Mathematics

Research article

Relationships between the discrete Riemann-Liouville and Liouville-Caputo fractional differences and their associated convexity results

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Abstract: In this study, we have presented two new alternative definitions corresponding to the basic definitions of the discrete delta and nabla fractional difference operators. These definitions and concepts help us in establishing a relationship between Riemann-Liouville and Liouville-Caputo fractional differences of higher orders for both delta and nabla operators. We then propose and analyse some convexity results for the delta and nabla fractional differences of the Riemann-Liouville type. We also derive similar results for the delta and nabla fractional differences of Liouville-Caputo

type by using the proposed relationships. Finally, we have presented two examples to confirm the main theorems.

1. Introduction

Discrete fractional operators represent several fundamental models in order to understand the discrete fractional calculus phenomena. Furthermore, the study of these operators and their properties has motivated the innovation of new mathematical tools which have provided useful insights for a number of problems arising in various fields of application. On the other hand, several tools and techniques borrowed from mathematical analysis [1–3], stability analysis [4–6], probability theory [7–9], geometry [10, 11], ecology [12, 13] and topology [14–16] have contributed to a better understanding of the properties of these discrete fractional operators.

In addition to many experiments, the rules of discrete fractional operators have also been under active investigations emerging from theoretical and computational discrete fractional calculus. The discrete fractional calculus theory has been successfully applied to analyze the positivity and monotonicity of discrete delta and nabla fractional operators of the Riemann-Liouville, Liouville-Caputo, Atangana-Baleanu and Caputo-Fabrizio types (see [17–21]).

While, in recent years, several results have been obtained for monotonicity and positivity analysis involving discrete fractional operators, fewer results are available in the convexity analysis setting (see [22–24]). In this study, at first we establish a relationship between the Δ fractional difference of order β of the Riemann-Liouville type $\left({}^{\text{RL}}_{t_0}\Delta^{\beta}g\right)(z)$ and the Liouville-Caputo type $\left({}^{\text{LC}}_{t_0}\Delta^{\beta}g\right)(z)$, and a relationship between the ∇ fractional difference of order β of Riemann-Liouville type $\left({}^{\text{RL}}_{t_0}\nabla^{\beta}g\right)(z)$ and Liouville-Caputo type $\left({}^{\text{LC}}_{t_0}\nabla^{\beta}g\right)(z)$ for each $\aleph - 1 < \beta < \aleph$ with $\aleph \in \mathbb{S}_1$. We then establish some convexity results for the Δ and ∇ fractional difference of the Riemann-Liouville type as well as for the Liouville-Caputo type by using our derived relationships. For a systematic investigation of fractional calculus and its widespread applications, we refer the reader to the monograph by Kilbas et al. [25] (see also [26–29] for some recent developments based upon the Riemann-Liouville and Liouville-Caputo fractional integrals and fractional derivatives as well as their associated dufference operators).

The remainder of the study is organized as follows. Section 2 presents a brief overview of the delta and nabla fractional differences and a description of the essentials of the discrete fractional calculus methods. It also gives alternative discrete fractional definitions equivalent to the standard definitions and provides the relationships between the fractional Riemann-Liouville and Liouville-Caputo difference operators. Section 3 establishes several convexity results for the fractional Riemann-Liouville and Liouville-Caputo differences. Section 4 contains the application examples. These are achieved by using two basic formulas for Δ^2 and ∇^2. Several concluding remarks and thoughts on open questions are offered in Section 5.

2. Basic tools and fractional difference relationships

In this section, we consider the discrete fractional sums and differences in both of Riemann-Liouville and Liouville-Caputo senses. The reader can refer to [30–33] for the relevant details about these definitions and many other properties associated with them.

Let us define the sets $\mathbb{S}_{t_0} := \{t_0, t_0 + 1, t_0 + 2, \ldots\}$ and $\mathscr{D}_{t_0}(g) := \{g : g : \mathbb{S}_{t_0} \to \mathcal{R} \text{ for } a \in \mathcal{R}\}$. For $g \in \mathscr{D}_{t_0+\beta}(g)$ with $\beta > 0$, the Δ fractional sum of order β can be expressed as follows:

$$\left(_{t_0}\Delta^{-\beta}g\right)(z) = \frac{1}{\Gamma(\beta)} \sum_{s=t_0}^{z-\beta} (z - s - 1)^{\underline{\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+\beta}. \tag{2.1}$$

Moreover, for $g \in \mathscr{D}_{t_0}(g)$, the ∇ fractional sum of order β can be expressed as follows:

$$\left(_{t_0}\nabla^{-\beta}g\right)(z) = \frac{1}{\Gamma(\beta)} \sum_{s=t_0+1}^{z} (z - s + 1)^{\overline{\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+1}. \tag{2.2}$$

Here, and in what follows, $z^{\underline{\beta}}$ and $z^{\overline{\beta}}$ are defined by

$$z^{\underline{\beta}} = \frac{\Gamma(z + 1)}{\Gamma(z + 1 - \beta)} \quad \text{and} \quad z^{\overline{\beta}} = \frac{\Gamma(z + \beta)}{\Gamma(z)}, \tag{2.3}$$

such that the right-hand sides of these identities are well defined. Besides, we use $z^{\underline{\beta}} = 0$ and $z^{\overline{\beta}} = 0$ when the numerators in each identities are well-defined, but the denominator is not defined. Further, we have

$$\Delta z^{\underline{\beta}} = \beta z^{\underline{\beta-1}} \quad \text{and} \quad \nabla z^{\overline{\beta}} = \beta z^{\overline{\beta-1}}. \tag{2.4}$$

Definition 2.1 (see [30–32]). Let $g \in \mathscr{D}_{t_0}(g)$. Then $(\Delta g)(z) := g(z + 1) - g(z)$, for $z \in \mathbb{S}_{t_0}$, is the Δ difference operator and $(\nabla g)(z) := g(z) - g(z - 1)$, for $z \in \mathbb{S}_{t_0+1}$, is the ∇ difference operator. In addition, the Δ fractional difference of order β ($\aleph - 1 < \beta < \aleph$) of the Riemann-Liouville type is defined by

$$\left(_{t_0}^{\mathrm{RL}}\Delta^{\beta}g\right)(z) = \frac{\Delta^{\aleph}}{\Gamma(\aleph - \beta)} \sum_{s=t_0}^{z+\beta-\aleph} (z - s - 1)^{\underline{\aleph-\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph-\beta}, \tag{2.5}$$

and the ∇ fractional difference of order β ($\aleph - 1 < \beta < \aleph$) of the Riemann-Liouville type is defined by

$$\left(_{t_0}^{\mathrm{RL}}\nabla^{\beta}g\right)(z) = \frac{\nabla^{\aleph}}{\Gamma(\aleph - \beta)} \sum_{s=t_0+1}^{z} (z - s + 1)^{\overline{\aleph-\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph}. \tag{2.6}$$

The following theorem is an alternative representation of the ∇ fractional difference (2.6).

Theorem 2.1 (see [21, Lemma 2.1]). *For $g \in \mathscr{D}_{t_0}(g)$ and $\aleph - 1 < \beta < \aleph$, the ∇ fractional difference of order β of the Riemann-Liouville type can be expressed as follows*:

$$\left(_{t_0}^{\mathrm{RL}}\nabla^{\beta}g\right)(z) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0+1}^{z} (z - s + 1)^{\overline{-\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph}. \tag{2.7}$$

Furthermore, the following theorem is an alternative representation of the Δ fractional difference (2.5), which is also the generalization of the result established for $0 < \beta < 1$ in [18].

Theorem 2.2. *For* $g \in \mathscr{D}_{t_0+\beta}(g)$ *with* $\aleph - 1 < \beta < \aleph$, *the Δ fractional difference of order β of the Riemann-Liouville type can be expressed as follows*:

$$\left({}^{\mathrm{RL}}_{t_0}\Delta^\beta g \right)(z) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z - s - 1)^{\underline{-\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph-\beta}. \tag{2.8}$$

Proof. The result was proved by Mohammed et al. in [18, Theorem 1] for $\aleph = 1$ (that is, for $0 < \beta < 1$), and their result is given below:

$$\left({}^{\mathrm{RL}}_{t_0}\Delta^\beta g \right)(z) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z - s - 1)^{\underline{-\beta-1}} g(s), \quad \text{for z in } \mathbb{S}_{t_0+1-\beta}. \tag{2.9}$$

For $\aleph = 2$ (that is, for $1 < \beta < 2$), by Definition (2.5) we find for each $z \in \mathbb{S}_{t_0+2-\beta}$ that

$$\left({}^{\mathrm{RL}}_{t_0}\Delta^\beta g \right)(z) = \Delta \left(\frac{\Delta}{\Gamma(-\beta+2)} \sum_{s=t_0}^{z+\beta-2} (z-s-1)^{\underline{1-\beta}} g(s) \right)$$

$$\overset{by}{\underset{(2.9)}{=}} \Delta \left(\frac{1}{\Gamma(-\beta+1)} \sum_{s=t_0}^{z+\beta-1} (z-s-1)^{\underline{-\beta}} g(s) \right)$$

$$= \frac{1}{\Gamma(-\beta+1)} \left\{ \sum_{s=t_0}^{z+\beta} (z-s)^{\underline{-\beta}} g(s) - \sum_{s=t_0}^{z+\beta-1} (z-s-1)^{\underline{-\beta}} g(s) \right\}$$

$$= \frac{1}{\Gamma(-\beta+1)} \left\{ \sum_{s=t_0}^{z+\beta} (z-s)^{\underline{-\beta}} g(s) - \sum_{s=t_0}^{z+\beta} (z-s-1)^{\underline{-\beta}} g(s) \right\}$$

$$= \frac{1}{\Gamma(-\beta+1)} \sum_{s=t_0}^{z+\beta} \Delta (z-s-1)^{\underline{-\beta}} g(s)$$

$$= \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z-s-1)^{\underline{-\beta-1}} g(s),$$

where we have first used (see [18, Lemma 1])

$$(-\beta - 1)^{\underline{-\beta}} = 0,$$

and then used

$$\Delta \left(z^{\underline{-\beta}} \right) = -\beta \, z^{\underline{-\beta-1}}.$$

The same procedure can be repeated $(\aleph-1)$ times to obtain the required result asserted by Theorem 2.2. $\square$

Definition 2.2 (see [30–32])**.** For g $\in \mathscr{D}_{t_0+\beta}(g)$, the Δ fractional difference of order β ($\aleph - 1 < \beta < \aleph$) of Liouville-Caputo type is defined by

$$\left({}_{t_0}^{LC}\Delta^\beta g\right)(z) = \frac{1}{\Gamma(\aleph - \beta)} \sum_{s=t_0}^{z+\beta-\aleph} (z - s - 1)^{\overline{\aleph-\beta-1}}(\Delta^\aleph g)(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph-\beta}, \tag{2.10}$$

and, for g $\in \mathscr{D}_{t_0}(g)$, the ∇ fractional difference of order β ($\aleph - 1 < \beta < \aleph$) of the Liouville-Caputo type is defined by

$$\left({}_{t_0}^{LC}\nabla^\beta g\right)(z) = \frac{1}{\Gamma(\aleph - \beta)} \sum_{s=t_0+1}^{z} (z - s + 1)^{\overline{\aleph-\beta-1}}(\nabla^\aleph g)(s), \quad \text{for z in } \mathbb{S}_{t_0+\aleph}. \tag{2.11}$$

The following proposition provides relationships between the Δ and ∇ fractional differences of the Riemann-Liouville and Liouville-Caputo types of the higher order β.

Proposition 2.1. *Let* $\beta \in (\aleph - 1, \aleph)$. *Then, for* g $\in \mathscr{D}_{t_0+\beta}(g)$,

$$\left({}_{t_0}^{LC}\Delta^\beta g\right)(z) = \left({}_{t_0}^{RL}\Delta^\beta g\right)(z) - \sum_{J=0}^{\aleph-1} \frac{(z - t_0)^{\overline{-\beta+J}}}{\Gamma(-\beta + J + 1)}(\Delta^J g)(t_0), \tag{2.12}$$

for z *in* $\mathbb{S}_{t_0+\aleph-\beta}$. *Moreover, for* g $\in \mathscr{D}_{t_0}(g)$, *it is asserted that*

$$\left({}_{t_0}^{LC}\nabla^\beta g\right)(z) = \left({}_{t_0}^{RL}\nabla^\beta g\right)(z) - \sum_{J=0}^{\aleph-1} \frac{(z - t_0)^{\overline{-\beta+J}}}{\Gamma(-\beta + J + 1)}(\nabla^J g)(t_0), \tag{2.13}$$

for z *in* $\mathbb{S}_{t_0+\aleph}$.

Proof. We only prove the first part and we omit the second part, because they have the same proof technique. Considering (2.10), for $\aleph = 1$ we have

$$\left({}^{LC}_{t_0}\Delta^\beta g\right)(z) = \frac{1}{\Gamma(-\beta + 1)} \sum_{s=t_0}^{z+\beta-1} (z - s - 1)^{\overline{-\beta}}(\Delta g)(s)$$

$$= \frac{1}{\Gamma(-\beta + 1)} \left\{ \sum_{s=t_0}^{z+\beta-1} (z - s - 1)^{\overline{-\beta}} g(s+1) - \sum_{s=t_0}^{z+\beta-1} (z - s - 1)^{\overline{-\beta}} g(s) \right\}$$

$$= \frac{1}{\Gamma(-\beta + 1)} \left\{ -(z - t_0)^{\overline{-\beta}} g(t_0) + \sum_{s=t_0}^{z+\beta} \Delta (z - s - 1)^{\overline{-\beta}} g(s) \right\}$$

$$= \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z - s - 1)^{\overline{-\beta-1}} g(s) - \frac{(z - t_0)^{\overline{-\beta}}}{\Gamma(-\beta + 1)} g(t_0), \tag{2.14}$$

where we have first used $(-\beta - 1)^{\overline{-\beta}} = 0$ and then used $\Delta \left(z^{\overline{-\beta}}\right) = -\beta\, z^{\overline{-\beta-1}}$ as above.

Now, for $\aleph = 2$, we have

$$\left({}^{\text{LC}}_{t_0}\Delta^{\beta}g\right)(z) = \frac{1}{\Gamma(-\beta + 2)} \sum_{s=t_0}^{z+\beta-2} (z - s - 1)^{\underline{1-\beta}}(\Delta^2 g)(s)$$

$$\overset{by}{\underset{(2.14)}{=}} \frac{1}{\Gamma(-\beta + 1)} \sum_{s=t_0}^{z+\beta-1} (z - s - 1)^{\underline{-\beta}}(\Delta g)(s) - \frac{(z - t_0)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)}(\Delta g)(t_0)$$

$$\overset{\text{again by}}{\underset{(2.14)}{=}} \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z - s - 1)^{\underline{-\beta-1}}g(s) - \frac{(z - t_0)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) - \frac{(z - t_0)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)}(\Delta g)(t_0).$$

We can repeat the same procedure $(\aleph - 1)$ times to get the required (2.12). Hence the proof is completed.
□

3. Convexity results

We start with two lemmas concerning the Δ^2 and ∇^2 fractional differences which will be useful in the sequel.

Lemma 3.1. *Let* $g \in \mathscr{D}_{t_0}(g), \beta \in (2, 3)$ *and* $\left({}^{\text{RL}}_{t_0}\Delta^{\beta}g\right)(z) \geqq 0$ *for* $z \in \mathbb{S}_{t_0+3-\beta}$. *Then, for* $z := t_0 + -\beta + 3 + \eta$ *with* $\eta \in \mathbb{S}_0$,

$$(\Delta^2 g)(t_0 + \eta + 1) \geqq -\frac{(-\beta + 3 + \eta)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) - \frac{(-\beta + 3 + \eta)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)}(\Delta g)(t_0)$$

$$- \frac{1}{\Gamma(-\beta + 2)} \sum_{\iota=0}^{\eta} (-\beta + \eta + 2 - \iota)^{\underline{1-\beta}}(\Delta^2 g)(t_0 + \iota), \tag{3.1}$$

where

$$\frac{(-\beta + \eta + 2 - \iota)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)} = \frac{(-\beta + 2)(-\beta + 3)\cdots(-\beta + \eta + 2 - \iota)}{(\eta - \iota + 1)!} < 0,$$

$$\frac{(-\beta + 3 + \eta)^{\underline{-\beta}}}{\Gamma(-\beta + 1)} > 0 \qquad and \qquad \frac{(-\beta + 3 + \eta)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)} < 0.$$

Proof. By considering Theorem 2.2 and (2.4), we have

$$\left({}^{\text{RL}}_{t_0}\Delta^{\beta}g\right)(z) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0}^{z+\beta} (z - s - 1)^{\underline{-\beta-1}}g(s)$$

$$= \frac{1}{\Gamma(-\beta + 1)} \sum_{s=t_0}^{z+\beta} \Delta (z - s - 1)^{\underline{-\beta}}g(s)$$

$$= \frac{(z - t_0)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) + \frac{1}{\Gamma(-\beta + 1)} \sum_{s=t_0}^{z+\beta} (z - s - 1)^{\underline{-\beta}}(\Delta g)(s),$$

where we have used $(-\beta - 1)^{\underline{-\beta}} = 0$. By the same technique as in above, we can deduce

$$\left({}^{\mathrm{RL}}_{t_0}\Delta^\beta g\right)(z) = \frac{(z - t_0)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) + \frac{1}{\Gamma(-\beta + 2)}\sum_{s=t_0}^{z+\beta}\Delta\,(z - s - 1)^{\underline{1-\beta}}(\Delta\,g)(s)$$

$$= \frac{(z - t_0)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) + \frac{(z - t_0)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)}(\Delta\,g)(t_0) + \frac{1}{\Gamma(-\beta + 2)}\sum_{s=t_0}^{z+\beta}(z - s - 1)^{\underline{1-\beta}}(\Delta^2\,g)(s), \quad (3.2)$$

where we have used $(-\beta - 1)^{\underline{1-\beta}} = 0$. Since $(z - s - 1)^{\underline{1-\beta}} = 0$ at $s = z + \beta, z + \beta - 1$, (3.2) becomes

$$\left({}^{\mathrm{RL}}_{t_0}\Delta^\beta g\right)(z) = \frac{(z - t_0)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) + \frac{(z - t_0)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)}(\Delta\,g)(t_0) + \frac{1}{\Gamma(-\beta + 2)}\sum_{s=t_0}^{z+\beta-2}(z - s - 1)^{\underline{1-\beta}}(\Delta^2 g)(s)$$

$$= (\Delta^2 g)(z + \beta - 2) + \frac{(z - t_0)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) + \frac{(z - t_0)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)}(\Delta\,g)(t_0)$$

$$+ \frac{1}{\Gamma(-\beta + 2)}\sum_{s=t_0}^{z+\beta-3}(z - s - 1)^{\underline{1-\beta}}(\Delta^2 g)(s).$$

Considering the assumption that $\left({}^{\mathrm{RL}}_{t_0}\Delta^\beta g\right)(z) \geqq 0$, it follows that

$$(\Delta^2 g)(z + \beta - 2) \geqq -\frac{(z - t_0)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) - \frac{(z - t_0)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)}(\Delta\,g)(t_0) - \frac{1}{\Gamma(-\beta + 2)}\sum_{s=t_0}^{z+\beta-3}(z - s - 1)^{\underline{1-\beta}}(\Delta^2 g)(s).$$

For $z := t_0 + -\beta + 3 + \eta$ for $\eta \in \mathbb{S}_0$, it becomes

$$(\Delta^2 g)(t_0 + \eta + 1) \geqq -\frac{(-\beta + 3 + \eta)^{\underline{-\beta}}}{\Gamma(-\beta + 1)}g(t_0) - \frac{(-\beta + 3 + \eta)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)}(\Delta\,g)(t_0)$$

$$- \frac{1}{\Gamma(-\beta + 2)}\sum_{\iota=0}^{\eta}(2 - \beta + \eta - \iota)^{\underline{1-\beta}}(\Delta^2 g)(t_0 + \iota),$$

which is the required (3.1). Now, it is clear for $2 < \beta < 3$ that

$$\frac{(-\beta + \eta + 2 - \iota)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)} = \frac{\Gamma(-\beta + 3 + \eta - \iota)}{\Gamma(-\beta + 2)\Gamma(2 + \eta - \iota)}$$

$$= \frac{(-\beta + 2)(-\beta + 3)\cdots(-\beta + \eta + 2 - \iota)}{(\eta - \iota + 1)!} < 0,$$

$$\frac{(-\beta + 3 + \eta)^{\underline{-\beta}}}{\Gamma(-\beta + 1)} = \frac{\Gamma(-\beta + 4 + \eta)}{\Gamma(4 + \eta)\Gamma(-\beta + 1)}$$

$$= \frac{(-\beta + 1)(-\beta + 2)(-\beta + 3)\cdots(\eta + 1 - \beta)(-\beta + \eta + 2)(-\beta + \eta + 3)}{(\eta + 3)!} > 0,$$

and

$$\frac{(-\beta + 3 + \eta)^{\underline{1-\beta}}}{\Gamma(-\beta + 2)} = \frac{\Gamma(-\beta + 4 + \eta)}{\Gamma(4 + \eta)\Gamma(-\beta + 2)} = \frac{(-\beta + 2)(-\beta + 3)\cdots(-\beta + \eta + 2)(-\beta + \eta + 3)}{(\eta + 3)!} < 0,$$

for $\iota = 0, 1, \ldots, \eta$ and $\eta \in \mathbb{S}_0$. Thus, our proof is complete. $\qquad\square$

Lemma 3.2. *Let* $g \in \mathscr{D}_{t_0+1}(g), \beta \in (2, 3)$ *and* $\left({}_{t_0}^{\mathrm{RL}}\nabla^\beta g \right)(z) \geqq 0$ *for* $z \in \mathbb{S}_{t_0+1}$. *Then, for* $z := t_0 + \eta$,

$$(\nabla^2 g)(t_0 + \eta) \geqq \frac{(\eta - 2)(\eta)^{\overline{-\beta}}}{\Gamma(-\beta + 1)} g(t_0 + 1) - \frac{1}{\Gamma(-\beta + 2)} \sum_{\iota=2}^{\eta-2} (\eta - \iota)^{\overline{1-\beta}} (\nabla^2 g)(t_0 + \iota + 1), \qquad (3.3)$$

where

$$\frac{(\eta - \iota)^{\overline{1-\beta}}}{\Gamma(-\beta + 2)} = \frac{(-\beta + 2)(-\beta + 3) \cdots (-\beta + \eta - \iota)(-\beta + \eta - \iota - 1)}{(\eta - \iota - 1)!} < 0$$

and

$$\frac{(\eta - 2)(\eta)^{\overline{-\beta}}}{\Gamma(-\beta + 1)} g(t_0 + 1) \geqq 0,$$

for $\eta \in \mathbb{S}_4$.

Proof. By making use of Theorem 2.1 and (2.4), we find for $z \in \mathbb{S}_{t_0+4}$ that

$$\left({}_{t_0}^{\mathrm{RL}}\nabla^\beta g \right)(z) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0+1}^{z} (z - s + 1)^{\overline{-\beta-1}} g(s)$$

$$= \frac{1}{\Gamma(-\beta + 1)} \sum_{s=t_0+1}^{z+\beta} \nabla (z - s + 1)^{\overline{-\beta}} g(s)$$

$$= \frac{(z - t_0)^{\overline{-\beta}}}{\Gamma(-\beta + 1)} g(t_0 + 1) + \frac{1}{\Gamma(-\beta + 1)} \sum_{s=t_0+2}^{z} (z - s + 1)^{\overline{-\beta}} (\nabla g)(s),$$

where we used $(0)^{\overline{-\beta}} = 0$. We can continue by the same technique to get

$$\left({}_{t_0}^{\mathrm{RL}}\nabla^\beta g \right)(z) = \frac{(z - t_0)^{\overline{-\beta}}}{\Gamma(-\beta + 1)} g(t_0) + \frac{1}{\Gamma(-\beta + 2)} \sum_{s=t_0+2}^{z} \nabla (z - s + 1)^{\overline{1-\beta}} (\nabla g)(s)$$

$$= \frac{(z - t_0)^{\overline{-\beta}}}{\Gamma(-\beta + 1)} g(t_0 + 1) + \frac{(z - t_0 - 1)^{\overline{1-\beta}}}{\Gamma(-\beta + 2)} (\nabla g)(t_0 + 2)$$

$$\quad + \frac{1}{\Gamma(-\beta + 2)} \sum_{s=t_0+3}^{z} (z - s + 1)^{\overline{1-\beta}} (\nabla^2 g)(s)$$

$$= (\nabla^2 g)(z) + \frac{(z - t_0)^{\overline{-\beta}}}{\Gamma(-\beta + 1)} g(t_0 + 1) + \frac{(z - t_0 - 1)^{\overline{1-\beta}}}{\Gamma(-\beta + 2)} (\nabla g)(t_0 + 2)$$

$$\quad + \frac{1}{\Gamma(-\beta + 2)} \sum_{s=t_0+3}^{z-1} (z - s + 1)^{\overline{1-\beta}} (\nabla^2 g)(s),$$

where this time we used $(0)^{\overline{1-\beta}} = 0$. By using the assumption that $\left({}_{t_0}^{\mathrm{RL}}\nabla^\beta g \right)(z) \geqq 0$, it follows that

$$(\nabla^2 g)(z) \geqq -\frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+1)} g(t_0+1)$$

$$-\frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)}(\nabla g)(t_0+2) - \frac{1}{\Gamma(-\beta+2)}\sum_{s=t_0+3}^{z-1}(z-s+1)^{\overline{1-\beta}}(\nabla^2 g)(s)$$

$$= (z-t_0-2+\beta)\frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+2)} g(t_0+1) - \frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} g(t_0+2)$$

$$-\frac{1}{\Gamma(-\beta+2)}\sum_{s=t_0+3}^{z-1}(z-s+1)^{\overline{1-\beta}}(\nabla^2 g)(s), \tag{3.4}$$

where we have used

$$\frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} - \frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+1)} = (z-t_0-2+\beta)\frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+2)}.$$

Now, by using the assumption $\left({}^{\mathrm{RL}}_{t_0}\nabla^\beta g\right)(z) \geqq 0$ at $z = t_0 + 1$ and $t_0 + 2$ in Theorem 2.1, we have

$$\left({}^{\mathrm{RL}}_{t_0}\nabla^\beta g\right)(t_0+1) = \frac{1}{\Gamma(-\beta)}\sum_{s=t_0+1}^{t_0+1}(t_0+2-s)^{\overline{-\beta-1}}g(s) = g(t_0+1) \geqq 0, \tag{3.5}$$

and

$$\left({}^{\mathrm{RL}}_{t_0}\nabla^\beta g\right)(t_0+2) = \frac{1}{\Gamma(-\beta)}\sum_{s=t_0+1}^{t_0+2}(t_0+3-s)^{\overline{-\beta-1}}g(s) = g(t_0+2) - \beta g(t_0+1) \geqq 0. \tag{3.6}$$

By using (3.6) in (3.4), we get

$$(\nabla^2 g)(z) \geqq (z-t_0-2+\beta)\frac{(z-t_0)^{\overline{-\beta}}}{\Gamma(-\beta+2)} g(t_0+1) - \beta\frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} g(t_0+1)$$

$$-\frac{1}{\Gamma(-\beta+2)}\sum_{s=t_0+3}^{z-1}(z-s+1)^{\overline{1-\beta}}(\nabla^2 g)(s)$$

$$= (z-t_0-2)\frac{\Gamma(z-t_0-\beta)}{\Gamma(-\beta+1)\Gamma(z-t_0)} g(t_0+1) - \frac{1}{\Gamma(-\beta+2)}\sum_{s=t_0+3}^{z-1}(z-s+1)^{\overline{1-\beta}}(\nabla^2 g)(s), \tag{3.7}$$

where we used

$$-\frac{(z-t_0-1)^{\overline{1-\beta}}}{\Gamma(-\beta+2)} = -\frac{(-\beta+2)(-\beta+3)\cdots(-\beta+z-t_0-2)(-\beta+z-t_0-1)}{(z-t_0-1)!} > 0, \quad \text{for } 2 < \beta < 3.$$

Set $z := t_0 + \eta$ for $\eta \in \mathbb{S}_4$ in (3.7), we can deduce

$$(\nabla^2 g)(t_0+\eta) \geqq (\eta-2)\frac{\Gamma(\eta-\beta)}{\Gamma(-\beta+1)\Gamma(\eta)} g(t_0+1) - \frac{1}{\Gamma(-\beta+2)}\sum_{\iota=2}^{\eta-2}(\eta-\iota)^{\overline{1-\beta}}(\nabla^2 g)(t_0+\iota+1),$$

which rearranges to the desired (3.3). Now, it is clear that

$$
\frac{(\eta - \iota)^{\overline{1-\beta}}}{\Gamma(-\beta + 2)} = \frac{\Gamma(\eta - \iota + 1 - \beta)}{\Gamma(-\beta + 2)\Gamma(\eta - \iota)}
$$
$$
= \frac{(-\beta + 2)(-\beta + 3)\cdots(-\beta + \eta - \iota)(-\beta + \eta - \iota - 1)}{(\eta - \iota - 1)!} < 0,
$$

and from (3.5), we see that

$$
\frac{(\eta - 2)(\eta)^{\overline{-\beta}}}{\Gamma(-\beta + 1)} g(t_0 + 1)
$$
$$
= (\eta - 2)\frac{(-\beta + 1)(-\beta + 2)(-\beta + 3)\cdots(-\beta + \eta - 3)(-\beta + \eta - 2)(-\beta + \eta - 1)}{(\eta - 1)!} g(t_0 + 1)
$$
$$
\geqq 0,
$$

for $2 < \beta < 3$, $\iota = 2, 3, \ldots, \eta - 2$ and $\eta \in \mathbb{S}_4$. Thus, the proof is done. $\qquad\square$

Based on the above lemmas, we can now present our Δ and ∇ convexity results.

Theorem 3.1. *If $\beta \in (2, 3)$ and $g \in \mathscr{D}_{t_0}(g)$ satisfies $\left(\begin{smallmatrix}\mathrm{RL}\\t_0\end{smallmatrix}\Delta^\beta g\right)(z) \geqq 0$, for all $z \in \mathbb{S}_{t_0+3-\beta}$, $g(t_0) \leqq 0$, $(\Delta g)(t_0) \geqq 0$ and $(\Delta^2 g)(t_0) \geqq 0$, then $(\Delta^2 g)(z) \geqq 0$ for $z \in \mathbb{S}_{t_0}$.*

Proof. By using strong induction we will done this proof. From the assumption, we know that $(\Delta^2 g)(t_0) \geqq 0$. We assume that $(\Delta^2 g)(t_0 + \iota) \geqq 0$ for $\iota = 0, 1, \ldots, \eta$. Then Lemma 3.1 gives that $(\Delta^2 g)(t_0 + \eta + 1) \geqq 0$. Thus, the proof is done. $\qquad\square$

Corollary 3.1. *If $\beta \in (2, 3)$ and $g \in \mathscr{D}_{t_0}(g)$ satisfies*

$$
\left(\begin{smallmatrix}\mathrm{LC}\\t_0\end{smallmatrix}\Delta^\beta g\right)(z) \geqq - \sum_{J=0}^{2} \frac{(z - t_0)^{\overline{-\beta+J}}}{\Gamma(-\beta + J + 1)}(\Delta^J g)(t_0), \quad \text{for all } z \in \mathbb{S}_{t_0+3-\beta},
$$

$g(t_0) \leqq 0$, $(\Delta g)(t_0) \geqq 0$ and $(\Delta^2 g)(t_0) \geqq 0$, then, $(\Delta^2 g)(z) \geqq 0$, for $z \in \mathbb{S}_{t_0}$.

Proof. The result follows immediately from (2.12) with $\aleph = 3$ and Theorem 3.1. $\qquad\square$

Theorem 3.2. *If $\beta \in (2, 3)$ and $g \in \mathscr{D}_{t_0+1}(g)$ satisfies $\left(\begin{smallmatrix}\mathrm{RL}\\t_0\end{smallmatrix}\nabla^\beta g\right)(z) \geqq 0$, for all $z \in \mathbb{S}_{t_0+1}$, then $(\nabla^2 g)(z) \geqq 0$ for $z \in \mathbb{S}_{t_0+3}$.*

Proof. The proof can be accomplished by using the principle of mathematical induction. Indeed, by using Theorem 2.1 with $z = t_0 + 3$, we see that

$$
\left(\begin{smallmatrix}\mathrm{RL}\\t_0\end{smallmatrix}\nabla^\beta g\right)(t_0 + 3) = \frac{1}{\Gamma(-\beta)} \sum_{s=t_0+1}^{t_0+3} (t_0 + 4 - s)^{\overline{-\beta-1}} g(s)
$$
$$
= \frac{\beta(\beta - 1)}{2} g(t_0 + 1) - \beta\, g(t_0 + 2) + g(t_0 + 3)
$$
$$
= (\nabla^2 g)(t_0 + 3) - (\beta - 2)g(t_0 + 2) - g(t_0 + 1) + \frac{\beta(\beta - 1)}{2}\, g(t_0 + 1) \geqq 0.
$$

Thus, upon solving for $(\nabla^2 g)(t_0 + 3)$, we have

$$
\begin{aligned}
(\nabla^2 g)(t_0 + 3) &\geqq (\beta - 2)g(t_0 + 2) - \frac{(\beta - 2)(\beta + 1)}{2} g(t_0 + 1) \\
&\overset{by}{\underset{(3.6)}{\geqq}} \beta(\beta - 2)g(t_0 + 1) - \frac{(\beta - 2)(\beta + 1)}{2} g(t_0 + 1) \\
&= \frac{(\beta - 2)(\beta - 1)}{2} g(t_0 + 1) \geqq 0.
\end{aligned}
$$

Now, we assume that $(\nabla^2 g)(t_0 + \iota) \geqq 0$ for $\iota = 3, 4, \ldots, \eta - 1$. Then Lemma 3.2 leads to $(\Delta^2 g)(t_0 + \eta) \geqq 0$. Hence, the proof is done. $\qquad\square$

Corollary 3.2. *If $\beta \in (2, 3)$ and $g \in \mathscr{D}_{t_0}(g)$ satisfies*

$$
\left({}^{LC}_{t_0}\nabla^\beta g \right)(z) \geqq -\sum_{J=0}^{2} \frac{(z - t_0)^{\overline{-\beta + J}}}{\Gamma(-\beta + J + 1)} (\nabla^J g)(t_0), \quad \text{for all } z \in \mathbb{S}_{t_0 + 1},
$$

then, $(\nabla^2 g)(z) \geqq 0$, for $z \in \mathbb{S}_{t_0 + 3}$.

Proof. The result follows immediately from (2.13) with $\aleph = 3$ and Theorem 3.2. $\qquad\square$

4. Applications

We try to apply our main results on the function $g(z) = z^2$ with $\beta = \frac{5}{2}$ and $t_0 = 0$. Firstly, from the definition of delta fractional difference (2.8), we have

$$
\begin{aligned}
\left({}^{RL}_{0}\Delta^\beta g \right)(3 - \beta) &= \frac{1}{\Gamma(-\beta)} \sum_{s=0}^{3} (2 - \beta - s)^{\underline{-\beta-1}} g(s) \\
&= \frac{1}{\Gamma(-\beta)} \left[(2 - \beta)^{\underline{-\beta-1}} g(0) + (1 - \beta)^{\underline{-\beta-1}} g(1) + (-\beta)^{\underline{-\beta-1}} g(2) + (-\beta - 1)^{\underline{-\beta-1}} g(3) \right] \\
&= \frac{7}{8} \geq 0,
\end{aligned}
$$

similarly, we can have

$$
\begin{aligned}
\left({}^{RL}_{0}\Delta^\beta g \right)(4 - \beta) &= \frac{1}{\Gamma(-\beta)} \sum_{s=0}^{4} (3 - \beta - s)^{\underline{-\beta-1}} g(s) \\
&= \frac{11}{16} \geq 0.
\end{aligned}
$$

We can continue by the same technique as above to get

$$
\left({}^{RL}_{0}\Delta^\beta g \right)(\eta - \beta) \geq 0,
$$

for each $\eta \in \mathbb{S}_3$. In addition, we have that

$$
g(0) = 0 \leq 0, \quad (\Delta g)(0) = 1 \geq 0, \quad (\Delta^2 g)(0) = 2 \geq 0.
$$

Hence, $g(z) = z^2$ is convex on $\mathbb{S}_0$ according to Theorem 3.1.

On the other hand, from the definition of nabla fractional difference (2.7), we have

$$
\begin{aligned}
\left({}^{RL}_0\nabla^\beta g\right)(1) &= \frac{1}{\Gamma(-\beta)} \sum_{s=1}^{1} (2-s)^{\overline{-\beta-1}} g(s) \\
&= \frac{1}{\Gamma(-\beta)} (1)^{\overline{-\beta-1}} g(1) \\
&= g(1) = 1 \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\left({}^{RL}_0\nabla^\beta g\right)(2) &= \frac{1}{\Gamma(-\beta)} \sum_{s=1}^{2} (3-s)^{\overline{-\beta-1}} g(s) \\
&= \frac{1}{\Gamma(-\beta)} \left[(2)^{\overline{-\beta-1}} g(1) + (1)^{\overline{-\beta-1}} g(2) \right] \\
&= \frac{3}{2} \geq 0.
\end{aligned}
$$

Proceeding by the same technique as above to get

$$
\left({}^{RL}_0\nabla^\beta g\right)(\eta) \geq 0,
$$

for all $\eta \in \mathbb{S}_1$. Therefore, $g(z) = z^2$ is convex on $\mathbb{S}_3$ by Theorem 3.2.

5. Conclusions and future directions

The outcomes of this article are briefly as follows:

(1) Alternative definitions have been presented to the basic definitions of the discrete delta and nabla fractional differences in Theorems 2.2 and 2.1, respectively.
(2) A relationship between the delta and nabla fractional differences of the Riemann-Liouville and Liouville-Caputo types of higher orders has been established in Proposition 2.1.
(3) The formulas for $(\Delta^2 g)$ and $(\nabla^2 g)$ have been represented in Lemmas 3.1 and 3.2, respectively.
(4) Based on Lemmas 3.1 and 3.2, some convexity results have been analyzed for the delta and nabla fractional differences of the Riemann-Liouville type in Theorems 3.1 and 3.2, respectively.
(5) Similar results have been obtained for the delta and nabla fractional differences of the Liouville-Caputo type by using the proposed relationships in Corollaries 3.1 and 3.2, respectively.

For the future direction of this work, we notice that the use of discrete fractional operators of the Caputo-Fabrizio and Atangana-Baleanu types to establish similar relationships and convexity results has the potential to lead to future investigations by the interested researchers (see [19, 32] for information about these discrete fractional operators).

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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