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Some new dynamic Gronwall–Bellman–Pachpatte type inequalities with delay on time scales and certain applications

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Abstract

The main objective of the present article is to prove some new delay nonlinear dynamic inequalities of Gronwall–Bellman–Pachpatte type on time scales. We introduce very important generalized results with the help of Leibniz integral rule on time scales. For some specific time scales, we further show some relevant inequalities as special cases: integral inequalities and discrete inequalities. Our results can be used as handy tools for the study of qualitative and quantitative properties of solutions of dynamic equations on time scales. Some examples are provided to demonstrate the applications of the results.

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Keywords: Dynamic Gronwall type inequality; Time scales; Leibniz integral rule on time scales

1 Introduction

In 1919 Thomas Gronwall [1] discovered a vital inequality, which can be used as an effective tool in the study of existence, uniqueness, boundedness, stability, and other qualitative properties of solutions of certain nonlinear differential and difference equations. The Gronwall inequality is stated as follows: If u is a continuous function defined on the interval $D = [a, a+h]$ and

$$0 \le u(t) \le \int_a^t \left[\zeta u(s) + \xi\right] ds, \quad \forall t \in D,$$

where a, ξ, ζ, and h are nonnegative constants, then

$$0 \le u(t) \le \xi h e^{\zeta h}, \quad \forall t \in D.$$

In 1943, Richard Bellman [2] proved the fundamental inequality, named Gronwall–Bellman's inequality, as a generalization for Gronwall's inequality. He proved that: If u and

f are continuous and nonnegative functions defined on $[a,b]$, and let c be a nonnegative constant, then the inequality

$$u(t) \leq c + \int_a^t f(s)u(s)\,ds, \quad t \in [a,b], \tag{1.1}$$

implies that

$$u(t) \leq c \exp\left(\int_a^t f(s)\,ds\right), \quad t \in [a,b].$$

As a generalization of (1.1), Bellman himself [3] proved that: If $u, f, a, \in C(\mathbb{R}_+, \mathbb{R}_+)$ and a is nondecreasing, then the inequality

$$u(t) \leq a(t) + \int_0^t f(s)u(s)\,ds, \quad \forall t \in \mathbb{R}_+, \tag{1.2}$$

implies

$$u(t) \leq a(t) \exp\left(\int_0^t f(s)\,ds\right), \quad \forall t \in \mathbb{R}_+.$$

The discrete version of (1.2) was studied by Pachpatte in [4]. In particular, he proved that: If $\Omega(n), f(n), \gamma(n)$ are nonnegative sequences defined for $n \in \mathbb{N}_0$, and $f(n)$ is nondecreasing for $n \in \mathbb{N}_0$, then

$$\Omega(n) \leq f(n) + \sum_{s=0}^{n-1} \gamma(s)\Omega(s), \quad n \in \mathbb{N}_0, \tag{1.3}$$

implies

$$\Omega(n) \leq f(n) \prod_{s=0}^{n-1} \bigl[1 + \gamma(s)\bigr], \quad n \in \mathbb{N}_0.$$

In [5], Pachpatte studied the following inequalities:

$$\Omega^p(t) \leq c^p(t) + b(t) \int_0^t \bigl[g(s)\Omega^p(s) + h(s)\Omega(s)\bigr]\,ds, \quad t \in \mathbb{R}_+, \tag{1.4}$$

$$\Omega^p(t) \leq a(t) + b(t) \int_0^t k(t,s)\bigl[g(s)\Omega^p(s) + h(s)\Omega(s)\bigr]\,ds, \quad t \in \mathbb{R}_+, \tag{1.5}$$

and

$$\Omega^p(t) \leq a(t) + b(t) \int_0^t f\bigl(s, \Omega(s)\bigr)\,ds, \quad t \in \mathbb{R}_+, \tag{1.6}$$

where Ω, a, b, g, h and $c \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$, $k(t,s)$ and its partial derivative $\frac{\partial k(t,s)}{\partial t}$ are real-valued nonnegative continuous functions for $0 \leq s \leq t \leq \infty$, $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, and $p > 1$ is a constant.

On the other hand, also in [5], Pachpatte investigated the following discrete analogues of (1.4), (1.5), and (1.6):

$$\Omega^p(n) \le c^p(n) + b(n)\sum_{s=n_0}^{n-1}\big[g(s)\Omega^p(s) + h(s)\Omega(s)\big], \quad n \in \mathbb{N}_0,$$

$$\Omega^p(n) \le a(n) + b(n)\sum_{s=n_0}^{n-1}k(n,s)\big[g(s)\Omega^p(s) + h(s)\Omega(s)\big], \quad n \in \mathbb{N}_0,$$

$$\Omega^p(n) \le a(n) + b(n)\sum_{s=n_0}^{n-1}F\big(s,\Omega(s)\big), \quad n \in \mathbb{N}_0,$$

where $\Omega(n)$, $a(n)$, $b(n)$, $g(n)$, $h(n)$, and $c(n)$ are real-valued nonnegative sequences, $F : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$, and $k(n,s)$, $\Delta_1 k(n,s)$ are real-valued nonnegative functions for $n_0 \le s \le n$, $n \in \mathbb{N}_0$.

In 2014, El-Owaidy et al. [6] proved the following new form:

$$\begin{aligned}\Omega(t) \le \gamma(t) &+ \int_a^{\alpha_1(t)} \epsilon(s)w_1\big(\Omega(s)\big)\,ds \\ &+ \int_a^{\alpha_2(t)} \varepsilon(s)w_2\big(\Omega(s)\big)\,ds \quad \text{for all } t \in I_1 = [a,b].\end{aligned} \tag{1.7}$$

In the same paper [6], the authors also studied the following inequality:

$$\Omega(t) \le \gamma(t) + \int_a^{\alpha(t)} \varepsilon(s)w\big(\Omega(s)\big)\,ds + \int_a^{\alpha(t)} k(t,s)w\big(\Omega(s)\big)\,ds \quad \text{for all } t \in I_1,$$

where $\Omega, \epsilon, \varepsilon \in \mathcal{C}(I_1,\mathbb{R}_+)$, $\alpha, \in \mathcal{C}^1(I_1,I_1)$ are nondecreasing functions, with $\alpha_i(t) \le t$, $\alpha_i(a) = a$, $\alpha_i'(t) \ge 0$, $i = 1,2$, and $w_i \in (\mathbb{R}_+,\mathbb{R}_+)$ is a nondecreasing function, and $k(t,s) \in \mathcal{C}(I_1 \times I_1,\mathbb{R}_+)$ with $\frac{\partial k(t,s)}{\partial t} \in \mathcal{C}(I_1 \times I_1,\mathbb{R}_+)$.

In 2015, Abdeldaim and El-Deeb [7] discussed the new form:

$$\Omega(t) \le \Omega_0 + \int_0^{\alpha(t)} \gamma(s)\varphi\big(\Omega(s)\big)\left[\varphi\big(\Omega(s)\big) + \int_0^s \epsilon(\lambda)\varphi\big(\Omega(\lambda)\big)\,d\lambda\right]ds \quad \text{for all } t \in \mathbb{R}_+,$$

where $\gamma, \epsilon \in \mathcal{C}(\mathbb{R}_+,\mathbb{R}_+)$ and $\varphi, \varphi', \alpha \in \mathcal{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ are increasing functions, with $\varphi'(t) \le k$, $\varphi > 0$, $\alpha(t) \le t$, $\alpha(0) = 0$ and k, Ω_0 are positive constants.

In the same paper [7], by using the composite function, the authors introduced a new inequality with a different kernel as follows:

$$\varphi_1\big(\Omega(t)\big) \le \Omega_0 + \int_0^{\alpha(t)} \gamma(s)\varphi_2\big(\Omega(s)\big)\left[\Omega(s) + \int_0^s \epsilon(\lambda)\varphi_1\big(\Omega(\lambda)\big)\,d\lambda\right]^p ds \quad \text{for all } t \in \mathbb{R}_+,$$

where $\varphi_1, \varphi_2, \alpha \in \mathcal{C}^1(\mathbb{R}_+,\mathbb{R}_+)$ are increasing functions with $\alpha(t) \le t$, $\varphi_i(t) > 0$, $i = 1,2$, $\alpha(0) = 0$ and $\varphi_1'(t) = \varphi_2(t)$, $p \ge 1$ and Ω_0 are positive constants.

In [8], one of the new generalizations of Gronwall type inequalities has been proved by Abdeldaim and El-Deeb, and it can be written as follows:

$$\varphi_1\big(\Omega(t)\big) \le \Omega_0 + \int_0^{\alpha(t)} \epsilon(s)\varphi_1\big(\Omega(s)\big)\,ds + \int_0^{\alpha(t)} \varepsilon(s)\varphi_2\big(\Omega(s)\big)\,ds \quad \text{for all } t \in \mathbb{R}_+,$$

with $\alpha(t) \leq t$, $\varphi_i(t) > 0$, $i = 1, 2$, $\alpha(0) = 0$, $\varphi_1'(t) = \varphi_2(t)$, and $\varphi_1^{-1}(t)$ is a submultiplicative function and Ω_0 is a positive constant.

Recently, in 2017, El-Deeb and Ahmed [9] studied the following inequality with retardation $\alpha(t) \leq t$:

$$\Omega^p(t) \leq c(t) + \int_a^{\alpha(t)} \gamma(s)\Omega(s)\,ds + \int_a^b \epsilon(s)\Omega^p(s)\,ds \quad \text{for all } t \in [a, b],$$

where Ω, γ, $\epsilon \in \mathcal{C}([a, b], \mathbb{R}_+)$ and α, $c \in \mathcal{C}^1([a, b], \mathbb{R}_+)$ with $\alpha(t) \leq t$, $\alpha(a) = 0$ and $p \geq 1$ is a constant.

Lately, in 2019, Li and Wang [10] established the following inequality:

$$\Omega(t) \leq a(t) + \int_{t_0}^{\alpha(t)} \gamma(s)\left[\Omega^m(s) + \int_{t_0}^{s} \epsilon(\tau)\Omega^n(\tau)\,d\tau\right]^p ds \quad \text{for all } t \in [t_0, +\infty),$$

where $\Omega, a, \gamma, \epsilon \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and α is a continuously differentiable nondecreasing function on $[t_0, +\infty)$ with $\alpha(t) \leq t$, $\alpha(t_0) = 0$ and p, m, $n \in (0, 1]$ are positive constants.

Many generalizations, refinements, and extensions of Gronwall–Bellman type inequalities can be found in [11–17].

Stefan Hilger was the first to discover the theory of time scales which he demonstrated in his PhD thesis [18]. For further information and details on the time scales, we refer the reader to books [19, 20]. Many dynamic inequalities have been investigated by different authors during the past decade (see [21–40] and the references cited therein). Throughout this paper, knowledge and understanding of the time scales notion and time scale calculus are assumed.

In [35, Theorem 6.4, page 256], Bohner and Peterson introduced a dynamic inequality on a time scale $\mathbb{T}$ which unifies the continuous version inequality (1.2) and the discrete version inequality (1.3) as follows: If Ω, ζ are right dense continuous functions and $\gamma \geq 0$ is a regressive and right-dense continuous function, then

$$\Omega(t) \leq \zeta(t) + \int_{t_0}^{t} \Omega(\eta)\gamma(\eta)\Delta\eta \quad \text{for all } t \in \mathbb{T}$$

implies

$$\Omega(t) \leq \zeta(t) + \int_{t_0}^{t} e_\gamma\bigl(t, \sigma(\eta)\bigr)\zeta(\eta)\gamma(\eta)\Delta\eta \quad \text{for all } t \in \mathbb{T}.$$

In this paper, motivated by the above-mentioned inequalities, we prove some new delay dynamic inequalities of Gronwall–Bellman–Pachpatte type on time scales. Some special cases of our results contain continuous Gronwall type inequalities and their discrete analogues. We also present some application examples to illustrate our results at the end. The paper is organized as follows: Sect. 2 contains the main results of this paper. In Sect. 3, an application to study some qualitative properties of the solutions of certain retarded dynamic equations are demonstrated. In Sect. 4, we state the conclusion.

Before we arrive at the main results in the next section, we need the following lemmas and essential relations on some time scales such as $\mathbb{R}$, $\mathbb{Z}$, $h\mathbb{Z}$, and $\overline{q^{\mathbb{Z}}}$. Note that:

(i) If $\mathbb{T}=\mathbb{R}$, then

$$\begin{aligned} &\sigma(\tau)=\tau, \qquad \mu(\tau)=0, \qquad f^{\Delta}(\tau)=f'(\tau), \\ &\int_a^b f(\tau)\Delta\tau=\int_a^b f(\tau)\,d\tau; \end{aligned} \tag{1.8}$$

(ii) If $\mathbb{T}=\mathbb{Z}$, then

$$\begin{aligned} &\sigma(\tau)=\tau+1, \qquad \mu(\tau)=1, \qquad f^{\Delta}(\tau)=\Delta f(\tau), \\ &\int_a^b f(\tau)\Delta\tau=\sum_{\tau=a}^{b-1} f(\tau); \end{aligned} \tag{1.9}$$

(iii) If $\mathbb{T}=\overline{q^{\mathbb{Z}}}=\{q^k:k\in\mathbb{Z}\}\cup\{0\}$, $q>1$, then

$$\begin{aligned} &\sigma(\tau)=q\tau, \qquad \mu(\tau)=(q-1)\tau, \\ &\int_a^b f(\tau)\Delta\tau=(q-1)\sum_{k=\log_q(a)}^{\log_q(b)-1} q^k f\bigl(q^k\bigr), \quad \forall a,b\in q^{\mathbb{N}_0}. \end{aligned} \tag{1.10}$$

If $\lambda\in C_{rd}(\mathbb{T})$ (see [35]), then the Cauchy integral $\Lambda(\tau):=\int_{\tau_0}^{\tau}\lambda(s)\Delta s$ exists, $\tau_0\in\mathbb{T}$, and satisfies $\Lambda^{\Delta}(\tau)=\lambda(\tau)$, $\tau\in\mathbb{T}$. An infinite integral follows

$$\int_a^{\infty}\Omega(\tau)\Delta\tau=\lim_{b\to\infty}\int_a^b\Omega(\tau)\Delta\tau.$$

The function $\eta:\mathbb{T}\to\mathbb{R}$ is called regressive provided $1+\mu(t)\eta(t)\neq 0$ for all $t\in\mathbb{T}^{\kappa}$. The set of all positively regressive elements of $\Re$ is $\Re^+=\{\eta\in\Re:1+\mu(t)\eta(t)>0,\forall t\in\mathbb{T}\}$. We form an Abelian group under the addition $\oplus$ by the set of all regressive functions on a time scale $\mathbb{T}$ by $\eta\oplus\zeta=\eta+\zeta+\mu\eta\zeta$. If $\eta\in\Re$, then the exponential function is defined by

$$e_{\eta}(t,s)=\exp\biggl(\int_s^t\hat{\xi}_{\mu(\tau)}\bigl(\eta(\tau)\bigr)\Delta\tau\biggr), \quad s,t\in\mathbb{T},$$

where $\hat{\xi}_{\hat{h}}(z)$ is the cylinder transformation, which is defined by

$$\hat{\xi}_{\hat{h}}(z)=\begin{cases}\frac{\operatorname{Log}(1+\hat{h}z)}{\hat{h}}, & \hat{h}\neq 0,\\ z, & \hat{h}=0.\end{cases}$$

If $\eta\in\Re$, then $e_{\eta}(\tau,s)$ is real-valued and nonzero on $\mathbb{T}$. If $\eta\in\Re^+$, then $e_{\eta}(\tau,\tau_0)$ is always positive.

Note that:

- If $\mathbb{T}=\mathbb{R}$, then

$$e_b(\tau,\tau_0)=\exp\biggl(\int_{\tau_0}^{\tau}b(s)\,ds\biggr). \tag{1.11}$$

- If $\mathbb{T} = \mathbb{Z}$, then

$$e_b(\tau,\tau_0) = \prod_{s=\tau_0}^{\tau-1}\bigl(1 + b(s)\bigr). \tag{1.12}$$

- If $\mathbb{T} = q^{\mathbb{N}_0}$, then

$$e_b(\tau,\tau_0) = \prod_{s=\tau_0}^{\tau-1}\bigl(1 + (q-1)sb(s)\bigr). \tag{1.13}$$

Lemma 1.1 ([41]) *If $\eta \in \Re$ and $a, b, d \in \mathbb{T}$, then*

1. $e_\eta(\tau,\tau) = 1$ *and* $e_0(\tau,s) = 1$;
2. $e_\eta(\sigma(\tau),s) = (1 + \mu(\tau)\eta(\tau))e_\eta(\tau,s)$;
3. *If* $\eta \in \Re^+$, *then* $e_\eta(\tau,\tau_0) > 0$, $\forall \tau \in \mathbb{T}$;
4. $\int_a^b \eta(\tau)e_\eta(d,\sigma(\tau))\Delta\tau = -\int_a^b [e_\eta(d,\cdot)]^\Delta \Delta\tau = e_\eta(d,a) - e_\eta(d,b)$.

Lemma 1.2 (See [41]) *Let $\chi : \mathbb{T} \to \mathbb{R}$ be a delta differentiable function. If $\eta \in \Re$ and fix $t_0 \in \mathbb{T}$, then the exponential function $e_\eta(t,t_0)$ is the unique solution of the following initial value problem*:

$$\begin{cases} \chi^\Delta(t) = \eta(t)\chi(t), \\ \chi(t_0) = 1. \end{cases} \tag{1.14}$$

Lemma 1.3 (See [41]) *Let $t_0 \in \mathbb{T}^\kappa$ and $\varsigma : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ be continuous at (t,t), where $t > t_0$ and $t \in \mathbb{T}^\kappa$. Assume that $\varsigma^\Delta(t,\cdot)$ is rd-continuous on $[t_0,\sigma(t)]_\mathbb{T}$. If for any $\varepsilon > 0$ there exists a neighborhood U of t, independent of $\lambda \in [t_0,\sigma(t)]_\mathbb{T}$, such that*

$$\bigl|\bigl[\varsigma\bigl(\sigma(t),\lambda\bigr) - \varsigma(s,\lambda)\bigr] - \varsigma^\Delta(t,\lambda)\bigl[\sigma(t) - s\bigr]\bigr| \le \varepsilon\bigl|\sigma(t) - s\bigr|, \quad \forall s \in U,$$

where ς^Δ denotes the derivative of ς with respect to the first variable, then

$$\chi(t) = \int_{t_0}^t \varsigma(t,\lambda)\Delta\lambda$$

implies

$$\chi^\Delta(t) = \int_{t_0}^t \varsigma^\Delta(t,\lambda)\Delta\lambda + \varsigma\bigl(\sigma(t),t\bigr).$$

Lemma 1.4 ([41]) *Suppose $\chi, b \in C_{rd}$, $a \in \Re^+$, then*

$$\chi^\Delta(t) \le a(t)\chi(t) + b(t), \quad t \ge t_0, t \in \mathbb{T}^\kappa,$$

implies

$$\chi(t) \le \chi(t_0)e_a(t,t_0) + \int_{t_0}^t e_a\bigl(t,\sigma(\tau)\bigr)b(\tau)\Delta\tau, \quad t \ge t_0, t \in \mathbb{T}^\kappa.$$

Lemma 1.5 ([42]) *If $x \geq 0$ and $p \geq 1$, then*

$$x^{1/p} \leq m_1 x + m_2, \tag{1.15}$$

where $m_1 = \frac{1}{p}K^{(1-p)/p}$, $m_2 = \frac{p-1}{p}K^{1/p}$, and $K > 0$.

Theorem 1.6 ([43, Leibniz integral rule on time scales]) *In the following, by $f^{\Delta}(t,s)$ we mean the delta derivative of $f(t,s)$ with respect to t. Similarly, $f^{\nabla}(t,s)$ is understood. If f, f^{Δ}, and f^{∇} are continuous and $u,h : \mathbb{T} \to \mathbb{T}$ are delta differentiable functions, then the following formulas hold $\forall t \in \mathbb{T}^{\kappa}$:*

(i) $[\int_{u(t)}^{h(t)} f(t,s)\Delta s]^{\Delta} = \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\Delta s + h^{\Delta}(t)f(\sigma(t),h(t)) - u^{\Delta}(t)f(\sigma(t),u(t))$;

(ii) $[\int_{u(t)}^{h(t)} f(t,s)\Delta s]^{\nabla} = \int_{u(t)}^{h(t)} f^{\nabla}(t,s)\Delta s + h^{\nabla}(t)f(\rho(t),h(t)) - u^{\nabla}(t)f(\rho(t),u(t))$;

(iii) $[\int_{u(t)}^{h(t)} f(t,s)\nabla s]^{\Delta} = \int_{u(t)}^{h(t)} f^{\Delta}(t,s)\nabla s + h^{\Delta}(t)f(\sigma(t),h(t)) - u^{\Delta}(t)f(\sigma(t),u(t))$;

(iv) $[\int_{u(t)}^{h(t)} f(t,s)\nabla s]^{\nabla} = \int_{u(t)}^{h(t)} f^{\nabla}(t,s)\nabla s + h^{\nabla}(t)f(\rho(t),h(t)) - u^{\nabla}(t)f(\rho(t),u(t))$.

2 Main results

In this section, the authors state and justify the main results and investigate some dynamic Gronwall–Bellman inequalities on time scales.

Theorem 2.1 *Let $a,b \in \mathbb{T}^k$ with $a < b$, and let $\Im, f, g, c \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_+)$ and $\alpha : \mathbb{T} \to \mathbb{T}$. Furthermore, assume that α and c are delta-differentiable on $\mathbb{T}$ with $c^{\Delta}(t) \geq 0$, $\alpha^{\Delta}(t) \geq 0$, $\alpha(t) \leq t$ and $\alpha(a) = a$. For any constant $p \geq 1$, if*

$$\Im^p(t) \leq c(t) + \int_a^{\alpha(t)} g(s)\Im(s)\Delta s + \int_a^b f(s)\Im^p(s)\Delta s \quad \text{for all } t \in [a,b]_{\mathbb{T}}, \tag{2.1}$$

then

$$\Im(t) \leq \left\{ \Lambda_1 e_{\ell_1}(t,a) + \int_a^t e_{\ell_1}\big(t,\sigma(s)\big)\Gamma_1(s)\Delta s \right\}^{1/p} \quad \text{for all } t \in [a,b]_{\mathbb{T}}, \tag{2.2}$$

where

$$\Lambda_1 = \frac{c(a) + \int_a^b f(s)(\int_a^s e_{\ell_1}(s,\sigma(\lambda))\Gamma_1(\lambda)\Delta\lambda)\Delta s}{1 - \int_a^b f(s)e_{\ell_1}(s,a)\Delta s}, \tag{2.3}$$

such that

$$\int_a^b f(s)e_{\ell_1}(s,a)\Delta s < 1,$$

and

$$\Gamma_1(t) = c^{\Delta}(t) + m_2\alpha^{\Delta}(t)g\big(\alpha(t)\big), \tag{2.4}$$

$$\ell_1(t) = m_1\alpha^{\Delta}(t)g\big(\alpha(t)\big), \tag{2.5}$$

where m_1, m_2 are defined as in Lemma 1.5.

Proof Define a function $\chi_1(t)$ by

$$\chi_1(t) = c(t) + \int_a^{\alpha(t)} g(s)\Im(s)\Delta s + \int_a^b f(s)\Im^p(s)\Delta s. \tag{2.6}$$

We notice that $\chi_1(t) \geq 0$ and nondecreasing on $[a,b]_{\mathbb{T}}$. Since $\alpha(a) = a$, we get that

$$\chi_1(a) = c(a) + \int_a^b f(s)\Im^p(s)\Delta s. \tag{2.7}$$

Then from (2.1), (2.6) and by using the monotonicity of $\chi_1(t)$, we get

$$\Im(t) \leq \chi_1^{1/p}(t),$$

which implies

$$\Im\big(\alpha(t)\big) \leq \chi_1^{1/p}\big(\alpha(t)\big) \leq \chi_1^{1/p}(t). \tag{2.8}$$

From (2.6), (2.8) and using Theorem 1.6, we have

$$\chi_1^\Delta(t) = c^\Delta(t) + \alpha^\Delta(t)g\big(\alpha(t)\big)\Im\big(\alpha(t)\big) \leq c^\Delta(t) + \alpha^\Delta(t)g\big(\alpha(t)\big)\chi_1^{1/p}(t). \tag{2.9}$$

Therefore, using (2.9) and Lemma 1.5, we get that

$$\begin{aligned}
\chi_1^\Delta(t) &\leq c^\Delta(t) + m_1\alpha^\Delta(t)g\big(\alpha(t)\big)\chi_1(t) + m_2 g(t) \\
&= m_1\alpha^\Delta(t)g\big(\alpha(t)\big)\chi_1(t) + \big[c^\Delta(t) + m_2\alpha^\Delta(t)g\big(\alpha(t)\big)\big] \\
&= \ell_1\chi_1(t) + \Gamma_1(t),
\end{aligned} \tag{2.10}$$

where $\Gamma_1(t)$ and $\ell_1(t)$ are defined as in (2.4) and (2.5), respectively.

Now an application of Lemma 1.4 to (2.10) yields

$$\chi_1(t) \leq \chi_1(a)e_{\ell_1}(t,a) + \int_a^t e_{\ell_1}\big(t,\sigma(s)\big)\Gamma_1(s)\Delta s. \tag{2.11}$$

From (2.8) and (2.11), we get that

$$\Im^p(t) \leq \chi_1(a)e_{\ell_1}(t,a) + \int_a^t e_{\ell_1}\big(t,\sigma(s)\big)\Gamma_1(s)\Delta s. \tag{2.12}$$

From (2.7) and (2.12), we have

$$\begin{aligned}
\chi_1(a) &= c(a) + \int_a^b f(s)\Im^p(s)\Delta s \\
&\leq c(a) + \int_a^b f(s)\left[\chi_1(a)e_{\ell_1}(s,a) + \int_a^s e_{\ell_1}\big(s,\sigma(\lambda)\big)\Gamma_1(\lambda)\Delta\lambda\right]\Delta s \\
&\leq c(a) + \chi_1(a)\int_a^b f(s)e_{\ell_1}(s,a)\Delta s
\end{aligned}$$

$$+\int_a^b f(s)\left(\int_a^s e_{\ell_1}\big(s,\sigma(\lambda)\big)\Gamma_1(\lambda)\Delta\lambda\right)\Delta s. \tag{2.13}$$

Thus, from (2.13), we obtain

$$\chi_1(a) \le \Lambda_1, \tag{2.14}$$

where Λ_1 is defined as in (2.3).

Then we get the desired inequality (2.2) by combining (2.12) and (2.14). This completes the proof. □

Remark 2.2 If we take $\mathbb{T} = \mathbb{R}$, $\alpha(t) = t$, and $p = 1$, then, using relations (1.8), Theorem 2.1 reduces to [44, Theorem 1.5.1].

Remark 2.3 If we take $\mathbb{T} = \mathbb{R}$ and $\alpha(t) = t$, then, using relations (1.8), Theorem 2.1 reduces to [45, Theorem 2.1].

Remark 2.4 If we take $\mathbb{T} = \mathbb{R}$, then, using relations (1.8), Theorem 2.1 reduces to [9, Theorem 2.1].

As a special case of Theorem 2.1, if we take $\mathbb{T} = \mathbb{Z}$ and the delay function $\alpha(n) = n - \tau$, where $\tau > 0$, and so $\Delta\alpha(n) = 1 > 0$, then, using relations (1.9) and (1.12), we obtain the following completely new discrete result.

Corollary 2.5 *Assume that $\Im(n)$, $g(n)$, $c(n)$, and $f(n)$ are nonnegative sequences defined for $n \in \mathbb{N}_0$, with $\Delta c(n) \ge 0$ for $n \in \mathbb{N}_0$. If $\Im(n)$ satisfies the following delay discrete inequality*:

$$\Im^p(n) \le c(n) + \sum_{s=a}^{n-\tau-1} g(s)\Im(s) + \sum_{s=a}^{b-1} f(s)\Im^p(s),$$

then

$$\Im(n) \le \left\{\tilde{\Lambda}_1 \prod_{s=a}^{n-1}\big(1+\hat{\ell}_1(s)\big) + \sum_{s=a}^{n-1}\hat{\Gamma}_1(s)\prod_{\lambda=s+1}^{n-1}\big(1+\hat{\ell}_1(\lambda)\big)\right\}^{1/p},$$

where

$$\hat{\Lambda}_1 = \frac{c(a) + \sum_{s=a}^{b-1} f(s)[\sum_{\lambda=a}^{s-1}\hat{\Gamma}_1(\lambda)\prod_{\upsilon=\lambda+1}^{s-1}(1+\hat{\ell}_1(\upsilon))]}{1 - \sum_{s=a}^{b-1} f(s)\prod_{\lambda=a}^{s-1}(1+\hat{\ell}_1(\lambda))},$$

such that

$$\sum_{s=a}^{b-1} f(s)\prod_{\lambda=a}^{s-1}\big(1+\hat{\ell}_1(\lambda)\big) < 1,$$

and

$$\hat{\Gamma}_1(n) = \Delta c(n) + m_2\Delta(n-\tau)g(n-\tau)$$

$$
\begin{aligned}
&= c(n+1) - c(n) + m_2 g(n-\tau), \\
\hat{\ell}_1(n) &= m_1 \Delta(n-\tau) g(n-\tau) \\
&= m_1 g(n-\tau).
\end{aligned}
$$

Theorem 2.6 *Let $a, b \in \mathbb{T}^k$ with $a < b$, and let $\Im$, $g, c \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_+)$ and $\alpha : \mathbb{T} \to \mathbb{T}$. Further, assume that α and c are delta-differentiable on $\mathbb{T}$ with $c^\Delta(t) \geq 0$, $\alpha^\Delta(t) \geq 0$, $\alpha(t) \leq t$, and $\alpha(a) = a$. Moreover, assume that $k(t,s), k^\Delta(t,s) \in C_{rd}([a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}, \mathbb{R}_+)$ for $a \leq s \leq t \leq b$. For any constant $p \geq 1$, if*

$$
\Im^p(t) \leq c(t) + \int_a^{\alpha(t)} k(t,s)\Im(s)\Delta s + \int_a^b g(s)\Im^p(s)\Delta s \quad \text{for all } t \in [a,b]_{\mathbb{T}}, \tag{2.15}
$$

then

$$
\Im(t) \leq \left\{ \Lambda_2 e_{\ell_2}(t,a) + \int_a^t \Gamma_2(s) e_{\ell_2}\bigl(t, \sigma(s)\bigr)\Delta s \right\}^{1/p} \quad \text{for all } t \in [a,b]_{\mathbb{T}}, \tag{2.16}
$$

where

$$
\Lambda_2 = \frac{c(a) + \int_a^b g(s)(\int_a^s e_{\ell_2}(s,\sigma(\tau))\Gamma_2(\tau)\Delta\tau)\Delta s}{1 - \int_a^b g(s) e_{\ell_2}(s,a)\Delta s},
$$

such that

$$
\int_a^b g(s) e_{\ell_2}(s,a)\Delta s < 1,
$$

and

$$
\Gamma_2(t) = c^\Delta(t) + m_2 \left[\alpha^\Delta(t) k\bigl(\sigma(t), \alpha(t)\bigr) + \int_a^{\alpha(t)} k^\Delta(t,\tau)\Delta\tau \right], \tag{2.17}
$$

$$
\ell_2(t) = m_1 \left[\alpha^\Delta(t) k\bigl(\sigma(t), \alpha(t)\bigr) + \int_a^{\alpha(t)} k^\Delta(t,\tau)\Delta\tau \right], \tag{2.18}
$$

where m_1, m_2 are defined as in Lemma 1.5.

Proof Define a function $\chi_2(t)$ by

$$
\chi_2(t) = c(t) + \int_a^{\alpha(t)} k(t,s)\Im(s)\Delta s + \int_a^b g(s)\Im^p(s)\Delta s. \tag{2.19}
$$

Clearly, $\chi_2(t)$ is nonnegative nondecreasing on $[a,b]_{\mathbb{T}}$. As $\alpha(a) = a$, we have

$$
\chi_2(a) = c(a) + \int_a^b f(s)\Im^p(s)\Delta s. \tag{2.20}
$$

Then from (2.15), (2.19) and by using the monotonicity of $\chi_2(t)$, we obtain

$$
\Im(t) \leq \chi_2^{1/p}(t), \qquad \Im\bigl(\alpha(t)\bigr) \leq \chi_2^{1/p}\bigl(\alpha(t)\bigr) \leq \chi_2^{1/p}(t). \tag{2.21}
$$

Using Theorem 1.6 to delta differentiating (2.19) and from (2.21), we get

$$\begin{aligned}\chi_2^{\Delta}(t) &= c^{\Delta}(t) + \alpha^{\Delta}(t)k\big(\sigma(t),\alpha(t)\big)\Im\big(\alpha(t)\big) + \int_a^{\alpha(t)} k^{\Delta}(t,\tau)\Im(\tau)\Delta\tau \\ &\le c^{\Delta}(t) + \alpha^{\Delta}(t)k\big(\sigma(t),\alpha(t)\big)\chi_2^{1/p}(t) + \int_a^{\alpha(t)} k^{\Delta}(t,\tau)\chi_2^{1/p}(\tau)\Delta\tau \\ &\le c^{\Delta}(t) + \left[\alpha^{\Delta}(t)k\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k^{\Delta}(t,\tau)\Delta\tau\right]\chi_2^{1/p}(t). \qquad (2.22)\end{aligned}$$

Using Lemma 1.5, inequality (2.22) can be rewritten as

$$\begin{aligned}\chi_2^{\Delta}(t) &\le c^{\Delta}(t) + m_1\left[\alpha^{\Delta}(t)k\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k^{\Delta}(t,\tau)\Delta\tau\right]\chi_2(t) \\ &\quad + m_2\left[\alpha^{\Delta}(t)k\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k^{\Delta}(t,\tau)\Delta\tau\right] \\ &= \ell_2(t)\chi_2(t) + \Gamma_2(t), \qquad (2.23)\end{aligned}$$

where $\ell_2(t)$ and $\Gamma_2(t)$ are defined as in (2.18) and (2.17), respectively.

Now, applying Lemma 1.4 to (2.23) yields

$$\chi_2(t) \le \chi_2(a)e_{\ell_2}(t,a) + \int_a^t e_{\ell_2}\big(t,\sigma(\tau)\big)\Gamma_2(s)\Delta\tau. \qquad (2.24)$$

From (2.21) and (2.24), we get that

$$\Im^p(t) \le \chi_2(a)e_{\ell_2}(t,a) + \int_a^t e_{\ell_2}\big(t,\sigma(\tau)\big)\Gamma_2(\tau)\Delta\tau. \qquad (2.25)$$

From (2.20) and (2.25), we have

$$\begin{aligned}\chi_2(a) &= c(a) + \int_a^b g(s)\Im^p(s)\Delta s \\ &\le c(a) + \int_a^b g(s)\left[\chi_2(a)e_{\ell_2}(s,a) + \int_a^s e_{\ell_2}\big(s,\sigma(\tau)\big)\Gamma_2(\tau)\Delta\tau\right]\Delta s \\ &\le c(a) + \chi_2(a)\int_a^b g(s)e_{\ell_2}(s,a)\Delta s \\ &\quad + \int_a^b g(s)\left(\int_a^s e_{\ell_2}\big(s,\sigma(\tau)\big)\Gamma_2(\tau)\Delta\tau\right)\Delta s. \qquad (2.26)\end{aligned}$$

Thus, from (2.26) we obtain

$$\chi_2(a) \le \frac{c(a) + \int_a^b g(s)(\int_a^s e_{\ell_2}(s,\sigma(\tau))\Gamma_2(\tau)\Delta\tau)\Delta s}{1 - \int_a^b g(s)e_{\ell_2}(s,a)\Delta s} = \Lambda_2. \qquad (2.27)$$

Our desired result (2.16) follows directly from (2.25) and (2.27). This concludes the proof. □

Remark 2.7 If we take $\mathbb{T} = \mathbb{R}$, $\alpha(t) = t$, and $p = 1$, then, using relations (1.8), Theorem 2.6 reduces to [44, Theorem 1.5.2 (b_1)].

Remark 2.8 If we take $\mathbb{T} = \mathbb{R}$ and $\alpha(t) = t$, then, using relations (1.8), Theorem 2.6 reduces to [45, Theorem 2.3].

Remark 2.9 If we take $\mathbb{T} = \mathbb{R}$, then, using relations (1.8), Theorem 2.6 reduces to [9, Theorem 2.2].

As a special case of Theorem 2.6, if we take $\mathbb{T} = \mathbb{Z}$ and the delay function $\alpha(n) = n - \tau$, where $\tau > 0$, and so $\Delta\alpha(n) = 1 > 0$, then, using relations (1.9) and (1.12), we obtain the following completely new discrete result.

Corollary 2.10 *Assume that $\Im(n)$, $g(n)$, $c(n)$ $\alpha(n)$, and $f(n)$ are nonnegative sequences defined for $t \in \mathbb{N}_0$, with $\Delta c(n) \geq 0$ and $k(n,s)$, $\Delta k(n,s)$ are nonnegative sequences defined on $E = \{(m,n) \in \mathbb{N}_0^2 : 0 \leq n \leq m < \infty\}$. If $\Im(n)$ satisfies the following delay discrete inequality*

$$\Im^p(n) \leq c(n) + \sum_{s=a}^{n-\tau-1} k(n,s)\Im(s) + \sum_{s=a}^{b-1} g(s)\Im^p(s),$$

then

$$\Im(n) \leq \left\{\hat{\Lambda}_2 \prod_{s=a}^{n-1}\bigl(1 + \hat{\ell}_2(s)\bigr) + \sum_{s=a}^{n-1} \hat{\Gamma}_2(s) \prod_{\tau=s+1}^{n-1} \bigl(1 + \hat{\ell}_2(\tau)\bigr)\right\}^{1/p},$$

where

$$\hat{\Lambda}_2 = \frac{c(a) + \sum_{s=a}^{b-1} g(s)[\sum_{\lambda=a}^{s-1} \hat{\Gamma}_2(\lambda) \prod_{\tau=\lambda+1}^{s-1}(1 + \hat{\ell}_2(\tau))]}{1 - \sum_{s=a}^{b-1} g(s) \prod_{\lambda=\upsilon}^{s-1}(1 + \hat{\ell}_2(\lambda))},$$

such that

$$\sum_{s=a}^{b-1} g(s) \prod_{\lambda=a}^{s-1}\bigl(1 + \hat{\ell}_2(\lambda)\bigr) < 1,$$

and

$$\begin{aligned}\hat{\Gamma}_2(t) &= \Delta c(n) + m_2\left[\Delta(n-\tau)k(n+1,n-\tau) + \sum_{s=a}^{n-\tau-1} \Delta k(n,s)\right] \\ &= c(n+1) - c(n) + m_2\Biggl\{k(n+1,n-\tau) \\ &\quad + \sum_{s=a}^{n-\tau-1} \bigl[k(n+1,s) - k(n,s)\bigr]\Biggr\},\end{aligned}$$

$$\begin{aligned}\hat{\ell}_2(t) &= m_1\left[\Delta(n-\tau)k(n+1,n-\tau)+\sum_{s=a}^{n-\tau-1}\Delta k(n,s)\right]\\ &= m_1\left\{k(n+1,n-\tau)+\sum_{s=a}^{n-\tau-1}\bigl[k(n+1,s)-k(n,s)\bigr]\right\}.\end{aligned}$$

Theorem 2.11 *Assume that $a,b\in\mathbb{T}^k$ with $a<b$, and let $\Im$, α, and c be defined as in Theorem 2.6. Further, suppose that $k_1(t,s)$, $k_2(t,s)$, $k_1^\Delta(t,s)$, and $k_2^\Delta(t,s)\in C_{rd}([a,b]_\mathbb{T}\times[a,b]_\mathbb{T},\mathbb{R}_+)$ for $a\le s\le t\le b$. For any constant $p\ge 1$, if*

$$\Im^p(t)\le c(t)+\int_a^{\alpha(t)}k_1(t,s)\Im(s)\Delta s+\int_a^b k_2(t,s)\Im^p(s)\Delta s\quad \text{for all } t\in[a,b]_\mathbb{T}, \tag{2.28}$$

then

$$\Im(t)\le\left\{\Lambda_3 e_{\ell_3}(t,a)+\int_a^t\Gamma_3(s)e_{\ell_3}\bigl(t,\sigma(s)\bigr)\Delta s\right\}^{1/p}\quad \text{for all } t\in[a,b]_\mathbb{T}, \tag{2.29}$$

where

$$\Lambda_3=\frac{c(a)+\int_a^b k_2(a,s)(\int_a^s\Gamma_3(\lambda)e_{\ell_3}(s,\sigma(\lambda))\Delta\lambda)\Delta s}{1-\int_a^b k_2(a,s)e_{\ell_3}(s,a)\Delta s}, \tag{2.30}$$

such that

$$\int_a^b k_2(s,a)e_{\ell_3}(s,a)\Delta s<1, \tag{2.31}$$

and

$$\begin{aligned}\ell_3(t)&=m_1\left[\alpha^\Delta(t)k_1\bigl(\sigma(t),\alpha(t)\bigr)+\int_a^{\alpha(t)}k_1^\Delta(t,s)\Delta s\right]+\int_a^b k_2^\Delta(t,s)\Delta s,\\ \Gamma_3(t)&=c^\Delta(t)+m_2\left[\alpha^\Delta(t)k_1\bigl(\sigma(t),\alpha(t)\bigr)+\int_a^{\alpha(t)}k_1^\Delta(t,s)\Delta s\right],\end{aligned}$$

where m_1, m_2 are defined as in Lemma 1.5.

Proof Define a function $\chi_3(t)$ by

$$\chi_3(t)=c(t)+\int_a^{\alpha(t)}k_1(t,s)\Im(s)\Delta s+\int_a^b k_2(t,s)\Im^p(s)\Delta s. \tag{2.32}$$

We notice that $\chi_3(t)$ is nonnegative nondecreasing on $[a,b]_\mathbb{T}$. Since $\alpha(a)=a$, we get that

$$\chi_3(a)=c(a)+\int_a^b f(s)\Im^p(s)\Delta s. \tag{2.33}$$

Then from (2.28), (2.32) and by using the monotonicity of $\chi_1(t)$, we obtain

$$\Im(t)\le\chi_3^{1/p}(t),$$

which implies

$$\Im\big(\alpha(t)\big) \le \chi_3^{1/p}\big(\alpha(t)\big) \le \chi_3^{1/p}(t). \tag{2.34}$$

From (2.32), (2.34) and by using Theorem 1.6, we have

$$\begin{aligned}
\chi_3^{\Delta}(t) &= c^{\Delta}(t) + \alpha^{\Delta}(t)k_1\big(\sigma(t),\alpha(t)\big)\Im\big(\alpha(t)\big) + \int_a^{\alpha(t)} k_1^{\Delta}(t,s)\Im(s)\Delta s \\
&\quad + \int_a^b k_2^{\Delta}(t,s)\Im^p(s)\Delta s \\
&\le c^{\Delta}(t) + \alpha^{\Delta}(t)k_1\big(\sigma(t),\alpha(t)\big)\chi_3^{1/p}(t) + \int_a^{\alpha(t)} k_1^{\Delta}(t,s)\chi_3^{1/p}(s)\Delta s \\
&\quad + \int_a^b k_2^{\Delta}(t,s)\chi_3(s)\Delta s \\
&\le c^{\Delta}(t) + \left[\alpha^{\Delta}(t)k_1\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k_1^{\Delta}(t,s)\Delta s\right]\chi_3^{1/p}(t) \\
&\quad + \left(\int_a^b k_2^{\Delta}(t,s)\Delta s\right)\chi_3(t).
\end{aligned} \tag{2.35}$$

By applying Lemma 1.5 to (2.35), we get

$$\begin{aligned}
\chi_3^{\Delta}(t) &\le c^{\Delta}(t) + m_1\left[\alpha^{\Delta}(t)k_1\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k_1^{\Delta}(t,s)\Delta s\right]\chi_3(t) \\
&\quad + m_2\left[\alpha^{\Delta}(t)k_1\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k_1^{\Delta}(t,s)\Delta s\right] \\
&\quad + \left(\int_a^b k_2^{\Delta}(t,s)\Delta s\right)\chi_3(t) \\
&\le \left\{m_1\left[\alpha^{\Delta}(t)k_1\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k_1^{\Delta}(t,s)\Delta s\right]\right. \\
&\quad \left. + \int_a^b k_2^{\Delta}(t,s)\Delta s\right\}\chi_3(t) \\
&\quad + c^{\Delta}(t) + m_2\left[\alpha^{\Delta}(t)k_1\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k_1^{\Delta}(t,s)\Delta s\right] \\
&= \ell_3(t)\chi_3(t) + \Gamma_3(t).
\end{aligned} \tag{2.36}$$

Therefore, using Lemma (1.4) in (2.36), we get that

$$\chi_3(t) \le \chi_3(a)e_{\ell_3}(t,a) + \int_a^t \Gamma_3(s)e_{\ell_3}\big(t,\sigma(s)\big)\Delta s. \tag{2.37}$$

Combining (2.34) and (2.37) yields

$$\Im^p(t) \le \chi_3(a)e_{\ell_3}(t,a) + \int_a^t \Gamma_3(s)e_{\ell_3}\big(t,\sigma(s)\big)\Delta s. \tag{2.38}$$

From (2.33) and (2.38), we have

$$\begin{aligned}\chi_3(a) &\le c(a) + \int_a^b k_2(a,s)\bigg[\chi_3(a)e_{\ell_3}(s,a) \\ &\quad + \int_a^s \Gamma_3(\lambda)e_{\ell_3}\big(s,\sigma(\lambda)\big)\Delta\lambda\bigg]\Delta s \\ &\le c(a) + \chi_3(a)\int_a^b k_2(a,s)e_{\ell_3}(s,a)\Delta s \\ &\quad + \int_a^b k_2(a,s)\bigg(\int_a^s \Gamma_3(\lambda)e_{\ell_3}\big(s,\sigma(\lambda)\big)\Delta\lambda\bigg)\Delta s. \end{aligned} \tag{2.39}$$

Therefore, from (2.39) we obtain

$$\chi_3(a) \le \Lambda_3, \tag{2.40}$$

where Λ_3 is defined as in (2.30).

We obtain the desired inequality (2.29) by combining (2.38) and (2.40). The proof is complete. □

Remark 2.12 If we take $\mathbb{T} = \mathbb{R}$, $\alpha(t) = t$, $p = 1$, then, using relations (1.8), Theorem 2.6 reduces to [44, Theorem 1.5.2 (b_2)].

Remark 2.13 If we take $\mathbb{T} = \mathbb{R}$ and $\alpha(t) = t$, then, using relations (1.8), Theorem 2.6 reduces to [45, Theorem 2.3].

Remark 2.14 If we take $\mathbb{T} = \mathbb{R}$, then, using relations (1.8), Theorem 2.6 reduces to [9, Theorem 2.3].

As a special case of Theorem 2.11, if we take $\mathbb{T} = \mathbb{Z}$ and the delay function $\alpha(n) = n - \tau$, where $\tau > 0$, and so $\Delta\alpha(n) = 1 > 0$, then, using relations (1.9) and (1.12), we obtain the following completely new discrete result.

Corollary 2.15 *Assume that $\Im(n)$, $g(n)$, $c(n)$, and $\alpha(n)$ are nonnegative sequences defined for $t \in \mathbb{N}_0$, with $\Delta c(n) \ge 0$ and $k_1(n,s)$, $k_2(n,s)$, $\Delta k_1(n,s)$, $\Delta k_2(n,s)$ are nonnegative sequences defined on $E = \{(m,n) \in \mathbb{N}_0^2 : 0 \le n \le m < \infty\}$. If $\Im(n)$ satisfies the following delay discrete inequality*:

$$\Im^p(n) \le c(n) + \sum_{s=a}^{n-1} k_1(n,s-\tau)\Im(s-\tau) + \sum_{s=a}^{b-1} k_2(n,s)\Im^p(s),$$

then

$$\Im(n) \le \left\{\hat{\Lambda}_3\prod_{s=a}^{n-1}\big(1+\hat{\ell}_3(s)\big)\big) + \sum_{s=a}^{n-1}\hat{\Gamma}_3(s)\prod_{\lambda=s+1}^{n-1}\big(1+\hat{\ell}_3(\lambda)\big)\right\}^{1/p},$$

where

$$\hat{\Lambda}_3 = \frac{c(a) + \sum_{s=a}^{b-1} k_2(a,s)[\sum_{\lambda=a}^{s-1}\hat{\Gamma}_3(\lambda)\prod_{\lambda=\upsilon+1}^{s-1}(1+\hat{\ell}_3(\lambda))]}{1 - \sum_{s=a}^{b-1} k_2(a,s)\prod_{\lambda=a}^{s-1}(1+\hat{\ell}_3(\lambda)},$$

such that

$$\sum_{s=a}^{b-1} k_2(a,s) \prod_{\lambda=a}^{s-1} \bigl(1 + \hat{\ell}_3(\lambda)\bigr) < 1,$$

and

$$\begin{aligned}
\hat{\ell}_3(n) &= m_1 \left[\Delta(n-\tau) k_1(n+1, n-\tau) + \sum_{s=a}^{n-1} \Delta k_1(n,s) \right] \\
&\quad + \sum_{s=a}^{b-1} \Delta k_2(n,s),, \\
&= m_1 \left[k_1(n+1, n-\tau) + \sum_{s=a}^{n-1} \bigl[k_1(n+1,s) - k_1(n,s) \bigr] \right] \\
&\quad + \sum_{s=a}^{b-1} k_2(n+1,s) - k_2(n,s), \\
\hat{\Gamma}_3(t) &= \Delta c(n) + m_2 \left[\Delta(n-\tau) k_1(n+1, n-\tau) + \sum_{s=a}^{n-1} \Delta k_1(n,s) \right] \\
&= c(n+1) - c(n) + m_2 \left[k_1(n+1, n-\tau) + \sum_{s=a}^{n-1} \bigl[k_1(n+1,s) - k_1(n,s) \bigr] \right].
\end{aligned}$$

3 Applications

In this section, by using Theorem 2.11, we demonstrate the global existence of solutions for a class of nonlinear retarded dynamic integral equations of the form

$$\begin{aligned}
&\Im^p(t) = h(t) + \Upsilon\left(t, \int_a^{\alpha(t)} \Psi_1\bigl(s, \Im(s), k_1\bigr) \Delta s, \int_a^b \Psi_2\bigl(s, \Im^p(s), k_2\bigr) \Delta s \right), \\
&\Im^p(a) = \tilde{r},
\end{aligned} \tag{3.1}$$

where $\Upsilon \in C_{rd}([a,b]_{\mathbb{T}} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$.

Now, in the following theorem, we obtain the explicit estimates for the solution of (3.1).

Theorem 3.1 *Consider the retarded dynamic integral equation* (3.1), *and assume the following*:

$$\begin{aligned}
&\bigl|h(t)\bigr| \le c(t), \\
&\bigl|\Upsilon_1(t, u, \tilde{\nu})\bigr| \le |u| + |\tilde{\nu}|, \\
&|\Psi_1| \le k_1(t,s) \Im(s), \\
&|\Psi_2| \le k_2(t,s) \Im(s),
\end{aligned} \tag{3.2}$$

where $\Im$, c, h, $\in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}_+)$, *c is delta-differentiable on* $\mathbb{T}^k$ *with* $c^{\Delta}(t) \ge 0$, $k_1(t,s)$, $k_1^{\Delta}(t,s)$, $k_2(t,s)$, $k_2^{\Delta}(t,s) \in C_{rd}([a,b]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}, \mathbb{R}_+)$ *for* $a \le s \le t \le b$ *and* $p \ge 1$ *is a constant*.

Then we have the explicit bound estimation for the solution $\Im$ *of* (3.1) *as follows*:

$$\Im(t) \le \left\{ \Lambda_3 e_{\ell_3}(t,a) + \int_a^t \Gamma_3(s) e_{\ell_3}\big(t,\sigma(s)\big)\Delta s \right\}^{1/p}, \tag{3.3}$$

where

$$\Lambda_3 = \frac{c(a) + \int_a^b k_2(a,s)(\int_a^s \Gamma_3(\lambda)e_{\ell_3}(s,\sigma(\lambda))\Delta\lambda)\Delta s}{1 - \int_a^b k_2(a,s)e_{\ell_3}(s,a)\Delta s},$$

such that

$$\int_a^b k_2(s,a)e_{\ell_3}(s,a)\Delta s < 1,$$

and

$$\ell_3(t) = m_1\left[\alpha^\Delta(t)k_1\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k_1^\Delta(t,s)\Delta s\right] + \int_a^b k_2^\Delta(t,s)\Delta s,$$

$$\Gamma_3(t) = c^\Delta(t) + m_2\left[\alpha^\Delta(t)k_1\big(\sigma(t),\alpha(t)\big) + \int_a^{\alpha(t)} k_1^\Delta(t,s)\Delta s\right],$$

where m_1, m_2 *are defined as in Lemma* 1.5.

Proof From (3.1) and (3.2), we have

$$\big|\Im(t)\big|^p \le c(t) + \int_a^t k_1(t,s)\big|\Im(s)\big|\Delta s + \int_a^b k_2(t,s)\big|\Im(s)\big|^p\Delta s. \tag{3.4}$$

Now, applying Theorem 2.11 to inequality (3.4), we get

$$\Im(t) \le \left\{ \Lambda_3 e_{\ell_3}(t,a) + \int_a^t \Gamma_3(s) e_{\ell_3}\big(t,\sigma(s)\big)\Delta s \right\}^{1/p},$$

which is the desired estimation in (3.3). This completes the proof. □

4 Conclusion

First, we introduced Theorem 1.6 which was needed in the proofs of the rest of results. Second, we generalized a number of Gronwall–Pachpatte type inequalities, in two independent variables, to a general time scale. We applied our results to study the uniqueness and global existence of solutions for a class of nonlinear retarded Volterra–Fredholm dynamic integral equations.

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