi MA. Solution of higher order nonlinear time-fractional reaction diffusion equation. *Entrophy*. 2016;18(9):329.
55. Pandey P, Kumar S, Gomez-Aguilar JF, Baleanu D. An efficient technique for solving the space–time fractional reaction–diffusion equation in porous media. *Chin J Phys*. 2020;68:483–492.
56. Rui W, Zhang H. Separation variable method combined with integral bifurcation method for solving time-fractional reaction–diffusion models. *J Comput Appl Math*. 2020;39:299.
57. Roy R, Akbar MA. A new approach to study nonlinear space–time fractional sine-Gordon and Burgers equations. *IOP SciNotes*. 2020;1:035003.
58. Ma WX, Wu HY, He JS. Partial differential equations possessing Frobenius integrable decompositions. *Phys Lett A*. 2007;364(1):29–32.