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Research article

Solving a Fredholm integral equation via coupled fixed point on bicomplex partial metric space

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Abstract: In this paper, we obtain some coupled fixed point theorems on a bicomplex partial metric space. An example and an application to support our result are presented.

Keywords: coupled fixed point; bicomplex partial metric space; Fredholm integral equations
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1. Introduction

Segre [1] made a pioneering attempt in the development of special algebra. He conceptualized the commutative generalization of complex numbers, bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Subsequently, in the 1930s, researchers contributed in this area [2–4]. The next fifty years failed to witness any advancement in this field. Later, Price [5] developed the bicomplex algebra and function theory. Recent works in this subject [6, 7] find some significant applications in different fields of mathematical sciences as well as other branches of science and technology. An impressive body of work has been developed by a number of researchers. Among these works, an important work on elementary functions of bicomplex numbers has been done by Luna-Elizaarrarás et al. [8]. Choi et al. [9] proved some common fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces. Jebril [10] proved some common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces. In 2017, Dhivya and Marudai [11] introduced the concept of a complex partial metric space, suggested a plan to expand the results and proved some common fixed point theorems under a rational expression contraction condition. In 2019, Mani and Mishra [12] proved coupled fixed point theorems on a complex partial metric space using different types of contractive conditions. In 2021, Gunaseelan et al. [13] proved common fixed point theorems on a complex partial metric space. In 2021, Beg et al. [14] proved fixed point theorems on a bicomplex valued metric space. In 2021, Zhaohui et al. [15] proved common fixed theorems on a bicomplex partial metric space. In this paper, we prove coupled fixed point theorems on a bicomplex partial metric space. An example is provided to verify the effectiveness and applicability of our main results. An application of these results to Fredholm integral equations and nonlinear integral equations is given.

2. Preliminaries

Throughout this paper, we denote the set of real, complex and bicomplex numbers, respectively, as $\mathscr{C}_0, \mathscr{C}_1$ and $\mathscr{C}_2$. Segre [1] defined the complex number as follows:

$$\mathfrak{z} = \vartheta_1 + \vartheta_2 i_1,$$

where $\vartheta_1, \vartheta_2 \in \mathscr{C}_0$, $i_1^2 = -1$. We denote the set of complex numbers $\mathscr{C}_1$ as:

$$\mathscr{C}_1 = \{\mathfrak{z} : \mathfrak{z} = \vartheta_1 + \vartheta_2 i_1, \vartheta_1, \vartheta_2 \in \mathscr{C}_0\}.$$

Let $\mathfrak{z} \in \mathscr{C}_1$; then, $|\mathfrak{z}| = (\vartheta_1^2 + \vartheta_2^2)^{\frac{1}{2}}$. The norm $||.||$ of an element in $\mathscr{C}_1$ is the positive real valued function $||.|| : \mathscr{C}_1 \to \mathscr{C}_0^+$ defined by

$$||\mathfrak{z}|| = (\vartheta_1^2 + \vartheta_2^2)^{\frac{1}{2}}.$$

Segre [1] defined the bicomplex number as follows:

$$\varsigma = \vartheta_1 + \vartheta_2 i_1 + \vartheta_3 i_2 + \vartheta_4 i_1 i_2,$$

where $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathscr{C}_0$, and independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We denote the set of bicomplex numbers $\mathscr{C}_2$ as:

$$\mathscr{C}_2 = \{\varsigma : \varsigma = \vartheta_1 + \vartheta_2 i_1 + \vartheta_3 i_2 + \vartheta_4 i_1 i_2, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathscr{C}_0\},$$

i.e.,

$$\mathscr{C}_2 = \{\varsigma : \varsigma = \mathfrak{z}_1 + i_2 \mathfrak{z}_2, \mathfrak{z}_1, \mathfrak{z}_2 \in \mathscr{C}_1\},$$

where $\mathfrak{z}_1 = \vartheta_1 + \vartheta_2 i_1 \in \mathscr{C}_1$ and $\mathfrak{z}_2 = \vartheta_3 + \vartheta_4 i_1 \in \mathscr{C}_1$. If $\varsigma = \mathfrak{z}_1 + i_2 \mathfrak{z}_2$ and $\eta = \omega_1 + i_2 \omega_2$ are any two bicomplex numbers, then the sum is $\varsigma \pm \eta = (\mathfrak{z}_1 + i_2 \mathfrak{z}_2) \pm (\omega_1 + i_2 \omega_2) = \mathfrak{z}_1 \pm \omega_1 + i_2(\mathfrak{z}_2 \pm \omega_2)$, and the product is $\varsigma.\eta = (\mathfrak{z}_1 + i_2 \mathfrak{z}_2)(\omega_1 + i_2 \omega_2) = (\mathfrak{z}_1 \omega_1 - \mathfrak{z}_2 \omega_2) + i_2(\mathfrak{z}_1 \omega_2 + \mathfrak{z}_2 \omega_1)$.

There are four idempotent elements in $\mathscr{C}_2$: They are $0, 1, \mathfrak{e}_1 = \frac{1+i_1 i_2}{2}, \mathfrak{e}_2 = \frac{1-i_1 i_2}{2}$ of which $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are nontrivial, such that $\mathfrak{e}_1 + \mathfrak{e}_2 = 1$ and $\mathfrak{e}_1\mathfrak{e}_2 = 0$. Every bicomplex number $\mathfrak{z}_1 + i_2\mathfrak{z}_2$ can be uniquely expressed as the combination of $\mathfrak{e}_1$ and $\mathfrak{e}_2$, namely

$$\varsigma = \mathfrak{z}_1 + i_2\mathfrak{z}_2 = (\mathfrak{z}_1 - i_1\mathfrak{z}_2)\mathfrak{e}_1 + (\mathfrak{z}_1 + i_1\mathfrak{z}_2)\mathfrak{e}_2.$$

This representation of ς is known as the idempotent representation of a bicomplex number, and the complex coefficients $\varsigma_1 = (\mathfrak{z}_1 - i_1\mathfrak{z}_2)$ and $\varsigma_2 = (\mathfrak{z}_1 + i_1\mathfrak{z}_2)$ are known as the idempotent components of the bicomplex number ς.

An element $\varsigma = \mathfrak{z}_1 + i_2\mathfrak{z}_2 \in \mathscr{C}_2$ is said to be invertible if there exists another element η in $\mathscr{C}_2$ such that $\varsigma\eta = 1$, and η is said to be inverse (multiplicative) of ς. Consequently, ς is said to be the inverse(multiplicative) of η. An element which has an inverse in $\mathscr{C}_2$ is said to be a non-singular element of $\mathscr{C}_2$, and an element which does not have an inverse in $\mathscr{C}_2$ is said to be a singular element of $\mathscr{C}_2$.

An element $\varsigma = \mathfrak{z}_1 + i_2\mathfrak{z}_2 \in \mathscr{C}_2$ is non-singular if and only if $\|\mathfrak{z}_1^2 + \mathfrak{z}_2^2\| \neq 0$ and singular if and only if $\|\mathfrak{z}_1^2 + \mathfrak{z}_2^2\| = 0$. When it exists, the inverse of ς is as follows.

$$\varsigma^{-1} = \eta = \frac{\mathfrak{z}_1 - i_2\mathfrak{z}_2}{\mathfrak{z}_1^2 + \mathfrak{z}_2^2}.$$

Zero is the only element in $\mathscr{C}_0$ which does not have a multiplicative inverse, and in $\mathscr{C}_1$, $0 = 0 + i_1 0$ is the only element which does not have a multiplicative inverse. We denote the set of singular elements of $\mathscr{C}_0$ and $\mathscr{C}_1$ by $\mathfrak{O}_0$ and $\mathfrak{O}_1$, respectively. However, there is more than one element in $\mathscr{C}_2$ which does not have a multiplicative inverse: for example, e_1 and e_2. We denote this set by $\mathfrak{O}_2$, and clearly $\mathfrak{O}_0 = \{0\} = \mathfrak{O}_1 \subset \mathfrak{O}_2$.

A bicomplex number $\varsigma = \vartheta_1 + \vartheta_2 i_1 + \vartheta_3 i_2 + \vartheta_4 i_1 i_2 \in \mathscr{C}_2$ is said to be degenerated (or singular) if the matrix

$$\begin{pmatrix} \vartheta_1 & \vartheta_2 \\ \vartheta_3 & \vartheta_4 \end{pmatrix}$$

is degenerated (or singular). The norm $\|.\|$ of an element in $\mathscr{C}_2$ is the positive real valued function $\|.\| : \mathscr{C}_2 \to \mathscr{C}_0^+$ defined by

$$\begin{aligned} \|\varsigma\| = \|\mathfrak{z}_1 + i_2\mathfrak{z}_2\| &= \{\|\mathfrak{z}_1^2\| + \|\mathfrak{z}_2^2\|\}^{\frac{1}{2}} \\ &= \left[\frac{|\mathfrak{z}_1 - i_1\mathfrak{z}_2|^2 + |\mathfrak{z}_1 + i_1\mathfrak{z}_2|^2}{2}\right]^{\frac{1}{2}} \\ &= (\vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2 + \vartheta_4^2)^{\frac{1}{2}}, \end{aligned}$$

where $\varsigma = \vartheta_1 + \vartheta_2 i_1 + \vartheta_3 i_2 + \vartheta_4 i_1 i_2 = \mathfrak{z}_1 + i_2\mathfrak{z}_2 \in \mathscr{C}_2$.

The linear space $\mathscr{C}_2$ with respect to a defined norm is a normed linear space, and $\mathscr{C}_2$ is complete. Therefore, $\mathscr{C}_2$ is a Banach space. If $\varsigma, \eta \in \mathscr{C}_2$, then $\|\varsigma\eta\| \leq \sqrt{2}\|\varsigma\|\|\eta\|$ holds instead of $\|\varsigma\eta\| \leq \|\varsigma\|\|\eta\|$, and therefore $\mathscr{C}_2$ is not a Banach algebra. For any two bicomplex numbers $\varsigma, \eta \in \mathscr{C}_2$, we can verify the following:

1. $\varsigma \preceq_{i_2} \eta \iff \|\varsigma\| \leq \|\eta\|$,
2. $\|\varsigma + \eta\| \leq \|\varsigma\| + \|\eta\|$,

3. $\|\vartheta\varsigma\| = |\vartheta|\|\varsigma\|$, where ϑ is a real number,

4. $\|\varsigma\eta\| \leq \sqrt{2}\|\varsigma\|\|\eta\|$, and the equality holds only when at least one of ς and η is degenerated,

5. $\|\varsigma^{-1}\| = \|\varsigma\|^{-1}$ if ς is a degenerated bicomplex number with $0 \prec \varsigma$,

6. $\|\frac{\varsigma}{\eta}\| = \frac{\|\varsigma\|}{\|\eta\|}$, if η is a degenerated bicomplex number.

The partial order relation $\preceq_{i_2}$ on $\mathscr{C}_2$ is defined as follows. Let $\mathscr{C}_2$ be the set of bicomplex numbers and $\varsigma = \mathfrak{z}_1 + i_2\mathfrak{z}_2$, $\eta = \omega_1 + i_2\omega_2 \in \mathscr{C}_2$. Then, $\varsigma \preceq_{i_2} \eta$ if and only if $\mathfrak{z}_1 \preceq \omega_1$ and $\mathfrak{z}_2 \preceq \omega_2$, i.e., $\varsigma \preceq_{i_2} \eta$ if one of the following conditions is satisfied:

1. $\mathfrak{z}_1 = \omega_1$, $\mathfrak{z}_2 = \omega_2$,

2. $\mathfrak{z}_1 \prec \omega_1$, $\mathfrak{z}_2 = \omega_2$,

3. $\mathfrak{z}_1 = \omega_1$, $\mathfrak{z}_2 \prec \omega_2$,

4. $\mathfrak{z}_1 \prec \omega_1$, $\mathfrak{z}_2 \prec \omega_2$.

In particular, we can write $\varsigma \precnsim_{i_2} \eta$ if $\varsigma \preceq_{i_2} \eta$ and $\varsigma \neq \eta$, i.e., one of 2, 3 and 4 is satisfied, and we will write $\varsigma \prec_{i_2} \eta$ if only 4 is satisfied.
Now, let us recall some basic concepts and notations, which will be used in the sequel.

Definition 2.1. [15] A bicomplex partial metric on a non-void set $\mathcal{U}$ is a function $\rho_{bcpms} : \mathcal{U} \times \mathcal{U} \to \mathscr{C}_2^+$, where $\mathscr{C}_2^+ = \{\varsigma : \varsigma = \vartheta_1 + \vartheta_2 i_1 + \vartheta_3 i_2 + \vartheta_4 i_1 i_2, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in \mathscr{C}_0^+\}$ and $\mathscr{C}_0^+ = \{\vartheta_1 \in \mathscr{C}_0 | \vartheta_1 \geq 0\}$ such that for all $\varphi, \zeta, \mathfrak{z} \in \mathcal{U}$:

1. $0 \preceq_{i_2} \rho_{bcpms}(\varphi, \varphi) \preceq_{i_2} \rho_{bcpms}(\varphi, \zeta)$ (small self-distances),

2. $\rho_{bcpms}(\varphi, \zeta) = \rho_{bcpms}(\zeta, \varphi)$ (symmetry),

3. $\rho_{bcpms}(\varphi, \varphi) = \rho_{bcpms}(\varphi, \zeta) = \rho_{bcpms}(\zeta, \zeta)$ if and only if $\varphi = \zeta$ (equality),

4. $\rho_{bcpms}(\varphi, \zeta) \preceq_{i_2} \rho_{bcpms}(\varphi, \mathfrak{z}) + \rho_{bcpms}(\mathfrak{z}, \zeta) - \rho_{bcpms}(\mathfrak{z}, \mathfrak{z})$ (triangularity).

A bicomplex partial metric space is a pair $(\mathcal{U}, \rho_{bcpms})$ such that $\mathcal{U}$ is a non-void set and ρ_{bcpms} is a bicomplex partial metric on $\mathcal{U}$.

Example 2.2. Let $\mathcal{U} = [0, \infty)$ be endowed with bicomplex partial metric space $\rho_{bcpms} : \mathcal{U} \times \mathcal{U} \to \mathscr{C}_2^+$ with $\rho_{bcpms}(\varphi, \zeta) = \max\{\varphi, \zeta\}e^{i_2\theta}$, where $e^{i_2\theta} = \cos\theta + i_2 \sin\theta$, for all $\varphi, \zeta \in \mathcal{U}$ and $0 \leq \theta \leq \frac{\pi}{2}$. Obviously, $(\mathcal{U}, \rho_{bcpms})$ is a bicomplex partial metric space.

Definition 2.3. [15] A bicomplex partial metric space $\mathcal{U}$ is said to be a T_0 space if for any pair of distinct points of $\mathcal{U}$, there exists at least one open set which contains one of them but not the other.

Theorem 2.4. *[15] Let $(\mathcal{U}, \rho_{bcpms})$ be a bicomplex partial metric space; then, $(\mathcal{U}, \rho_{bcpms})$ is T_0.*

Definition 2.5. [15] Let $(\mathcal{U}, \rho_{bcpms})$ be a bicomplex partial metric space. A sequence $\{\varphi_\tau\}$ in $\mathcal{U}$ is said to be convergent and converges to $\varphi \in \mathcal{U}$ if for every $0 \prec_{i_2} \epsilon \in \mathscr{C}_2^+$ there exists $\mathcal{N} \in \mathbb{N}$ such that $\varphi_\tau \in \mathfrak{B}_{\rho_{bcpms}}(\varphi, \epsilon) = \{\omega \in \mathcal{U} : \rho_{bcpms}(\varphi, \omega) < \epsilon + \rho_{bcpms}(\varphi, \varphi)\}$ for all $\tau \geq \mathcal{N}$, and it is denoted by $\lim_{\tau\to\infty} \varphi_\tau = \varphi$.

Lemma 2.6. *[15] Let $(\mathcal{U}, \rho_{bcpms})$ be a bicomplex partial metric space. A sequence $\{\varphi_\tau\} \in \mathcal{U}$ is converges to $\varphi \in \mathcal{U}$ iff $\rho_{bcpms}(\varphi, \varphi) = \lim_{\tau \to \infty} \rho_{bcpms}(\varphi, \varphi_\tau)$.*

Definition 2.7. [15] Let $(\mathcal{U}, \rho_{bcpms})$ be a bicomplex partial metric space. A sequence $\{\varphi_\tau\}$ in $\mathcal{U}$ is said to be a Cauchy sequence in $(\mathcal{U}, \rho_{bcpms})$ if for any $\epsilon > 0$ there exist $\vartheta \in \mathscr{C}_2^+$ and $\mathcal{N} \in \mathbb{N}$ such that $\|\rho_{bcpms}(\varphi_\tau, \varphi_\upsilon) - \vartheta\| < \epsilon$ for all $\tau, \upsilon \geq \mathcal{N}$.

Definition 2.8. [15] Let $(\mathcal{U}, \rho_{bcpms})$ be a bicomplex partial metric space. Let $\{\varphi_\tau\}$ be any sequence in $\mathcal{U}$. Then,

1. If every Cauchy sequence in $\mathcal{U}$ is convergent in $\mathcal{U}$, then $(\mathcal{U}, \rho_{bcpms})$ is said to be a complete bicomplex partial metric space.
2. A mapping $\mathcal{S} : \mathcal{U} \to \mathcal{U}$ is said to be continuous at $\varphi_0 \in \mathcal{U}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mathcal{S}(\mathfrak{B}_{\rho_{bcpms}}(\varphi_0, \delta)) \subset \mathfrak{B}_{\rho_{bcpms}}(\mathcal{S}(\varphi_0, \epsilon))$.

Lemma 2.9. *[15] Let $(\mathcal{U}, \rho_{bcpms})$ be a bicomplex partial metric space and $\{\varphi_\tau\}$ be a sequence in $\mathcal{U}$. Then, $\{\varphi_\tau\}$ is a Cauchy sequence in $\mathcal{U}$ iff $\lim_{\tau, \upsilon \to \infty} \rho_{bcpms}(\varphi_\tau, \varphi_\upsilon) = \rho_{bcpms}(\varphi, \varphi)$.*

Definition 2.10. Let $(\mathcal{U}, \rho_{bcpms})$ be a bicomplex partial metric space. Then, an element $(\varphi, \zeta) \in \mathcal{U} \times \mathcal{U}$ is said to be a coupled fixed point of the mapping $\mathcal{S} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ if $\mathcal{S}(\varphi, \zeta) = \varphi$ and $\mathcal{S}(\zeta, \varphi) = \zeta$.

Theorem 2.11. *[15] Let $(\mathcal{U}, \rho_{bcpms})$ be a complete bicomplex partial metric space and $\mathcal{S}, \mathcal{T} : \mathcal{U} \to \mathcal{U}$ be two continuous mappings such that*

$$\begin{aligned}\rho_{bcpms}(\mathcal{S}\varphi, \mathcal{T}\zeta) \preceq_{i_2} \mathfrak{l} \max\{&\rho_{bcpms}(\varphi, \zeta), \rho_{bcpms}(\varphi, \mathcal{S}\varphi), \rho_{bcpms}(\zeta, \mathcal{T}\zeta), \\ &\frac{1}{2}(\rho_{bcpms}(\varphi, \mathcal{T}\zeta) + \rho_{bcpms}(\zeta, \mathcal{S}\varphi))\},\end{aligned}$$

for all $\varphi, \zeta \in \mathcal{U}$, where $0 \leq \mathfrak{l} < 1$. Then, the pair $(\mathcal{S}, \mathcal{T})$ has a unique common fixed point, and $\rho_{bcpms}(\varphi^*, \varphi^*) = 0$.

Inspired by Theorem 2.11, here we prove coupled fixed point theorems on a bicomplex partial metric space with an application.

3. Main results

Theorem 3.1. *Let $(\mathcal{U}, \rho_{bcpms})$ be a complete bicomplex partial metric space. Suppose that the mapping $\mathcal{S} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ satisfies the following contractive condition:*

$$\rho_{bcpms}(\mathcal{S}(\varphi, \zeta), \mathcal{S}(\nu, \mu)) \preceq_{i_2} \lambda \rho_{bcpms}(\mathcal{S}(\varphi, \zeta), \varphi) + \mathfrak{l} \rho_{bcpms}(\mathcal{S}(\nu, \mu), \nu),$$

for all $\varphi, \zeta, \nu, \mu \in \mathcal{U}$, where $\lambda, \mathfrak{l}$ are nonnegative constants with $\lambda + \mathfrak{l} < 1$. Then, $\mathcal{S}$ has a unique coupled fixed point.

Proof. Choose $\nu_0, \mu_0 \in \mathcal{U}$ and set $\nu_1 = \mathcal{S}(\nu_0, \mu_0)$ and $\mu_1 = \mathcal{S}(\mu_0, \nu_0)$. Continuing this process, set $\nu_{\tau+1} = \mathcal{S}(\nu_\tau, \mu_\tau)$ and $\mu_{\tau+1} = \mathcal{S}(\mu_\tau, \nu_\tau)$. Then,

$$\rho_{bcpms}(\nu_\tau, \nu_{\tau+1}) = \rho_{bcpms}(\mathcal{S}(\nu_{\tau-1}, \mu_{\tau-1}), \mathcal{S}(\nu_\tau, \mu_\tau))$$

$$
\begin{aligned}
&\preceq_{i_2} \lambda\rho_{bcpms}(\mathcal{S}(\nu_{\tau-1},\mu_{\tau-1}),\nu_{\tau-1}) + ɩ\rho_{bcpms}(\mathcal{S}(\nu_{\tau},\mu_{\tau}),\nu_{\tau}) \\
&= \lambda\rho_{bcpms}(\nu_{\tau},\nu_{\tau-1}) + ɩ\rho_{bcpms}(\nu_{\tau+1},\nu_{\tau}) \\
\rho_{bcpms}(\nu_{\tau},\nu_{\tau+1}) &\preceq_{i_2} \frac{\lambda}{1-ɩ}\rho_{bcpms}(\nu_{\tau},\nu_{\tau-1}),
\end{aligned}
$$

which implies that

$$\|\rho_{bcpms}(\nu_{\tau},\nu_{\tau+1})\| \leq ȝ\|\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau})\| \tag{3.1}$$

where $ȝ = \frac{\lambda}{1-ɩ} < 1$. Similarly, one can prove that

$$\|\rho_{bcpms}(\mu_{\tau},\mu_{\tau+1})\| \leq ȝ\|\rho_{bcpms}(\mu_{\tau-1},\mu_{\tau})\|. \tag{3.2}$$

From (3.1) and (3.2), we get

$$
\begin{aligned}
\|\rho_{bcpms}(\nu_{\tau},\nu_{\tau+1})\| + \|\rho_{bcpms}(\mu_{\tau},\mu_{\tau+1})\| \leq ȝ(&\|\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau})\| \\
&+ \|\rho_{bcpms}(\mu_{\tau-1},\mu_{\tau})\|),
\end{aligned}
$$

where $ȝ < 1$.
Also,

$$\|\rho_{bcpms}(\nu_{\tau+1},\nu_{\tau+2})\| \leq ȝ\|\rho_{bcpms}(\nu_{\tau},\nu_{\tau+1})\| \tag{3.3}$$
$$\|\rho_{bcpms}(\mu_{\tau+1},\mu_{\tau+2})\| \leq ȝ\|\rho_{bcpms}(\mu_{\tau},\mu_{\tau+1})\|. \tag{3.4}$$

From (3.3) and (3.4), we get

$$
\begin{aligned}
\|\rho_{bcpms}(\nu_{\tau+1},\nu_{\tau+2})\| + \|\rho_{bcpms}(\mu_{\tau+1},\mu_{\tau+2})\| \leq ȝ(&\|\rho_{bcpms}(\nu_{\tau},\nu_{\tau+1})\| \\
&+ \|\rho_{bcpms}(\mu_{\tau},\mu_{\tau+1})\|).
\end{aligned}
$$

Repeating this way, we get

$$
\begin{aligned}
\|\rho_{bcpms}(\nu_{\tau},\nu_{\tau+1})\| + \|\rho_{bcpms}(\mu_{\tau},\mu_{\tau+1})\| &\leq ȝ(\|\rho_{bcpms}(\mu_{\tau-1},\mu_{\tau})\| + \|\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau})\|) \\
&\leq ȝ^2(\|\rho_{bcpms}(\mu_{\tau-2},\mu_{\tau-1})\| \\
&+ \|\rho_{bcpms}(\nu_{\tau-2},\nu_{\tau-1})\|) \\
&\leq \cdots \leq ȝ^{\tau}(\|\rho_{bcpms}(\mu_0,\mu_1)\| \\
&+ \|\rho_{bcpms}(\nu_0,\nu_1)\|).
\end{aligned}
$$

Now, if $\|\rho_{bcpms}(\nu_{\tau},\nu_{\tau+1})\| + \|\rho_{bcpms}(\mu_{\tau},\mu_{\tau+1})\| = \gamma_{\tau}$, then

$$\gamma_{\tau} \leq ȝ\gamma_{\tau-1} \leq \cdots \leq ȝ^{\tau}\gamma_0. \tag{3.5}$$

If $\gamma_0 = 0$, then $\|\rho_{bcpms}(\nu_0,\nu_1)\| + \|\rho_{bcpms}(\mu_0,\mu_1)\| = 0$. Hence, $\nu_0 = \nu_1 = \mathcal{S}(\nu_0,\mu_0)$ and $\mu_0 = \mu_1 = \mathcal{S}(\mu_0,\mu_0)$, which implies that (ν_0,μ_0) is a coupled fixed point of $\mathcal{S}$. Let $\gamma_0 > 0$. For each $\tau \geq \upsilon$, we have

$$\rho_{bcpms}(\nu_{\tau},\nu_{\upsilon}) \preceq_{i_2} \rho_{bcpms}(\nu_{\tau},\nu_{\tau-1}) + \rho_{bcpms}(\nu_{\tau-1},\nu_{\tau-2}) - \rho_{bcpms}(\nu_{\tau-1},\nu_{\tau-1})$$

$$+\rho_{bcpms}(\nu_{\tau-2},\nu_{\tau-3})+\rho_{bcpms}(\nu_{\tau-3},\nu_{\tau-4})-\rho_{bcpms}(\nu_{\tau-3},\nu_{\tau-3})$$
$$+\cdots+\rho_{bcpms}(\nu_{\upsilon+2},\nu_{\upsilon+1})+\rho_{bcpms}(\nu_{\upsilon+1},\nu_{\upsilon})-\rho_{bcpms}(\nu_{\upsilon+1},\nu_{\upsilon+1})$$
$$\preceq_{i_2}\rho_{bcpms}(\nu_{\tau},\nu_{\tau-1})+\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau-2})+\cdots+\rho_{bcpms}(\nu_{\upsilon+1},\nu_{\upsilon}),$$

which implies that

$$\begin{aligned}\|\rho_{bcpms}(\nu_{\tau},\nu_{\upsilon})\| \leq\ & \|\rho_{bcpms}(\nu_{\tau},\nu_{\tau-1})\|+\|\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau-2})\|\\ &+\cdots+\|\rho_{bcpms}(\nu_{\upsilon+1},\nu_{\upsilon})\|.\end{aligned}$$

Similarly, one can prove that

$$\begin{aligned}\|\rho_{bcpms}(\mu_{\tau},\mu_{\upsilon})\| \leq\ & \|\rho_{bcpms}(\mu_{\tau},\mu_{\tau-1})\|+\|\rho_{bcpms}(\mu_{\tau-1},\mu_{\tau-2})\|\\ &+\cdots+\|\rho_{bcpms}(\mu_{\upsilon+1},\mu_{\upsilon})\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|\rho_{bcpms}(\nu_{\tau},\nu_{\upsilon})\|+\|\rho_{bcpms}(\mu_{\tau},\mu_{\upsilon})\| &\leq \gamma_{\tau-1}+\gamma_{\tau-2}+\gamma_{\tau-3}+\cdots+\gamma_{\upsilon}\\ &\leq (\mathfrak{z}^{\tau-1}+\mathfrak{z}^{\tau-2}+\cdots+\mathfrak{z}^{\upsilon})\gamma_0\\ &\leq \frac{\mathfrak{z}^{\upsilon}}{1-\mathfrak{z}}\gamma_0 \to 0 \text{ as } \upsilon\to\infty,\end{aligned}$$

which implies that $\{\nu_\tau\}$ and $\{\mu_\tau\}$ are Cauchy sequences in $(\mathcal{U},\rho_{bcpms})$. Since the bicomplex partial metric space $(\mathcal{U},\rho_{bcpms})$ is complete, there exist $\nu,\mu\in\mathcal{U}$ such that $\{\nu_\tau\}\to\nu$ and $\{\mu_\tau\}\to\mu$ as $\tau\to\infty$, and

$$\rho_{bcpms}(\nu,\nu)=\lim_{\tau\to\infty}\rho_{bcpms}(\nu,\nu_\tau)=\lim_{\tau,\upsilon\to\infty}\rho_{bcpms}(\nu_\tau,\nu_\upsilon)=0,$$
$$\rho_{bcpms}(\mu,\mu)=\lim_{\tau\to\infty}\rho_{bcpms}(\mu,\mu_\tau)=\lim_{\tau,\upsilon\to\infty}\rho_{bcpms}(\mu_\tau,\mu_\upsilon)=0.$$

We now show that $\nu=\mathcal{S}(\nu,\mu)$. We suppose on the contrary that $\nu\neq\mathcal{S}(\nu,\mu)$ and $\mu\neq\mathcal{S}(\mu,\nu)$, so that $0\prec_{i_2}\rho_{bcpms}(\nu,\mathcal{S}(\nu,\mu))=\mathfrak{l}_1$ and $0\prec_{i_2}\rho_{bcpms}(\mu,\mathcal{S}(\mu,\nu))=\mathfrak{l}_2$. Then,

$$\begin{aligned}\mathfrak{l}_1=\rho_{bcpms}(\nu,\mathcal{S}(\nu,\mu)) &\preceq_{i_2}\rho_{bcpms}(\nu,\nu_{\tau+1})+\rho_{bcpms}(\nu_{\tau+1},\mathcal{S}(\nu,\mu))\\ &=\rho_{bcpms}(\nu,\nu_{\tau+1})+\rho_{bcpms}(\mathcal{S}(\nu_\tau,\mu_\tau),\mathcal{S}(\nu,\mu))\\ &\preceq_{i_2}\rho_{bcpms}(\nu,\nu_{\tau+1})+\lambda\rho_{bcpms}(\nu_{\tau-1},\nu_\tau)+\mathfrak{l}\rho_{bcpms}(\mathcal{S}(\nu,\mu),\nu)\\ &\preceq_{i_2}\frac{1}{1-\mathfrak{l}}\rho_{bcpms}(\nu,\nu_{\tau+1})+\frac{\lambda}{1-\mathfrak{l}}\rho_{bcpms}(\nu_{\tau-1},\nu_\tau),\end{aligned}$$

which implies that

$$\|\mathfrak{l}_1\|\leq\frac{1}{1-\mathfrak{l}}\|\rho_{bcpms}(\nu,\nu_{\tau+1})\|+\frac{\lambda}{1-\mathfrak{l}}\|\rho_{bcpms}(\nu_{\tau-1},\nu_\tau)\|.$$

As $\tau\to\infty$, $\|\mathfrak{l}_1\|\leq 0$. This is a contradiction, and therefore $\|\rho_{bcpms}(\nu,\mathcal{S}(\nu,\mu))\|=0$ implies $\nu=\mathcal{S}(\nu,\mu)$. Similarly, we can prove that $\mu=\mathcal{S}(\mu,\nu)$. Thus (ν,μ) is a coupled fixed point of $\mathcal{S}$. Now, if $(\mathfrak{g},\mathfrak{h})$ is another coupled fixed point of $\mathcal{S}$, then

$$\begin{aligned}\rho_{bcpms}(\nu,\mathfrak{g})=\rho_{bcpms}(\mathcal{S}(\nu,\mu),\mathcal{S}(\mathfrak{g},\mathfrak{h})) &\preceq_{i_2}\lambda\rho_{bcpms}(\mathcal{S}(\nu,\mu),\nu)+\mathfrak{l}\rho_{bcpms}(\mathcal{S}(\mathfrak{g},\mathfrak{h}),\mathfrak{g})\\ &=\lambda\rho_{bcpms}(\nu,\nu)+\mathfrak{l}\rho_{bcpms}(\mathfrak{g},\mathfrak{g})=0.\end{aligned}$$

Thus, we have $\mathfrak{g}=\nu$. Similarly, we get $\mathfrak{h}=\mu$. Therefore $\mathcal{S}$ has a unique coupled fixed point. □

Corollary 3.2. *Let $(\mathcal{U}, \rho_{bcpms})$ be a complete bicomplex partial metric space. Suppose that the mapping $\mathcal{S} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ satisfies the following contractive condition:*

$$\rho_{bcpms}(\mathcal{S}(\varphi, \zeta), \mathcal{S}(\nu, \mu)) \preceq_{i_2} \lambda(\rho_{bcpms}(\mathcal{S}(\varphi, \zeta), \varphi) + \rho_{bcpms}(\mathcal{S}(\nu, \mu), \nu)), \tag{3.6}$$

for all $\varphi, \zeta, \nu, \mu \in \mathcal{U}$, where $0 \leq \lambda < \frac{1}{2}$. Then, $\mathcal{S}$ has a unique coupled fixed point.

Theorem 3.3. *Let $(\mathcal{U}, \rho_{bcpms})$ be a complete complex partial metric space. Suppose that the mapping $\mathcal{S} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ satisfies the following contractive condition:*

$$\rho_{bcpms}(\mathcal{S}(\varphi, \zeta), \mathcal{S}(\nu, \mu)) \preceq_{i_2} \lambda \rho_{bcpms}(\varphi, \nu) + \text{ɩ} \rho_{bcpms}(\zeta, \mu),$$

for all $\varphi, \zeta, \nu, \mu \in \mathcal{U}$, where $\lambda, \text{ɩ}$ are nonnegative constants with $\lambda + \text{ɩ} < 1$. Then, $\mathcal{S}$ has a unique coupled fixed point.

Proof. Choose $\nu_0, \mu_0 \in \mathcal{U}$ and set $\nu_1 = \mathcal{S}(\nu_0, \mu_0)$ and $\mu_1 = \mathcal{S}(\mu_0, \nu_0)$. Continuing this process, set $\nu_{\tau+1} = \mathcal{S}(\nu_\tau, \mu_\tau)$ and $\mu_{\tau+1} = \mathcal{S}(\mu_\tau, \nu_\tau)$. Then,

$$\begin{aligned}\rho_{bcpms}(\nu_\tau, \nu_{\tau+1}) &= \rho_{bcpms}(\mathcal{S}(\nu_{\tau-1}, \mu_{\tau-1}), \mathcal{S}(\nu_\tau, \mu_\tau)) \\ &\preceq_{i_2} \lambda \rho_{bcpms}(\nu_{\tau-1}, \nu_\tau) + \text{ɩ} \rho_{bcpms}(\mu_{\tau-1}, \mu_\tau),\end{aligned}$$

which implies that

$$\|\rho_{bcpms}(\nu_\tau, \nu_{\tau+1})\| \leq \lambda \|\rho_{bcpms}(\nu_{\tau-1}, \nu_\tau)\| + \text{ɩ} \|\rho_{bcpms}(\mu_{\tau-1}, \mu_\tau)\|. \tag{3.7}$$

Similarly, one can prove that

$$\|\rho_{bcpms}(\mu_\tau, \mu_{\tau+1})\| \leq \lambda \|\rho_{bcpms}(\mu_{\tau-1}, \mu_\tau)\| + \text{ɩ} \|\rho_{bcpms}(\nu_{\tau-1}, \nu_\tau)\|. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\begin{aligned}\|\rho_{bcpms}(\nu_\tau, \nu_{\tau+1})\| + \|\rho_{bcpms}(\mu_\tau, \mu_{\tau+1})\| &\leq (\lambda + \text{ɩ})(\|\rho_{bcpms}(\mu_{\tau-1}, \mu_\tau)\| \\ &\quad + \|\rho_{bcpms}(\nu_{\tau-1}, \nu_\tau)\|) \\ &= \alpha(\|\rho_{bcpms}(\mu_{\tau-1}, \mu_\tau)\| \\ &\quad + \|\rho_{bcpms}(\nu_{\tau-1}, \nu_\tau)\|),\end{aligned}$$

where $\alpha = \lambda + \text{ɩ} < 1$. Also,

$$\|\rho_{bcpms}(\nu_{\tau+1}, \nu_{\tau+2})\| \leq \lambda \|\rho_{bcpms}(\nu_\tau, \nu_{\tau+1})\| + \text{ɩ} \|\rho_{bcpms}(\mu_\tau, \mu_{\tau+1})\| \tag{3.9}$$

$$\|\rho_{bcpms}(\mu_{\tau+1}, \mu_{\tau+2})\| \leq \lambda \|\rho_{bcpms}(\mu_\tau, \mu_{\tau+1})\| + \text{ɩ} \|\rho_{bcpms}(\nu_\tau, \nu_{\tau+1})\|. \tag{3.10}$$

From (3.9) and (3.10), we get

$$\begin{aligned}\|\rho_{bcpms}(\nu_{\tau+1}, \nu_{\tau+2})\| + \|\rho_{bcpms}(\mu_{\tau+1}, \mu_{\tau+2})\| &\leq (\lambda + \text{ɩ})(\|\rho_{bcpms}(\mu_\tau, \mu_{\tau+1})\| \\ &\quad + \|\rho_{bcpms}(\nu_\tau, \nu_{\tau+1})\|) \\ &= \alpha(\|\rho_{bcpms}(\mu_\tau, \mu_{\tau+1})\| \\ &\quad + \|\rho_{bcpms}(\nu_\tau, \nu_{\tau+1})\|).\end{aligned}$$

Repeating this way, we get

$$\begin{aligned}\|\rho_{bcpms}(\nu_\tau,\nu_{n+1})\|+\|\rho_{bcpms}(\mu_\tau,\mu_{\tau+1})\| &\leq \alpha(\|\rho_{bcpms}(\mu_{\tau-1},\mu_\tau)\| \\ &\quad+\|\rho_{bcpms}(\nu_{\tau-1},\nu_\tau)\|) \\ &\leq \alpha^2(\|\rho_{bcpms}(\mu_{\tau-2},\mu_{\tau-1})\| \\ &\quad+\|\rho_{bcpms}(\nu_{\tau-2},\nu_{\tau-1})\|) \\ &\leq \cdots \leq \alpha^\tau(\|\rho_{bcpms}(\mu_0,\mu_1)\| \\ &\quad+\|\rho_{bcpms}(\nu_0,\nu_1)\|).\end{aligned}$$

Now, if $\|\rho_{bcpms}(\nu_\tau,\nu_{\tau+1})\|+\|\rho_{bcpms}(\mu_\tau,\mu_{\tau+1})\|=\gamma_\tau$, then

$$\gamma_\tau \leq \alpha\gamma_{\tau-1} \leq \cdots \leq \alpha^\tau\gamma_0. \tag{3.11}$$

If $\gamma_0 = 0$, then $\|\rho_{bcpms}(\nu_0,\nu_1)\|+\|\rho_{bcpms}(\mu_0,\mu_1)\| = 0$. Hence, $\nu_0=\nu_1=\mathcal{S}(\nu_0,\mu_0)$ and $\mu_0=\mu_1=\mathcal{S}(\mu_0,\nu_0)$, which implies that (ν_0,μ_0) is a coupled fixed point of $\mathcal{S}$. Let $\gamma_0>0$. For each $\tau\geq\upsilon$, we have

$$\begin{aligned}\rho_{bcpms}(\nu_\tau,\nu_\upsilon) &\preceq_{i_2} \rho_{bcpms}(\nu_\tau,\nu_{\tau-1})+\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau-2})-\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau-1}) \\ &\quad+\rho_{bcpms}(\nu_{\tau-2},\nu_{\tau-3})+\rho_{bcpms}(\nu_{\tau-3},\nu_{\tau-4})-\rho_{bcpms}(\nu_{\tau-3},\nu_{\tau-3}) \\ &\quad+\cdots+\rho_{bcpms}(\nu_{\upsilon+2},\nu_{\upsilon+1})+\rho_{bcpms}(\nu_{\upsilon+1},\nu_\upsilon)-\rho_{bcpms}(\nu_{\upsilon+1},\nu_{\upsilon+1}) \\ &\preceq_{i_2} \rho_{bcpms}(\nu_\tau,\nu_{\tau-1})+\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau-2})+\cdots+\rho_{bcpms}(\nu_{\upsilon+1},\nu_\upsilon),\end{aligned}$$

which implies that

$$\begin{aligned}\|\rho_{bcpms}(\nu_\tau,\nu_\upsilon)\| &\leq \|\rho_{bcpms}(\nu_\tau,\nu_{\tau-1})\|+\|\rho_{bcpms}(\nu_{\tau-1},\nu_{\tau-2})\| \\ &\quad+\cdots+\|\rho_{bcpms}(\nu_{\upsilon+1},\nu_\upsilon)\|.\end{aligned}$$

Similarly, one can prove that

$$\begin{aligned}\|\rho_{bcpms}(\mu_\tau,\mu_\upsilon)\| &\leq \|\rho_{bcpms}(\mu_\tau,\mu_{\tau-1})\|+\|\rho_{bcpms}(\mu_{\tau-1},\mu_{\tau-2})\| \\ &\quad+\cdots+\|\rho_{bcpms}(\mu_{\upsilon+1},\mu_\upsilon)\|.\end{aligned}$$

Thus,

$$\begin{aligned}\|\rho_{bcpms}(\nu_\tau,\nu_\upsilon)\|+\|\rho_{bcpms}(\mu_\tau,\mu_\upsilon)\| &\leq \gamma_{\tau-1}+\gamma_{\tau-2}+\gamma_{\tau-3}+\cdots+\gamma_\upsilon \\ &\leq (\alpha^{\tau-1}+\alpha^{\tau-2}+\cdots+\alpha^\upsilon)\gamma_0 \\ &\leq \frac{\alpha^\upsilon}{1-\alpha}\gamma_0 \text{ as } \tau\to\infty,\end{aligned}$$

which implies that $\{\nu_\tau\}$ and $\{\mu_\tau\}$ are Cauchy sequences in $(\mathcal{U},\rho_{bcpms})$. Since the bicomplex partial metric space $(\mathcal{U},\rho_{bcpms})$ is complete, there exist $\nu,\mu\in\mathcal{U}$ such that $\{\nu_\tau\}\to\nu$ and $\{\mu_\tau\}\to\mu$ as $\tau\to\infty$, and

$$\begin{aligned}\rho_{bcpms}(\nu,\nu) &= \lim_{\tau\to\infty}\rho_{bcpms}(\nu,\nu_\tau) = \lim_{\tau,\upsilon\to\infty}\rho_{bcpms}(\nu_\tau,\nu_\upsilon)=0, \\ \rho_{bcpms}(\mu,\mu) &= \lim_{\tau\to\infty}\rho_{bcpms}(\mu,\mu_\tau) = \lim_{\tau,\upsilon\to\infty}\rho_{bcpms}(\mu_\tau,\mu_\upsilon)=0.\end{aligned}$$

Therefore ,

$$\begin{aligned}\rho_{bcpms}(\mathcal{S}(\nu,\mu),\nu) &\le \rho_{bcpms}(\mathcal{S}(\nu,\mu),\nu_{\tau+1}) + \rho_{bcpms}(\nu_{\tau+1},\nu) - \rho_{bcpms}(\nu_{\tau+1},\nu_{\tau+1}),\\ &\le \rho_{bcpms}(\mathcal{S}(\nu,\mu)),\mathcal{S}(\nu_\tau,\mu_\tau) + \rho_{bcpms}(\nu_{\tau+1},\nu)\\ &\le \lambda\rho_{bcpms}(\nu_\tau,\nu) + \mathfrak{l}\rho_{bcpms}(\mu_\tau,\mu) + \rho_{bcpms}(\nu_{\tau+1},\nu).\end{aligned}$$

As $\tau \to \infty$, from (3.6) and (3.12) we obtain $\rho_{bcpms}(\mathcal{S}(\nu,\mu),\nu) = 0$. Therefore $\mathcal{S}(\nu,\mu) = \nu$. Similarly, we can prove $\mathcal{S}(\mu,\nu) = \mu$, which implies that (ν,μ) is a coupled fixed point of $\mathcal{S}$. Now, if $(\mathfrak{g}_1,\mathfrak{h}_1)$ is another coupled fixed point of $\mathcal{S}$, then

$$\begin{aligned}\rho_{bcpms}(\mathfrak{g}_1,\nu) &= \rho_{bcpms}(\mathcal{S}(\mathfrak{g}_1,\mathfrak{h}_1),\mathcal{S}(\nu,\mu)) \preceq_{i_2} \lambda\rho_{bcpms}(\mathfrak{g}_1,\nu) + \mathfrak{l}\rho_{bcpms}(\mathfrak{h}_1,\mu),\\ \rho_{bcpms}(\mathfrak{h}_1,\mu) &= \rho_{bcpms}(\mathcal{S}(\mathfrak{h}_1,\mathfrak{g}_1),\mathcal{S}(\mu,\nu)) \preceq_{i_2} \lambda\rho_{bcpms}(\mathfrak{h}_1,\mu) + \mathfrak{l}\rho_{bcpms}(\mathfrak{g}_1,\nu),\end{aligned}$$

which implies that

$$\|\rho_{bcpms}(\mathfrak{g}_1,\nu)\| \le \lambda\|\rho_{bcpms}(\mathfrak{g}_1,\nu)\| + \mathfrak{l}\|\rho_{bcpms}(\mathfrak{h}_1,\mu)\|, \tag{3.12}$$

$$\|\rho_{bcpms}(\mathfrak{h}_1,\mu)\| \le \lambda\|\rho_{bcpms}(\mathfrak{h}_1,\mu)\| + \mathfrak{l}\|\rho_{bcpms}(\mathfrak{g}_1,\nu)\|. \tag{3.13}$$

From (3.12) and (3.13), we get

$$\|\rho_{bcpms}(\mathfrak{g}_1,\nu)\| + \|\rho_{bcpms}(\mathfrak{h}_1,\mu)\| \le (\lambda + \mathfrak{l})[\|\rho_{bcpms}(\mathfrak{g}_1,\nu)\| + \|\rho_{bcpms}(\mathfrak{h}_1,\mu)\|].$$

Since $\lambda + \mathfrak{l} < 1$, this implies that $\|\rho_{bcpms}(\mathfrak{g}_1,\nu)\| + \|\rho_{bcpms}(\mathfrak{h}_1,\mu)\| = 0$. Therefore, $\nu = \mathfrak{g}_1$ and $\mu = \mathfrak{h}_1$. Thus, $\mathcal{S}$ has a unique coupled fixed point. □

Corollary 3.4. *Let $(\mathcal{U},\rho_{bcpms})$ be a complete bicomplex partial metric space. Suppose that the mapping $\mathcal{S} : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$ satisfies the following contractive condition:*

$$\rho_{bcpms}(\mathcal{S}(\varphi,\zeta),\mathcal{S}(\nu,\mu)) \preceq_{i_2} \lambda(\rho_{bcpms}(\varphi,\nu) + \rho_{bcpms}(\zeta,\mu)), \tag{3.14}$$

for all $\varphi,\zeta,\nu,\mu \in \mathcal{U}$, where $0 \le \lambda < \frac{1}{2}$. Then, $\mathcal{S}$ has a unique coupled fixed point.

Example 3.5. Let $\mathcal{U} = [0,\infty)$ and define the bicomplex partial metric $\rho_{bcpms} : \mathcal{U}\times\mathcal{U} \to \mathscr{C}_2^+$ defined by

$$\rho_{bcpms}(\varphi,\zeta) = \max\{\varphi,\zeta\}e^{i_2\theta},\ 0 \le \theta \le \frac{\pi}{2}.$$

We define a partial order $\preceq$ in $\mathscr{C}_2^+$ as $\varphi \preceq \zeta$ iff $\varphi \le \zeta$. Clearly, $(\mathcal{U},\rho_{bcpms})$ is a complete bicomplex partial metric space.

Consider the mapping $\mathcal{S} : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$ defined by

$$\mathcal{S}(\varphi,\zeta) = \frac{\varphi+\zeta}{4}\ \ \forall\varphi,\zeta \in \mathcal{U}.$$

Now,

$$\rho_{bcpms}(\mathcal{S}(\varphi,\zeta),\mathcal{S}(\nu,\mu)) = \rho_{bcpms}\Big(\frac{\varphi+\zeta}{4},\frac{\nu+\mu}{4}\Big)$$

$$
\begin{aligned}
&= \frac{1}{4}\max\{\varphi+\zeta, \nu+\mu\}e^{i_2\theta}\\
&\preceq_{i_2} \frac{1}{4}\Big[\max\{\varphi,\nu\}+\max\{\zeta,\mu\}\Big]e^{i_2\theta}\\
&= \frac{1}{4}\Big[\rho_{bcpms}(\varphi,\nu)+\rho_{bcpms}(\zeta,\mu)\Big]\\
&= \lambda\Big(\rho_{bcpms}(\varphi,\nu)+\rho_{bcpms}(\zeta,\mu)\Big),
\end{aligned}
$$

for all $\varphi,\zeta,\nu,\mu \in \mathcal{U}$, where $0 \leq \lambda = \frac{1}{4} < \frac{1}{2}$. Therefore, all the conditions of Corollary 3.4 are satisfied, then the mapping $\mathcal{S}$ has a unique coupled fixed point $(0,0)$ in $\mathcal{U}$.

4. Applications to integral equations

As an application of Theorem 3.3, we find an existence and uniqueness result for a type of the following system of nonlinear integral equations:

$$
\begin{aligned}
\varphi(\mu) &= \int_0^{\mathcal{M}} \kappa(\mu,\mathfrak{p})[\mathcal{G}_1(\mathfrak{p},\varphi(\mathfrak{p}))+\mathcal{G}_2(\mathfrak{p},\zeta(\mathfrak{p}))]d\mathfrak{p}+\delta(\mu),\\
\zeta(\mu) &= \int_0^{\mathcal{M}} \kappa(\mu,\mathfrak{p})[\mathcal{G}_1(\mathfrak{p},\zeta(\mathfrak{p}))+\mathcal{G}_2(\mathfrak{p},\varphi(\mathfrak{p}))]d\mathfrak{p}+\delta(\mu),\ \ \mu,\in[0,\mathcal{M}], \mathcal{M}\geq 1.
\end{aligned}
\tag{4.1}
$$

Let $\mathcal{U} = C([0,\mathcal{M}],\mathbb{R})$ be the class of all real valued continuous functions on $[0,\mathcal{M}]$. We define a partial order $\preceq$ in $\mathscr{C}_2^+$ as $x \preceq y$ iff $x \leq y$. Define $\mathcal{S}:\mathcal{U}\times\mathcal{U}\to\mathcal{U}$ by

$$
\mathcal{S}(\varphi,\zeta)(\mu) = \int_0^{\mathcal{M}} \kappa(\mu,\mathfrak{p})[\mathcal{G}_1(\mathfrak{p},\varphi(\mathfrak{p}))+\mathcal{G}_2(\mathfrak{p},\zeta(\mathfrak{p}))]d\mathfrak{p}+\delta(\mu).
$$

Obviously, $(\varphi(\mu),\zeta(\mu))$ is a solution of system of nonlinear integral equations (4.1) iff $(\varphi(\mu),\zeta(\mu))$ is a coupled fixed point of $\mathcal{S}$. Define $\rho_{bcpms}:\mathcal{U}\times\mathcal{U}\to\mathscr{C}_2$ by

$$
\rho_{bcpms}(\varphi,\zeta) = (|\varphi-\zeta|+1)e^{i_2\theta},
$$

for all $\varphi,\zeta\in\mathcal{U}$, where $0\leq\theta\leq\frac{\pi}{2}$. Now, we state and prove our result as follows.

Theorem 4.1. *Suppose the following:*

1. *The mappings* $\mathcal{G}_1:[0,\mathcal{M}]\times\mathbb{R}\to\mathbb{R}$, $\mathcal{G}_2:[0,\mathcal{M}]\times\mathbb{R}\to\mathbb{R}$, $\delta:[0,\mathcal{M}]\to\mathbb{R}$ *and* $\kappa:[0,\mathcal{M}]\times\mathbb{R}\to[0,\infty)$ *are continuous.*
2. *There exists* $\eta>0$, *and* $\lambda,\mathfrak{l}$ *are nonnegative constants with* $\lambda+\mathfrak{l}<1$, *such that*

$$
\begin{aligned}
|\mathcal{G}_1(\mathfrak{p},\varphi(\mathfrak{p}))-\mathcal{G}_1(\mathfrak{p},\zeta(\mathfrak{p}))| &\preceq_{i_2} \eta\lambda(|\varphi-\zeta|+1)-\frac{1}{2},\\
|\mathcal{G}_2(\mathfrak{p},\zeta(\mathfrak{p}))-\mathcal{G}_2(\mathfrak{p},\varphi(\mathfrak{p}))| &\preceq_{i_2} \eta\mathfrak{l}(|\zeta-\varphi|+1)-\frac{1}{2}.
\end{aligned}
$$

3. $\int_0^{\mathcal{M}}\eta|\kappa(\mu,\mathfrak{p})|d\mathfrak{p}\preceq_{i_2} 1.$

Then, the integral equation (4.1) *has a unique solution in* $\mathcal{U}$.

Proof. Consider

$$\begin{aligned}
\rho_{bcpms}(\mathcal{S}(\varphi,\zeta),\mathcal{S}(\nu,\Phi)) &= (|\mathcal{S}(\varphi,\zeta)-\mathcal{S}(\nu,\Phi)|+1)e^{i_2\theta}\\
&= \Big(|\int_0^{\mathcal{M}}\kappa(\mu,\mathfrak{p})[\mathcal{G}_1(\mathfrak{p},\varphi(\mathfrak{p}))+\mathcal{G}_2(\mathfrak{p},\zeta(\mathfrak{p}))]d\mathfrak{p}+\delta(\mu)\\
&-\Big(\int_0^{\mathcal{M}}\kappa(\mu,\mathfrak{p})[\mathcal{G}_1(\mathfrak{p},\nu(\mathfrak{p}))+\mathcal{G}_2(\mathfrak{p},\Phi(\mathfrak{p}))]d\mathfrak{p}+\delta(\mu)\Big)|+1\Big)e^{i_2\theta}\\
&= \Big(|\int_0^{\mathcal{M}}\kappa(\mu,\mathfrak{p})[\mathcal{G}_1(\mathfrak{p},\varphi(\mathfrak{p}))-\mathcal{G}_1(\mathfrak{p},\nu(\mathfrak{p}))\\
&+\mathcal{G}_2(\mathfrak{p},\zeta(\mathfrak{p}))-\mathcal{G}_2(\mathfrak{p},\Phi(\mathfrak{p}))]d\mathfrak{p}|+1\Big)e^{i_2\theta}\\
&\preceq_{i_2}\Big(\int_0^{\mathcal{M}}|\kappa(\mu,\mathfrak{p})|[|\mathcal{G}_1(\mathfrak{p},\varphi(\mathfrak{p}))-\mathcal{G}_1(\mathfrak{p},\nu(\mathfrak{p}))|\\
&+|\mathcal{G}_2(\mathfrak{p},\zeta(\mathfrak{p}))-\mathcal{G}_2(\mathfrak{p},\Phi(\mathfrak{p}))|]d\mathfrak{p}+1\Big)e^{i_2\theta}\\
&\preceq_{i_2}\Big(\int_0^{\mathcal{M}}|\kappa(\mu,\mathfrak{p})|d\mathfrak{p}(\eta\lambda(|\varphi-\nu|+1)-\frac{1}{2}\\
&+\eta\mathfrak{l}(|\zeta-\Phi|+1)-\frac{1}{2})+1\Big)e^{i_2\theta}\\
&= \Big(\int_0^{\mathcal{M}}\eta|\kappa(\mu,\mathfrak{p})|d\mathfrak{p}(\lambda(|\varphi-\nu|+1)\\
&+\mathfrak{l}(|\zeta-\Phi|+1))\Big)e^{i_2\theta}\\
&\preceq_{i_2}\Big(\lambda(|\varphi-\nu|+1)+\mathfrak{l}(|\zeta-\Phi|+1)\Big)e^{i_2\theta}\\
&= \lambda\rho_{bcpms}(\varphi,\nu)+\mathfrak{l}\rho_{bcpms}(\zeta,\Phi)
\end{aligned}$$

for all $\varphi,\zeta,\nu,\Phi\in\mathcal{U}$. Hence, all the hypotheses of Theorem 3.3 are verified, and consequently, the integral equation (4.1) has a unique solution. □

Example 4.2. Let $\mathcal{U}=C([0,1],\mathbb{R})$. Now, consider the integral equation in $\mathcal{U}$ as

$$\begin{aligned}
\varphi(\mu) &= \int_0^1\frac{\mu\mathfrak{p}}{23(\mu+5)}\Big[\frac{1}{1+\varphi(\mathfrak{p})}+\frac{1}{2+\zeta(\mathfrak{p})}\Big]d\mathfrak{p}+\frac{6\mu^2}{5}\\
\zeta(\mu) &= \int_0^1\frac{\mu\mathfrak{p}}{23(\mu+5)}\Big[\frac{1}{1+\zeta(\mathfrak{p})}+\frac{1}{2+\varphi(\mathfrak{p})}\Big]d\mathfrak{p}+\frac{6\mu^2}{5}.
\end{aligned}\tag{4.2}$$

Then, clearly the above equation is in the form of the following equation:

$$\begin{aligned}
\varphi(\mu) &= \int_0^{\mathcal{M}}\kappa(\mu,\mathfrak{p})[\mathcal{G}_1(\mathfrak{p},\varphi(\mathfrak{p}))+\mathcal{G}_2(\mathfrak{p},\zeta(\mathfrak{p}))]d\mathfrak{p}+\delta(\mu),\\
\zeta(\mu) &= \int_0^{\mathcal{M}}\kappa(\mu,\mathfrak{p})[\mathcal{G}_1(\mathfrak{p},\zeta(\mathfrak{p}))+\mathcal{G}_2(\mathfrak{p},\varphi(\mathfrak{p}))]d\mathfrak{p}+\delta(\mu),\ \ \mu,\in[0,\mathcal{M}],
\end{aligned}\tag{4.3}$$

where $\delta(\mu) = \frac{6\mu^2}{5}$, $\kappa(\mu, \mathfrak{p}) = \frac{\mu\mathfrak{p}}{23(\mu+5)}$, $\mathcal{G}_1(\mathfrak{p}, \mu) = \frac{1}{1+\mu}$, $\mathcal{G}_2(\mathfrak{p}, \mu) = \frac{1}{2+\mu}$ and $\mathcal{M} = 1$. That is, (4.2) is a special case of (4.1) in Theorem 4.1. Here, it is easy to verify that the functions $\delta(\mu)$, $\kappa(\mu, \mathfrak{p})$, $\mathcal{G}_1(\mathfrak{p}, \mu)$ and $\mathcal{G}_2(\mathfrak{p}, \mu)$ are continuous. Moreover, there exist $\eta = 10$, $\lambda = \frac{1}{3}$ and $\mathfrak{l} = \frac{1}{4}$ with $\lambda + \mathfrak{l} < 1$ such that

$$|\mathcal{G}_1(\mathfrak{p}, \varphi) - \mathcal{G}_1(\mathfrak{p}, \zeta)| \leq \eta\lambda(|\varphi - \zeta| + 1) - \frac{1}{2},$$
$$|\mathcal{G}_2(\mathfrak{p}, \zeta) - \mathcal{G}_2(\mathfrak{p}, \varphi)| \leq \eta\mathfrak{l}(|\zeta - \varphi| + 1) - \frac{1}{2}$$

and $\int_0^{\mathcal{M}} \eta|\kappa(\mu, \mathfrak{p})|d\mathfrak{p} = \int_0^1 \frac{\eta\mu\mathfrak{p}}{23(\mu+5)}d\mathfrak{p} = \frac{\mu\eta}{23(\mu+5)} < 1$. Therefore, all the conditions of Theorem 3.3 are satisfied. Hence, system (4.2) has a unique solution (φ^*, ζ^*) in $\mathcal{U} \times \mathcal{U}$.

As an application of Corollary 3.4, we find an existence and uniqueness result for a type of the following system of Fredholm integral equations:

$$\varphi(\mu) = \int_{\mathcal{E}} \mathcal{G}(\mu, \mathfrak{p}, \varphi(\mathfrak{p}), \zeta(\mathfrak{p}))d\mathfrak{p} + \delta(\mu), \ \ \mu, \mathfrak{p} \in \mathcal{E},$$
$$\zeta(\mu) = \int_{\mathcal{E}} \mathcal{G}(\mu, \mathfrak{p}, \zeta(\mathfrak{p}), \varphi(\mathfrak{p}))d\mathfrak{p} + \delta(\mu), \ \ \mu, \mathfrak{p} \in \mathcal{E}, \tag{4.4}$$

where $\mathcal{E}$ is a measurable, $\mathcal{G} : \mathcal{E} \times \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\delta \in \mathcal{L}^\infty(\mathcal{E})$. Let $\mathcal{U} = \mathcal{L}^\infty(\mathcal{E})$. We define a partial order $\preceq$ in $\mathscr{C}_2^+$ as $x \preceq y$ iff $x \leq y$. Define $\mathcal{S} : \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ by

$$\mathcal{S}(\varphi, \zeta)(\mu) = \int_{\mathcal{E}} \mathcal{G}(\mu, \mathfrak{p}, \varphi(\mathfrak{p}), \zeta(\mathfrak{p}))d\mathfrak{p} + \delta(\mu).$$

Obviously, $(\varphi(\mu), \zeta(\mu))$ is a solution of the system of Fredholm integral equations (4.4) iff $(\varphi(\mu), \zeta(\mu))$ is a coupled fixed point of $\mathcal{S}$. Define $\rho_{bcpms} : \mathcal{U} \times \mathcal{U} \to \mathscr{C}_2$ by

$$\rho_{bcpms}(\varphi, \zeta) = (|\varphi - \zeta| + 1)e^{i_2\theta},$$

for all $\varphi, \zeta \in \mathcal{U}$, where $0 \leq \theta \leq \frac{\pi}{2}$. Now, we state and prove our result as follows.

Theorem 4.3. *Suppose the following:*

1. *There exists a continuous function* $\kappa : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *such that*

$$|\mathcal{G}(\mu, \mathfrak{p}, \varphi(\mathfrak{p}), \zeta(\mathfrak{p})) - \mathcal{G}(\mu, \mathfrak{p}, \nu(\mathfrak{p}), \Phi(\mathfrak{p}))| \preceq_{i_2} |\kappa(\mu, \mathfrak{p})|(|\varphi(\mathfrak{p}) - \nu(\mathfrak{p})| + |\zeta(\mathfrak{p}) - \Phi(\mathfrak{p})| - 2),$$

for all $\varphi, \zeta, \nu, \Phi \in \mathcal{U}$, $\mu, \mathfrak{p} \in \mathcal{E}$.

2. $\int_{\mathcal{E}} |\kappa(\mu, \mathfrak{p})|d\mathfrak{p} \preceq_{i_2} \frac{1}{4} \preceq_{i_2} 1$.

Then, the integral equation (4.4) *has a unique solution in* $\mathcal{U}$.

Proof. Consider

$$\rho_{bcpms}(\mathcal{S}(\varphi, \zeta), \mathcal{S}(\nu, \Phi)) = (|\mathcal{S}(\varphi, \zeta) - \mathcal{S}(\nu, \Phi)| + 1)e^{i_2\theta}$$

$$
\begin{aligned}
&= \Big(| \int_{\mathcal{E}} \mathcal{G}(\mu, \mathfrak{p}, \varphi(\mathfrak{p}), \zeta(\mathfrak{p}))d\mathfrak{p} + \delta(\mu) \\
&- \Big(\int_{\mathcal{E}} \mathcal{G}(\mu, \mathfrak{p}, \nu(\mathfrak{p}), \Phi(\mathfrak{p}))d\mathfrak{p} + \delta(\mu)\Big)| + 1\Big)e^{i_2\theta} \\
&= \Big(| \int_{\mathcal{E}} \Big(\mathcal{G}(\mu, \mathfrak{p}, \varphi(\mathfrak{p}), \zeta(\mathfrak{p})) \\
&- \mathcal{G}(\mu, \mathfrak{p}, \nu(\mathfrak{p}), \Phi(\mathfrak{p}))\Big)d\mathfrak{p}| + 1\Big)e^{i_2\theta} \\
&\preceq_{i_2} \Big(\int_{\mathcal{E}} |\mathcal{G}(\mu, \mathfrak{p}, \varphi(\mathfrak{p}), \zeta(\mathfrak{p})) - \mathcal{G}(\mu, \mathfrak{p}, \nu(\mathfrak{p}), \Phi(\mathfrak{p}))|d\mathfrak{p} + 1\Big)e^{i_2\theta} \\
&\preceq_{i_2} \Big(\int_{\mathcal{E}} |\kappa(\mu, \mathfrak{p})|(|\varphi(\mathfrak{p}) - \nu(\mathfrak{p})| + |\zeta(\mathfrak{p}) - \Phi(\mathfrak{p})| - 2)d\mathfrak{p} + 1\Big)e^{i_2\theta} \\
&\preceq_{i_2} \Big(\int_{\mathcal{E}} |\kappa(\mu, \mathfrak{p})|d\mathfrak{p}(|\varphi(\mathfrak{p}) - \nu(\mathfrak{p})| + |\zeta(\mathfrak{p}) - \Phi(\mathfrak{p})| - 2) + 1\Big)e^{i_2\theta} \\
&\preceq_{i_2} \frac{1}{4}\Big(|\varphi(\mathfrak{p}) - \nu(\mathfrak{p})| + |\zeta(\mathfrak{p}) - \Phi(\mathfrak{p})| - 2 + 4\Big)e^{i_2\theta} \\
&\preceq_{i_2} \frac{1}{4}(\rho_{bcpms}(\varphi, \nu) + \rho_{bcpms}(\zeta, \Phi)) \\
&= \lambda(\rho_{bcpms}(\varphi, \nu) + \rho_{bcpms}(\zeta, \Phi)),
\end{aligned}
$$

for all $\varphi, \zeta, \nu, \Phi \in \mathcal{U}$, where $0 \leq \lambda = \frac{1}{4} < \frac{1}{2}$. Hence, all the hypotheses of Corollary 3.4 are verified, and consequently, the integral equation (4.4) has a unique solution. □

5. Conclusions

In this paper, we proved coupled fixed point theorems on a bicomplex partial metric space. An illustrative example and an application on a bicomplex partial metric space were given.

Conflicts of interest

The authors declare no conflict of interest.

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