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Some new dynamic inequalities with several functions of Hardy type on time scales

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Abstract

The aim of this article is to prove some new dynamic inequalities of Hardy type on time scales with several functions. Our results contain some results proved in the literature, which are deduced as limited cases, and also improve some obtained results by using weak conditions. In order to do so, we utilize Hölder's inequality, the chain rule, and the formula of integration by parts on time scales.

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1 Introduction

Let $z > 0, \alpha > 0$ be real numbers. If a function f is nonnegative, integrable over finite interval $(0, z)$, and the integral of f^α over $(0, \infty)$ converges, then Hardy's inequality [1] is given as follows:

$$\int_0^\infty \frac{1}{z^\alpha} \left(\int_0^z f(s)\,ds \right)^\alpha dz \leq \left(\frac{\alpha}{\alpha - 1} \right)^\alpha \int_0^\infty f^\alpha(z)\,dz, \tag{1}$$

and the equality holds iff $f = 0$ almost everywhere. On the other hand, the constant $(\frac{\alpha}{\alpha-1})^\alpha$ is optimal. Hardy proved this inequality in 1925 [1] and the discrete version in 1920 [2]. The discrete version of (1) is given by

$$\sum_{k=1}^\infty \left(\frac{1}{k} \sum_{j=1}^k \alpha_j \right)^\alpha \leq \left(\frac{\alpha}{\alpha - 1} \right)^\alpha \sum_{k=1}^\infty \alpha_k^\alpha, \quad (\alpha_k > 0, \alpha > 1). \tag{2}$$

These two inequalities are known in the literature as Hardy–Hilbert type inequalities. Since the invention of these inequalities, plenty of papers containing new proofs, various extensions, and generalizations have appeared. Inequality (1) was extended in [3], where it was proved that, if $\alpha > 1$ and $f > 0$ are integrable on $(0, \infty)$, then

$$\int_0^\infty \frac{1}{z^m} \left(\int_0^z f(s)\,ds \right)^\alpha dz \leq \left(\frac{\alpha}{m - 1} \right)^\alpha \int_0^\infty \frac{f^\alpha(z)}{z^{m-\alpha}}\,dz, \quad m > 1, \tag{3}$$

and

$$\int_0^\infty \frac{1}{z^m} \left(\int_z^\infty f(s)\,ds \right)^\alpha dz \le \left(\frac{\alpha}{1-m} \right)^\alpha \int_0^\infty \frac{f^\alpha(z)}{z^{m-\alpha}}\,dz, \quad m < 1. \tag{4}$$

The study of Hardy's inequalities (discrete and continuous) focused on the investigations of new inequalities with weighted functions. These results are of interest and importance in analysis because the size of weight classes cannot be improved and the weight conditions themselves are interesting. These inequalities have applications in diverse fields of mathematics (spectral theory, PDEs theory, ODEs theory, etc.). These inequalities lead to a large number of impressive connections between different branches of mathematics. This explored area of mathematical analysis generates the publications of various monographs and research papers. We refer the reader to [4–13] and the references therein.

Over the last decades a lot of considerable effort has been devoted to improve and generalize Hardy's inequalities (1) and (2). In what follows, we introduce some of these improvements that motivated the content of this paper. Levinson in [14] expanded inequality (1) using Jensen's inequality. Under the following conditions:

- λ, f are positive functions;
- there exists a constant $K > 0$ having the following property:

$$\alpha - 1 + \frac{\Lambda(z)\lambda'(z)}{\lambda^2(z)} \ge \frac{\alpha}{K} \quad \text{for all } z > 0,$$

where $\Lambda(z) = \int_0^z \lambda(s)\,ds$ and $\alpha > 1$;
- Φ is a real-valued convex function such that $\Phi(v) > 0$ for $v > 0$,

Levinson proved that

$$\int_0^\infty \Phi \left(\frac{1}{\Lambda(z)} \int_0^z f(s)\lambda(s)\,ds \right) dz \le K^\alpha \int_0^\infty \Phi\big(f(z)\big)\,dz. \tag{5}$$

In [15] Copson showed that if $1 < \gamma, 1 \le \alpha$, then

$$\int_0^\infty \frac{\lambda(z)}{\Lambda^\gamma(z)} \Phi^\alpha(z)\,dz \le \left(\frac{\alpha}{\gamma - 1} \right)^\alpha \int_0^\infty \frac{\lambda(z)f^\alpha(z)}{\Lambda^{\gamma-\alpha}(z)}\,dz, \tag{6}$$

where $\Lambda(z) = \int_0^z \lambda(\xi)\,d\xi$ and $\Phi(z) = \int_0^z f(\xi)\lambda(\xi)\,d\xi$. Hwang and Yang, in [16], extended inequality (5) and derived that if λ, q, f are nonnegative functions, $\alpha > 1$, and K is a positive constant having the following propriety:

$$\alpha - 1 + \frac{\Lambda(z)q'(z)}{q^2(z)\lambda(z)} \ge \frac{\alpha}{K}, \quad \forall z > 0, \tag{7}$$

then

$$\int_0^\infty \left(\frac{\Phi(z)}{\Lambda(z)} \right)^\alpha \lambda(z)\,dz \le K^\alpha \int_0^\infty f^\alpha(z)\lambda(z)\,dz, \tag{8}$$

where

$$\Lambda(z) = \int_0^z f(\xi)\lambda(\xi)\,d\xi, \Phi(z) = \int_0^z f(\xi)q(\xi)\lambda(\xi)\,d\xi.$$

The authors in [17] proved that, for $i = 1, \ldots, n$, if $\alpha_i > \beta_i > 0$, $m_i > \beta_i$ are real numbers with $\sum_{i=1}^{n} \beta_i = 1$ and $m = \sum_{i=1}^{n} m_i$, and if

$$1 + \left(\frac{\alpha_i}{m_i - \beta_i} \right) \frac{x f_i'(x)}{f_i(x)} \geq \frac{1}{\kappa_i} > 0 \quad \text{for some constants } \kappa_i > 0,$$

then

$$\int_0^\infty \prod_{i=1}^{n} [\phi_i^{\alpha_i}(x)] x^{-m} \, dx \leq \left(\prod_{j=1}^{n} C_j^{-\alpha_j} \right) \sum_{i=1}^{n} \beta_i C_i^{\frac{\alpha_i}{\beta_i}} \left[\frac{\alpha_i \kappa_i}{m_i - \beta_i} \right]^{\frac{\alpha_i}{\beta_i}} \int_0^\infty g_i^{\frac{\alpha_i}{\beta_i}}(x) x^{-\frac{m_i}{\beta_i}} \, dx \qquad (9)$$

for any constant $C_i > 0$, where

$$\phi_i(x) = \frac{1}{f_i(x)} \int_0^x \frac{f_i(s) g_i(s)}{s} \, ds, \quad x \in (0, \infty).$$

A time scale is a closed subset of real numbers denoted by $\mathbb{T}$. The main objective is to demonstrate some results in dynamic inequalities where the involved functions are defined on an arbitrary time scale $\mathbb{T}$ domain. These results involve the classical discrete and continuous inequalities ($\mathbb{T} = \mathbb{N}$, $\mathbb{T} = \mathbb{R}$) and can be expanded to different inequalities on different time scales like $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$, $\mathbb{T} = h\mathbb{N}$, $h > 0$, etc.

For wholeness, the main results of dynamic inequalities inspiring the subject of this article are mentioned. Using Elliott's technique [18], Řehak in [19] found the time scale version of Hardy's inequality. Particularly, Řehak derived, for $\alpha > 1$ and f a positive function such that $\int_a^\infty (f(s))^\alpha \, \Delta s < \infty$, that

$$\int_a^\infty \frac{1}{(\check{\sigma}(\eta) - a)^\alpha} \left(\int_a^{\check{\sigma}(\eta)} f(s) \, \Delta s \right)^\alpha \Delta \eta \leq \left(\frac{\alpha}{\alpha - 1} \right)^\alpha \int_a^\infty f^\alpha(s) \, \Delta s. \qquad (10)$$

Additionally, if $\nu(s)/s \to 0$ as $s \to \infty$, then $(\frac{\alpha}{\alpha-1})^\alpha$ is the optimal constant. Nevertheless, to determine whether the constant in inequality (10) is optimal also on all time scales or just those fulfilling the condition $\lim_{s \to \infty} (\nu(s)/s) = 0$ is still an open problem.

Özkan and Yildirim [20] found a novel inequality with weight functions that can be thought of as a time scale Hardy–Knopp type inequality proved by Kaijser et al. in [21] of the form

$$\int_0^\infty \Phi \left(\frac{1}{z} \int_0^z f(s) \, ds \right) \frac{dz}{z} \leq \int_0^\infty \frac{\Phi(f(z))}{z} \, dz, \qquad (11)$$

where Φ is a convex function on $(0, \infty)$.

Authors of [22] derived the time scale analogue of (3), that is,

$$\int_0^\infty \frac{1}{\eta^\gamma} \left(\int_0^{\check{\sigma}(\eta)} f(s) \, ds \right)^\alpha \Delta \eta \leq \left(\frac{\alpha K^\gamma}{\gamma - 1} \right)^\alpha \int_0^\infty \frac{f^\alpha(\eta)}{\eta^{\gamma - \alpha}} \, \Delta \eta, \qquad (12)$$

where $\gamma > 1$, $\alpha > 1$ with the existence of a positive constant K having the following propriety: $1/K \leq \frac{s}{\check{\sigma}(s)}$ for $s \in \mathbb{T}$.

The authors in [23] generalized inequality (10) and showed that if $\gamma, \alpha > 1$, then

$$\int_c^\infty \frac{1}{(\check{\sigma}(\xi) - c)^\gamma} \left(\int_c^{\check{\sigma}(\xi)} f(s) \, \Delta s \right)^\alpha \Delta \xi$$

$$\leq \left(\frac{\alpha}{\gamma - 1} \right)^{\alpha} \int_{c}^{\infty} f^{\alpha}(\xi) \frac{(\check{\sigma}(\xi) - c)^{(\alpha - 1)\gamma}}{(\xi - c)^{\alpha(\gamma - 1)}} \, \Delta \xi. \tag{13}$$

Saker et al. [24] proved Copson inequalities (6) on time scales. In particular, it has been proved that if $\gamma, \alpha > 1$, then

$$\int_{c}^{\infty} \left(\Phi^{\check{\sigma}}(\xi) \right)^{\alpha} \frac{\lambda(\xi)}{(\Lambda^{\check{\sigma}}(\xi))^{\gamma}} \, \Delta \xi \leq \left(\frac{\alpha}{\gamma - 1} \right)^{\alpha} \int_{c}^{\infty} f^{\alpha}(\xi) \lambda(\xi) \frac{(\Lambda^{\check{\sigma}}(\xi))^{\gamma(\alpha - 1)}}{(\Lambda(\xi))^{\alpha(\gamma - 1)}} \, \Delta \xi, \tag{14}$$

where

$$\Lambda(\xi) = \int_{c}^{\xi} \lambda(\eta) \, \Delta \eta, \Phi(\xi) = \int_{c}^{\eta} f(\eta) \lambda(\eta) \, \Delta \eta.$$

In addition, some generalizations of the inequalities of Bennett and Leindler type on time scales have been proved. The authors demonstrated that if $1 > \gamma > 0, 1 \leq \alpha$, then

$$\int_{a}^{\infty} \left(\Phi^{\check{\sigma}}(\xi) \right)^{\alpha} \frac{\lambda(\xi)}{(\Omega(\xi))^{\gamma}} \, \Delta \xi \leq \left(\frac{\alpha}{1 - \gamma} \right)^{\alpha} \int_{a}^{\infty} f^{\alpha}(\xi) \frac{\lambda(\xi)}{(\Omega(\xi))^{\gamma - \alpha}} \, \Delta \xi \tag{15}$$

with

$$\Omega(\xi) = \int_{\xi}^{\infty} \lambda(\eta) \, \Delta \eta.$$

The aim of this article is to prove some new Hardy-type inequalities on time scales involving many functions which generalize and improve some of the above results and also improve some other already proved results in [25]. The manuscript is arranged as follows: In the preliminaries section, we recall a few elementary results and definitions concerning the delta calculus on time scale. In the main results section, we prove our results that cover a wide spectrum of previously proved inequalities.

2 Preliminaries

This section is devoted to presenting some basic definitions as well as some basic results on delta calculus on time scales that will be used in the sequel; for more details, see [26]. The backward jump operator and the forward jump operator are defined by $\varrho(s) := \sup\{\eta \in \mathbb{T} : s > \eta\}$ and $\check{\sigma}(s) := \inf\{\eta \in \mathbb{T} : \eta > s\}$, respectively, where $\sup \emptyset = \inf \mathbb{T}$. The forward graininess function $\nu : \mathbb{T} \to [0, \infty)$ is given by $\nu(s) := \check{\sigma}(s) - s$. A point $s \in \mathbb{T}$ is called:

- right-dense if $\check{\sigma}(s) = s$,
- left-dense if $\inf \mathbb{T} < s$ and $\varrho(s) = s$,
- right-scattered if $s < \check{\sigma}(s)$,
- left-scattered if $s > \varrho(s)$.

$u : \mathbb{T} \to \mathbb{R}$ is a right-dense continuous (noted rd-continuous) function if u is continuous at right-dense points and its left-hand limits are finite at left-dense points in $\mathbb{T}$. We denote by $C_{rd}(\mathbb{T})$ the set of rd-continuous functions.

Without loss of generality, we assume that $\sup \mathbb{T}$ is equal to ∞. We note $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$ the time scale interval. Throughout this paper, $\mathbb{T}$ is provided with the topology induced by the standard topology on $\mathbb{R}$ (see, for instance, [26]).

If u is defined on $\mathbb{T}$, then as an abbreviation $u(\check{\sigma}(t)) = u^{\check{\sigma}}(t)$. The derivative of UV and U/V of two delta-differentiable functions u and v is given by

$$(UV)^\Delta = U^{\check{\sigma}} V^\Delta + U^\Delta V = U^\Delta V^{\check{\sigma}} + UV^\Delta, \qquad \left(\frac{U}{V}\right)^\Delta = \frac{VU^\Delta - UV^\Delta}{V^{\check{\sigma}} V}. \qquad (16)$$

On the other hand, the Δ-integral on $\mathbb{T}$ is characterized by the following: If $\Gamma^\Delta(s) = \gamma(s)$, then $\int_a^s \gamma(t)\,\Delta t = \Gamma(s) - \Gamma(a)$ is the Cauchy Δ-integral of γ. The discrete time scale integration formula is given by

$$\int_a^b \gamma(\xi)\,\Delta\xi = \sum_{\xi \in [a,b)} \nu(\xi)\gamma(\xi),$$

while the infinite integral is defined as $\int_a^\infty \gamma(\xi)\,\Delta\xi = \lim_{b\to\infty} \int_a^b \gamma(\xi)\,\Delta\xi$. The chain rule for functions $U : \mathbb{R} \to \mathbb{R}$, which is continuously differentiable, and $V : \mathbb{T} \to \mathbb{R}$, which is delta-differentiable, is given by

$$(U \circ V)^\Delta(\eta) = U'\big(V(c)\big)v^\Delta(\eta) \quad \text{for } c \in \big[\eta, \check{\sigma}(\eta)\big],$$

and this rule leads to the useful form

$$\big(V^\alpha\big)^\Delta(\eta) = \alpha V^\Delta(\eta)V^{\alpha-1}(\xi) \quad \text{for } c \in \big[\eta, \check{\sigma}(\eta)\big]. \qquad (17)$$

Another formula to the chain rule is given by

$$(U \circ V)^\Delta(\eta) = \int_0^1 U'\big(V(\eta) + \xi\nu(\eta)V^\Delta(\eta)\big)\,d\xi\, V^\Delta(\eta),$$

which provides us with the following useful form:

$$\big(V^\alpha\big)^\Delta(\eta) = \alpha \int_0^1 \big(\xi V^{\check{\sigma}}(\eta) + (1-\xi)V(\eta)\big)^{\alpha-1}\,d\xi\, V^\Delta(\eta).$$

For $U, V \in C_{rd}(\mathbb{T})$ and $t_1, t_2 \in \mathbb{T}$, the following expression

$$\int_{t_1}^{t_2} U^{\check{\sigma}}(s)V^\Delta(s)\,\Delta s = UV(t_2) - UVg(t_1) - \int_{t_1}^{t_2} U^\Delta(s)V(s)\,\Delta s \qquad (18)$$

is known as the integration by parts formula.

Hölder's inequality on time scales is written as follows:

$$\int_{t_1}^{t_2} |U(s)V(s)|\,\Delta s \leq \left(\int_{t_1}^{t_2} |U(s)|^q\,\Delta s\right)^{\frac{1}{q}} \left(\int_{t_1}^{t_2} |V(s)|^p\,\Delta s\right)^{\frac{1}{p}},$$

where $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Throughout this paper, we make the following assumptions:

- the integrals exist (finite),
- $\mathbb{T}$ denotes the time scale set and $a \in [a, \infty)_{\mathbb{T}} := [a, \infty) \cap \mathbb{T}$,
- all functions appearing in the assumptions of theorems are positive and rd-continuous on $[a, \infty)_{\mathbb{T}}$.

3 Main results

First, we prove some new Hardy-type inequalities with several functions which cover a wide spectrum of previously proved inequalities. Then, we improve some inequalities showed in [25] by removing the imposed conditions on the functions. It will be convenient to use the convention $0.\infty = 0$ and $0/0 = 0$. To attain the first objective in this paper, we define the operators Λ and ϕ by

$$\Lambda(\xi) = \int_a^\xi \lambda(\eta)\,\Delta\eta \quad \text{and} \quad \phi(\xi) = h(\xi)\int_a^\xi g(\eta)f(\eta)\,\Delta\eta, \quad \forall \xi \in [a,\infty)_\mathbb{T}.$$

Theorem 1 *Let h be nondecreasing on $[a,\infty)_\mathbb{T}$ and $r > 1$, $\alpha > \beta > 0$ be real numbers. If there exists a positive constant κ having the following propriety*:

$$\lambda(\xi) - \left(\frac{\alpha}{\beta(r-1)}\right)\frac{h^\Delta \Lambda(\xi)}{h(\xi)} \geq \frac{1}{\kappa} > 0, \tag{19}$$

then

$$\int_a^\infty \Lambda^{-r}(\xi)\big[\phi^{\check{\sigma}}(\xi)\big]^{\frac{\alpha}{\beta}}\,\Delta\xi \leq \left[\frac{p\kappa}{\beta(r-1)}\right]^{\frac{\beta}{\beta}}\int_a^\infty \Lambda^{\frac{\alpha}{\beta}-r}(\xi)\big[h^{\check{\sigma}}gf\big]^{\frac{\alpha}{\beta}}(\xi)\,\Delta\xi. \tag{20}$$

Proof We define ν for any $\xi \in [a,\infty)_\mathbb{T}$ by

$$\nu(\xi) = \int_\xi^\infty \Lambda^{-r}(\eta)\lambda(\eta)\,\Delta\eta.$$

Using the formula of integration by parts (18), $\phi(a) = 0$, and $\nu(\infty) = 0$, we get

$$\int_a^\infty \lambda(\xi)\Lambda^{-r}(\xi)\big[\phi^{\check{\sigma}}(\xi)\big]^{\frac{\alpha}{\beta}}\,\Delta\eta = \int_a^\infty \big(\phi^{\frac{\alpha}{\beta}}(\xi)\big)^\Delta \nu(\xi)\,\Delta\eta.$$

By utilizing the chain rule (17), we observe that

$$\big(\phi^{\frac{\alpha}{\beta}}(\xi)\big)^\Delta = \frac{\alpha}{\beta}\phi^\Delta(\xi)\phi^{\frac{\alpha}{\beta}-1}(c) \quad \text{for } c \in \big[\xi,\check{\sigma}(\xi)\big] \tag{21}$$

and

$$\big(\Lambda^{1-r}(\xi)\big)^\Delta = (1-r)\Lambda^{-r}(c)\Lambda^\Delta(\xi) = (1-r)\Lambda^{-r}(c)\lambda(\xi), \quad c \in \big[\xi,\check{\sigma}(\xi)\big]. $$

Apply the derivative of the product formula (16) on $\phi(\xi)$ to obtain that

$$\phi^\Delta(\xi) = h^{\check{\sigma}}(\xi)f(\xi)g(\xi) + \frac{(\phi h^\Delta)(\xi)}{h(\xi)}. \tag{22}$$

Employing the assumption h is nondecreasing, we conclude that $\phi^\Delta(\xi) \geq 0$, and thus

$$\big(\phi^{\frac{\alpha}{\beta}}(\xi)\big)^\Delta \leq \frac{\alpha}{\beta}\big[\phi^{\check{\sigma}}(\xi)\big]^{\frac{\nu p}{\beta}-1}\phi^\Delta(\xi). \tag{23}$$

In addition, as $\Lambda^\Delta(\xi) = \lambda(\xi)$ is positive, we get

$$\left(\Lambda^{1-r}(\xi)\right)^\Delta \le (1-r)\lambda(\xi)\Lambda^{-r}(\xi),$$

integrating both sides gives that

$$\int_\xi^\infty \left(\Lambda^{1-r}(s)\right)^\Delta \Delta s \le (1-r)\int_\xi^\infty \Lambda^{-r}(s)\lambda(s)\,\Delta s. \qquad (24)$$

Hence, we have

$$\nu(\xi) \le \frac{1}{1-r}\int_\xi^\infty \left(\Lambda^{1-r}(s)\right)^\Delta \Delta s = \frac{1}{1-r}\left(\Lambda^{1-r}(\infty) - \Lambda^{1-r}(\xi)\right)$$

$$= \frac{-1}{1-r}\left(\Lambda^{1-r}(\xi) - \Lambda^{1-r}(\infty)\right) \le \frac{\Lambda^{1-r}(\xi)}{r-1}. \qquad (25)$$

By combining (22), (23), and (25), we find that

$$\int_a^\infty \Lambda^{-r}(\xi)\left[\phi^{\check\sigma}(\xi)\right]^{\frac{\alpha}{\beta}}\left(\lambda(\xi) - \frac{\alpha}{\beta(r-1)}\frac{\Lambda(\xi)h^\Delta(\xi)}{h(\xi)}\right)\Delta\eta$$

$$\le \frac{\alpha}{\beta(r-1)}\int_a^\infty \Lambda^{1-r}(\xi)\left[\phi^{\check\sigma}(\xi)\right]^{\frac{\alpha}{\beta}-1}h^{\check\sigma}(\xi)g(\xi)f(\xi)\,\Delta\eta.$$

Utilizing assumption (19) leads to

$$\int_a^\infty \Lambda^{-r}(\xi)\left[\phi^{\check\sigma}(\xi)\right]^{\frac{\alpha}{\beta}}\frac{1}{\kappa}\,\Delta\eta$$

$$\le \frac{\alpha}{\beta(r-1)}\int_a^\infty \left[\Lambda^{-\frac{r(\alpha-\beta)}{\alpha}}(\xi)\left[\phi^{\check\sigma}(\xi)\right]^{\frac{\alpha-\beta}{\beta}}\right]\left[\Lambda^{1-\frac{r\beta}{\alpha}}(\xi)g(\xi)f(\xi)h^{\check\sigma}(\xi)\right]\Delta\eta.$$

Applying Hölder's inequality with exponents $\frac{\alpha}{\beta}$ and $\frac{\alpha}{(\alpha-\beta)}$ produces

$$\int_a^\infty \Lambda^{-r}(\xi)\left[\phi^{\check\sigma}(\xi)\right]^{\frac{\alpha}{\beta}}\,\Delta\xi$$

$$\le \frac{\alpha\kappa}{\beta(r-1)}\left[\int_a^\infty \Lambda^{-r}(\xi)\left[\phi^{\check\sigma}(\xi)\right]^{\frac{\alpha}{\beta}}\,\Delta\xi\right]^{\frac{\alpha-\beta}{\alpha}}$$

$$\times \left[\int_a^\infty \Lambda^{\frac{\alpha}{\beta}-r}(\xi)\left(h^{\check\sigma}(\xi)g(\xi)f(\xi)\right)^{\frac{\alpha}{\beta}}\,\Delta\xi\right]^{\frac{\beta}{\alpha}}. \qquad (26)$$

Inequality (26) directly yields

$$\int_a^\infty \Lambda^{-r}(\xi)\left[\phi^{\check\sigma}(\xi)\right]^{\frac{\alpha}{\beta}}\,\Delta\xi \le \left[\frac{\alpha\kappa}{\beta(r-1)}\right]^{\frac{\alpha}{\beta}}\int_a^\infty \Lambda^{\frac{\alpha}{\beta}-r}(\xi)\left(h^{\check\sigma}(\xi)g(\xi)f(\xi)\right)^{\frac{\alpha}{\beta}}\,\Delta\xi. \qquad \square$$

Remark 1 Let $\mathbb{T} = \mathbb{R}$. In Theorem 1, setting $h(\xi) = \lambda(\xi) = g(\xi) = 1, \beta = 1, \kappa = 1, a = 0$ leads to inequality (1).

Remark 2 In Theorem 1, by putting $h(\xi) = \lambda(\xi) = g(\xi) = 1, \beta = 1$, and $r = \alpha$ in (20), we obtain

$$\int_a^\infty (\xi - a)^{-\alpha} \left[\int_a^{\check\sigma(\xi)} f(\eta) \, \Delta\eta \right]^p \Delta\xi \le \left[\frac{\alpha\kappa}{\alpha - 1} \right]^\alpha \int_a^\infty f^\alpha(\xi) \, \Delta\xi.$$

Since $\xi \le \check\sigma(\xi)$, then $(\xi - a)^{-\alpha} \ge (\check\sigma(\xi) - a)^{-\alpha}$, which implies along with the above inequality that

$$\int_a^\infty \left[\frac{1}{\check\sigma(s) - a} \int_a^{\check\sigma(s)} f(\xi) \, \Delta\xi \right]^\alpha \Delta s \le \left[\frac{\alpha\kappa}{\alpha - 1} \right]^\alpha \int_a^\infty f^\alpha(\xi) \, \Delta\xi.$$

Letting $\kappa = 1$, we obtain inequality (10).

In order to prove our next result, which is a new generalization of a Copson-type inequality, we define

$$\Phi(\xi) := \int_a^\xi g(\eta)\lambda(\eta) \, \Delta\eta \quad \text{for all } \xi \in [a,b]_\mathbb{T},$$

where $\mathbb{T}$ is a time scale, and assume that there exists $m \ge 1$ such that

$$\Phi^{\check\sigma}(s) \le m\Phi(s), \quad \forall s \in [a,b]_\mathbb{T}. \tag{27}$$

Theorem 2 *Let $a, b \in \mathbb{T}, \alpha \ge \beta \ge 1$, and $\alpha > m$. If g is an increasing function, then*

$$\int_a^b \lambda(\xi)\left(\Lambda^{\check\sigma}(\xi)\right)^{\frac{\beta}{\alpha} - 1}\left(F^{\check\sigma}(\xi)\right)^\beta \Delta\xi \le \left(\frac{m\alpha}{\alpha - m} \right)^\beta \int_a^b \left(\Lambda^{\check\sigma}(\xi)\right)^{\frac{p-q}{\alpha(\beta-1)}} \lambda(\xi) g^\beta(\xi) \, \Delta\xi, \tag{28}$$

where $F(\xi) := \frac{\Phi(\xi)}{\Lambda(\xi)}$ and Λ is defined as in Theorem 1.

Proof For any $\xi \in [a,b]_\mathbb{T}$, we define u, v by

$$u(\xi) = \int_a^\xi \left(\Lambda^{\check\sigma}(\eta)\right)^{\frac{\beta}{\alpha} - 1}\lambda(\eta) \, \Delta\eta \quad \text{and} \quad \upsilon(\xi) = F^\beta(\xi).$$

Integrate the L.H.S of (28) by parts to obtain

$$\int_a^b \lambda(\xi)\left(\Lambda^{\check\sigma}(\xi)\right)^{\frac{\beta}{\alpha} - 1}\left(F^{\check\sigma}(\xi)\right)^q \Delta\xi = u(b)F^\beta(b) + \int_a^b \left(F^\beta(\xi)\right)^\Delta \left(-u(\xi)\right) \Delta\xi. \tag{29}$$

By the chain rule, we have (note that $(\beta/\alpha) - 1 < 0$ and $\Lambda^\Delta(\xi) = \lambda(\xi) > 0$)

$$\left(\Lambda^{\beta/\alpha}(\xi)\right)^\Delta = \frac{\beta}{\alpha} \int_0^1 \left[\xi\Lambda^{\check\sigma}(\xi) + (1 - \xi)\Lambda(\xi)\right]^{\frac{\beta}{\alpha} - 1} d\xi \, \Lambda^\Delta(\xi)$$

$$\ge \frac{\beta}{\alpha}\left(\Lambda^{\check\sigma}(\xi)\right)^{(\frac{\beta}{\alpha} - 1)}\lambda(\xi).$$

Therefore

$$-u(\xi) = -\int_a^\xi \left(\Lambda^{\check\sigma}(\tau)\right)^{\frac{\beta}{\alpha} - 1}\lambda(\tau) \, \Delta\tau \ge -\frac{\alpha}{\beta}\Lambda^{\frac{\beta}{\alpha}}(\xi) \ge -\frac{\alpha}{\beta}\left(\Lambda^{\check\sigma}(\xi)\right)^{\frac{\beta}{\alpha}}. \tag{30}$$

Combining (29) and (30) gives

$$\int_a^b \lambda(\xi)\bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{\frac{\beta}{\alpha}-1}\bigl(F^{\check{\sigma}}(\xi)\bigr)^\beta \Delta\xi \geq \frac{\alpha}{\beta}\int_a^b \bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{\frac{\beta}{\alpha}}\bigl(-F^\beta(\xi)\bigr)^\Delta \Delta\xi + u(b)F^\beta(b). \quad (31)$$

From the quotient rule, we get (noting that g increasing and so $\Phi(\xi) < \Lambda(\xi)g(\xi)$) that

$$F^\Delta(\xi) = \left(\frac{\Phi(\xi)}{\Lambda(\xi)}\right)^\Delta = \frac{(\Lambda\lambda g - \Phi\lambda)(\xi)}{(\Lambda\Lambda^{\check{\sigma}})(\xi)} = \frac{\lambda(\xi)(\Lambda g - \Phi)(\xi)}{(\Lambda\Lambda^{\check{\sigma}})(\xi)} > 0.$$

By the chain rule (17), inequality (27), and $\Lambda^{\check{\sigma}} \geq \Lambda$, we have: for $d \in [\xi, \check{\sigma}(\xi)]$,

$$-\bigl(F^\beta(\xi)\bigr)^\Delta = -\beta F^{\beta-1}(d)F^\Delta(\xi) \geq -\beta\bigl(F^{\check{\sigma}}(\xi)\bigr)^{\beta-1}F^\Delta(\xi) \quad (32)$$

$$= q\bigl(F^{\check{\sigma}}(\xi)\bigr)^{\beta-1}\frac{\lambda(\xi)\Phi(\xi)}{\Lambda(\xi)\Lambda^{\check{\sigma}}(\xi)} - \frac{\lambda(\xi)g(\xi)}{\Lambda^{\check{\sigma}}(\xi)}\beta\bigl(F^{\check{\sigma}}(\xi)\bigr)^{\beta-1} \quad (33)$$

$$\geq \frac{\beta}{m}\frac{\lambda(\xi)}{\Lambda^{\check{\sigma}}(\xi)}\bigl(F^{\check{\sigma}}(\xi)\bigr)^\beta - \frac{\lambda(\xi)g(\xi)}{\Lambda^{\check{\sigma}}(\xi)}\beta\bigl(F^{\check{\sigma}}(\xi)\bigr)^{\beta-1}. \quad (34)$$

Inequalities (34) and (31) give after some simplifications

$$\left(\frac{\alpha}{m}-1\right)\int_a^b \lambda(\xi)\bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{(\frac{\beta}{\alpha}-1)}\bigl(F^{\check{\sigma}}(\xi)\bigr)^\beta \Delta\xi$$

$$\leq \alpha\int_a^b \bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{(\frac{\beta}{\alpha}-1)}\lambda(\xi)g(\xi)\bigl(F^{\check{\sigma}}(\xi)\bigr)^{\beta-1} \Delta\xi.$$

Apply Hölder's inequality with exponents β and $\beta/(\beta-1)$ to obtain

$$\left(\frac{p}{m}-1\right)\int_a^b \lambda(\xi)\bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{(\frac{\beta}{\alpha}-1)}\bigl(F^{\check{\sigma}}(\xi)\bigr)^\beta \Delta\xi$$

$$\leq \alpha\left\{\int_a^b \bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{\frac{\alpha-\beta}{\alpha(\beta-1)}}\lambda(\xi)g^\beta(\xi) \Delta\xi\right\}^{\frac{1}{\beta}} \times \left\{\int_a^b \bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{(\frac{\beta}{\alpha}-1)}\lambda(\xi)\bigl(F^{\check{\sigma}}(\xi)\bigr)^q \Delta\xi\right\}^{\frac{\beta-1}{\beta}}.$$

This gives (noting that $(\alpha/m)-1>0$) that

$$\left\{\int_a^b \lambda(\xi)\bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{\frac{\beta}{\alpha}-1}\bigl(F^{\check{\sigma}}(\xi)\bigr)^q \Delta\xi\right\}^{\frac{1}{\beta}}$$

$$\leq \left(\frac{m\alpha}{\alpha-m}\right)\left\{\int_a^b \bigl(\Lambda^{\check{\sigma}}(\xi)\bigr)^{\frac{\beta}{\alpha}-1}\lambda(\xi)g^\alpha(\xi) \Delta\xi\right\}^{\frac{1}{\beta}}. \qquad \square$$

With simple modifications on the proof of Theorem 1, the following result holds.

Lemma 1 *Let h be a nondecreasing function on $[a,\infty)_{\mathbb{T}}$. If there exist a positive constant κ and real numbers $r > 1$, $\alpha > \beta > 0$ such that*

$$1 - \left(\frac{\alpha}{(r-1)\beta}\right)\frac{\Lambda(\xi)h^\Delta(\xi)}{h(\xi)\lambda(\xi)} \geq \frac{1}{\kappa} > 0 \quad \textit{for all } \xi \in [a,\infty)_{\mathbb{T}},$$

then

$$\int_a^\infty \frac{\lambda(\xi)}{\Lambda^r(\xi)} \big[\phi^{\check\sigma}(\xi)\big]^{\frac{\alpha}{\beta}} \, \Delta\xi \le \left[\frac{\alpha\kappa}{\beta(r-1)}\right]^{\frac{\alpha}{\beta}} \int_a^\infty \frac{\lambda^{1-\frac{\alpha}{\beta}}(\xi)}{\Lambda^{r-\frac{\alpha}{\beta}}(\xi)} \big(h^{\check\sigma} g f\big)^{\frac{\alpha}{\beta}}(\xi) \, \Delta\xi.$$

Remark 3 In Lemma 1, taking $g(\xi) = \lambda(\xi), h(\xi) = 1$, and $\beta = 1 = \kappa$ leads to $\Lambda(\xi) = \int_a^\xi \lambda(\eta) \, \Delta\eta$, $\phi(\xi) = \int_a^\xi f(\eta)\lambda(\eta) \, \Delta\eta$. Therefore,

$$\int_a^\infty \frac{\lambda(\xi)}{\Lambda^r(\xi)} \left(\int_a^{\check\sigma(\xi)} f(s)\lambda(s) \, \Delta s\right)^\alpha \Delta\xi \le \left(\frac{\alpha}{r-1}\right)^\alpha \int_a^\infty \frac{\lambda(\xi)}{\Lambda^{r-\alpha}(\xi)} f^\alpha(\xi) \, \Delta\xi. \qquad (35)$$

In inequality (35) if $\mathbb{T} = \mathbb{R}$, we obtain inequality (6). On the other hand, since $\Lambda(\xi) \le \Lambda^{\check\sigma}(\xi)$ (Λ is increasing), $\alpha > 1$, and $r > 1$, then

$$\Lambda^{\alpha-r}(\xi) \le \frac{(\Lambda^{\check\sigma}(\xi))^{r(\alpha-1)}}{(\Lambda(\xi))^{\alpha(r-1)}},$$

therefore, inequality (35) becomes

$$\int_a^\infty \frac{\lambda(\xi)}{(\Lambda^{\check\sigma}(\xi))^r} \left(\int_a^{\check\sigma(\xi)} f(s)\lambda(s) \, \Delta s\right)^\alpha \Delta\xi \le \left(\frac{\alpha}{r-1}\right)^\alpha \int_a^\infty f^\alpha(\xi) \frac{(\Lambda^{\check\sigma}(\xi))^{r(\alpha-1)}}{(\Lambda(\xi))^{\alpha(r-1)}} \lambda(\xi) \, \Delta\xi,$$

and this is inequality (14).

Remark 4 In Lemma 1, choosing $g(\xi) = h(\xi) = \lambda(\xi) = 1$, and $\beta = 1 = \kappa$ leads to $\phi(\xi) = \int_a^\xi f(\eta) \, \Delta\eta$ and $\Lambda(\xi) = \xi - a$. This implies that

$$\int_a^\infty \left(\frac{1}{\xi-a}\right)^r \left(\int_a^{\check\sigma(\xi)} f(\eta) \, \Delta\eta\right)^\alpha \Delta\xi \le \left(\frac{\alpha}{r-1}\right)^\alpha \int_a^\infty (\xi-a)^{\alpha-r} f^\alpha(\xi) \, \Delta\xi.$$

Since $\xi \le \check\sigma(\xi)$, then $\frac{1}{\check\sigma(\xi)-a} \le \frac{1}{\xi-a}$, which along with the above inequality gives that

$$\int_a^\infty \frac{1}{(\check\sigma(\xi)-a)^r} \left(\int_a^{\check\sigma(\xi)} f(\eta) \, \Delta\eta\right)^\alpha \Delta\xi \le \left(\frac{\alpha}{r-1}\right)^\alpha \int_a^\infty f^\alpha(\xi) \frac{(\check\sigma(\xi)-a)^{r(\alpha-1)}}{(\xi-a)^{\alpha(r-1)}} \, \Delta\xi,$$

and this is inequality (13).

   In the following theorem, we try to give an answer to the remark "It would be interesting to prove some new results by excluding the condition that has been proposed on $q(t)$" given in [25, Remark 2.16]. So, we derive some already obtained results (see [25]) that are related to Hardy's inequality. However, we soften the conditions imposed on the functions in [25]. In other words, we prove the following result assuming that the function $q(t)$ is bounded, while the authors in [25] proposed it is an increasing function. We introduce new operators: for $\xi \in [a,\infty)_{\mathbb{T}}$,

$$\phi(\xi) = \int_\xi^\infty f(\eta)q(\eta)\vartheta(\eta) \, \Delta\eta \quad \text{and} \quad \Theta(\xi) = \int_a^\xi q(\eta)\vartheta(\eta) \, \Delta\eta$$

with $\int_a^\infty \vartheta(s)(\Theta^{\check\sigma}(s))^{-\gamma} \, \Delta s < \infty$.

Theorem 3 *Let $M > m > 0, \alpha > 1, 0 \leq \gamma < 1$ be real numbers, and a bounded function $q(\xi)$, i.e., $m \leq q(\xi) \leq M$. Then*

$$\int_a^\infty \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma}\phi^\alpha(\xi)\,\Delta\xi \leq \left(\frac{Mp}{m(1-\gamma)}\right)^\alpha \int_a^\infty \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{\alpha-\gamma}f^\alpha(\xi)\,\Delta\xi.$$

Proof It follows from using integration by parts on time scales with

$$u(\xi) = \phi^\alpha(\xi), \qquad v^\Delta(\xi) = \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma} \quad \text{and} \quad v(\xi) = \int_a^\xi \vartheta(s)\big(\Theta^{\check{\sigma}}(s)\big)^{-\gamma}\,\Delta s,$$

and $\phi(\infty) = v(a) = 0$, that

$$\int_a^\infty \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma}\phi^\alpha(\xi)\,\Delta\xi = v(\xi)\phi^\alpha(\xi)\big|_a^\infty - \int_a^\infty v^{\check{\sigma}}(\xi)\big(\phi^p(\xi)\big)^\Delta\,\Delta\xi$$

$$= \int_a^\infty v^{\check{\sigma}}(\xi)\big(-\phi^\alpha(t)\big)^\Delta\,\Delta\xi.$$

Applying the chain rule gives

$$\big(\Theta^{1-\gamma}(\xi)\big)^\Delta = (1-\gamma)\int_0^1 \big[\xi\Theta^{\check{\sigma}}(\xi) + (1-\xi)\Theta(t)\big]^{-\gamma}\,d\xi\,\Theta^\Delta(\xi)$$

$$= (1-\gamma)\int_0^1 \frac{1}{[\xi\Theta^{\check{\sigma}}(\xi) + (1-\xi)\Theta(\xi)]^\gamma}\,d\xi\,\Theta^\Delta(\xi).$$

Observing that $\Theta^\Delta(\xi) = q(\xi)\vartheta(\xi) > 0$ enables us to write

$$\big(\Theta^{1-\gamma}(\xi)\big)^\Delta \geq (1-\gamma)\int_0^1 \frac{1}{[\xi\Theta^{\check{\sigma}}(\xi) + (1-\xi)\Theta^{\check{\sigma}}(\xi)]^\gamma}\,d\xi\lambda(\xi)q(\xi)$$

$$= (1-\gamma)\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma}\vartheta(\xi)q(\xi).$$

Using the assumption $q(\xi) \geq m$ returns

$$\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma}\vartheta(\xi) \leq \frac{1}{m(1-\gamma)}\big(\Theta^{1-\gamma}(\xi)\big)^\Delta,$$

so

$$v^{\check{\sigma}}(\xi) = \int_a^{\check{\sigma}(\xi)} \vartheta(s)\big(\Theta^{\check{\sigma}}(s)\big)^{-\gamma}\,\Delta s \leq \frac{1}{m(1-\gamma)}\int_a^{\check{\sigma}(\xi)}\big(\Theta^{1-\gamma}(s)\big)^\Delta\,\Delta s$$

$$= \frac{1}{m(1-\gamma)}\big(\Theta^{\check{\sigma}}(\xi)\big)^{1-\gamma}. \tag{36}$$

Furthermore, by using the chain rule, we get

$$\big(-\phi^\alpha(\xi)\big)^\Delta = -\alpha\phi^\Delta(\xi)\phi^{\alpha-1}(c); \quad c \text{ in } \big[\xi, \check{\sigma}(\xi)\big], \tag{37}$$

and since $\phi^\Delta(\xi) = -f(\xi)q(\xi)\vartheta(\xi)$ is negative, we get

$$\big(-\phi^\alpha(\xi)\big)^\Delta \leq \alpha\vartheta(\xi)q(\xi)f(\xi)\phi^{\alpha-1}(c). \tag{38}$$

From (36), (37), (38), and the assumption $q(\xi) \leq M$, we get

$$\int_a^\infty \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma} \phi^p(\xi)\,\Delta\xi$$

$$= \int_a^\infty \big(-\phi^\alpha(\xi)\big)^\Delta v^{\check{\sigma}}(\xi)\,\Delta\xi$$

$$\leq \frac{M\alpha}{m(1-\gamma)} \int_a^\infty \big(\Theta^{\check{\sigma}}(\xi)\big)^{1-\gamma} \vartheta(\xi) f(\xi) \phi^{\alpha-1}(\xi)\,\Delta\xi$$

$$\leq \left(\frac{M\alpha}{m(1-\gamma)}\right) \int_a^\infty \Big[\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma\frac{\alpha-1}{\alpha}} \vartheta^{\frac{\alpha-1}{\alpha}}(\xi) \phi^{\alpha-1}(\xi)\Big]$$

$$\times \Big[\big(\Theta^{\check{\sigma}}(\xi)\big)^{1-\gamma}\big(\Theta^{\check{\sigma}}(\xi)\big)^{\gamma\frac{\alpha-1}{\alpha}} f(\xi) \lambda^{\frac{1}{\alpha}}(\xi)\Big]\,\Delta\xi$$

$$\leq \left(\frac{M\alpha}{m(1-\gamma)}\right)\left[\int_a^\infty \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma} \phi^\alpha(\xi)\,\Delta\xi\right]^{\frac{\alpha-1}{\alpha}}$$

$$\times \left[\int_a^\infty \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{\alpha-\gamma} f^\alpha(\xi)\,\Delta\xi\right]^{\frac{1}{\alpha}}.$$

The last inequality holds by applying Hölder's inequality with exponents α and $\frac{\alpha}{\alpha-1}$. By simple simplification, we have

$$\int_a^\infty \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{-\gamma} \phi^\alpha(\xi)\,\Delta\xi \leq \left(\frac{M\alpha}{m(1-\gamma)}\right)^\alpha \int_a^\infty \vartheta(\xi)\big(\Theta^{\check{\sigma}}(\xi)\big)^{\alpha-\gamma} f^\alpha(\xi)\,\Delta\xi. \qquad \square$$

Theorem 4 *Suppose that $\alpha_i > \beta_i > 0$, $m_i > \beta_i$ for $i = 1,\dots,n$ are real numbers such that $\sum_i^n \beta_i = 1$. Furthermore, assume that $h_i(t), f_i(t), g_i(t)$ are nonnegative functions and $h_i(t)$ is a nondecreasing function for $i = 1,\dots,n$ and define*

$$\phi_i(t) = h_i(t)\int_a^t f_i(s) g_i(s)\,\Delta s, \qquad \Theta_i(t) = \int_a^t \vartheta_i(s)\,\Delta s \quad \text{for } t \in [a,\infty)_{\mathbb{T}}.$$

If there exist positive constants κ_i satisfying

$$\vartheta_i(t) - \left(\frac{\alpha_i}{\beta_i(m_i-1)}\right)\frac{\Theta_i(t) h_i^\Delta(t)}{h_i(t)} \geq \frac{1}{\kappa_i} > 0,$$

then

$$\int_a^\infty \prod_{i=1}^n \big[\Theta_i^{-m_i}(t) \phi_i^{\check{\sigma}}(t)^{\alpha_i}\big]\,\Delta t$$

$$\leq \left(\prod_{j=1}^n C_j^{-\alpha_j}\right)\sum_{i=1}^n K_i \int_a^\infty \Theta_i^{\frac{\alpha_i-m_i}{q_i}}(t)\big(h_i^{\check{\sigma}}(t) f_i(t) g_i(t)\big)^{\frac{\alpha_i}{\beta_i}}\,\Delta t,$$

where

$$K_i = \beta_i C_i^{\frac{\alpha_i}{\beta_i}}\left(\frac{\alpha_i \kappa_i}{m_i-\beta_i}\right)^{\frac{\alpha_i}{\beta_i}} > 0 \quad and \quad C_i > 0, \quad \forall i = 1,\dots,n.$$

Proof We define the function ψ_i by

$$\psi_i(t) = h_i(t) \int_a^t f_i(s) g_i(s) \, \Delta s \quad \text{for } t \in [a, \infty)_{\mathbb{T}}.$$

Then, by Theorem 1, we find

$$\int_a^\infty \Theta_i^{-\frac{m_i}{\beta_i}}(t) \big[\psi_i^{\check{\sigma}}(t) \big]^{\frac{\alpha_i}{\beta_i}} \, \Delta t \le \left[\frac{\alpha_i \kappa_i}{m_i - \beta_i} \right]^{\frac{\alpha_i}{\beta_i}} \int_a^\infty \Theta_i^{\frac{\alpha_i}{\beta_i} - \frac{m_i}{\beta_i}}(t) \big(h_i^{\check{\sigma}}(t) f_i(t) g_i(t) \big)^{\frac{\alpha_i}{\beta_i}} \, \Delta t.$$

For some constants $C_i > 0$, and by using the arithmetic-geometric inequality [27], we get

$$\prod_{i=1}^n \big[\Theta_i^{-m_i}(t) \psi_i^{\check{\sigma}}(t)^{\alpha_i} \big] = \prod_{i=1}^n C_i^{-\alpha_i} \big[\big(\Theta_i^{\frac{-m_i}{\alpha_i}}(t) C_i \psi_i^{\check{\sigma}}(t) \big)^{\frac{\alpha_i}{\beta_i}} \big]^{\beta_i}$$

$$= \left(\prod_{j=1}^n C_j^{-\alpha_j} \right) \prod_{i=1}^n \big[\big(\Theta_i^{\frac{-m_i}{\alpha_i}}(t) C_i \psi_i^{\check{\sigma}}(t) \big)^{\frac{\alpha_i}{\beta_i}} \big]^{\beta_i}$$

$$\le \left(\prod_{j=1}^n C_j^{-\alpha_j} \right) \sum_{i=1}^n \beta_i \Theta_i^{\frac{-m_i}{\beta_i}}(t) C_i^{\frac{\alpha_i}{\beta_i}} \psi_i^{\check{\sigma}}(t)^{\frac{\alpha_i}{\beta_i}}.$$

Therefore,

$$\int_a^\infty \prod_{i=1}^n \big[\Theta_i^{-m_i}(t) \psi_i^{\check{\sigma}}(t)^{p_i} \big] \, \Delta t$$

$$\le \left(\prod_{j=1}^n C_j^{-\alpha_j} \right) \sum_{i=1}^n \beta_i \int_a^\infty \Theta_i^{\frac{-m_i}{\beta_i}}(t) C_i^{\frac{\alpha_i}{\beta_i}} \psi_i^{\check{\sigma}}(t)^{\frac{\alpha_i}{\beta_i}} \, \Delta t$$

$$\le \left(\prod_{j=1}^n C_j^{-\alpha_j} \right) \sum_{i=1}^n K_i \int_0^\infty \Theta_i^{\frac{\alpha_i - m_i}{\beta_i}}(t) \big(h_i^{\check{\sigma}}(t) f_i(t) g_i(t) \big)^{\frac{\alpha_i}{\beta_i}} \, \Delta t,$$

where $K_i = \beta_i \big[\frac{C_i \alpha_i \kappa_i}{(m_i - \beta_i)} \big]^{\frac{\alpha_i}{\beta_i}}$. $\qquad \square$

Remark 5 In Theorem 4, if $n = 1, \alpha_i = \alpha, \beta_i = 1, \lambda_i = 1$ with $h(t) = g(t) = 1$, then we get

$$\int_a^\infty t^{-m} \left(\int_a^{\check{\sigma}(t)} f(s) \, \Delta s \right)^\alpha \Delta t \le C^{-\alpha} K \int_a^\infty t^{\alpha - m} f^\alpha(t) \, \Delta t,$$

where

$$C^{-\alpha} K = C^{-\alpha} C^\alpha \left(\frac{\alpha \kappa}{m - 1} \right)^\alpha = \left(\frac{\alpha \kappa}{m - 1} \right)^\alpha.$$

Then we have

$$\int_a^\infty \frac{1}{t^m} \left(\int_a^{\check{\sigma}(t)} f(s) \, \Delta s \right)^\alpha \Delta t \le \left(\frac{\alpha \kappa}{m - 1} \right)^\alpha \int_a^\infty \frac{f^\alpha(t)}{t^{m - \alpha}} \, \Delta t,$$

as stated in relation (12).

Theorem 5 *Suppose that $\alpha_i > \beta_i > 0$, $m_i > \beta_i$ for $i = 1,\dots,n$ are real numbers such that $\sum_i^n \beta_i = 1$. Furthermore, assume that $h_i(t), f_i(t), g_i(t)$ are nonnegative functions and $h_i(t)$ is nondecreasing for $i = 1,\dots,n$, and define*

$$\phi_i(t) = h_i(t) \int_0^t f_i(s)g_i(s)\,\Delta s, \qquad \Theta_i(t) = \int_a^t \vartheta_i(s)\,\Delta s \quad \text{for } t \in [a,\infty)_{\mathbb{T}}.$$

If there exist positive constants κ_i satisfying

$$1 - \left(\frac{\alpha_i}{\beta_i(m_i - 1)}\right)\frac{\Theta_i(t)h_i^\Delta(t)}{\vartheta_i(t)h_i(t)} \geq \frac{1}{\kappa_i} > 0.$$

Then

$$\int_0^\infty \prod_{i=1}^n \left[\vartheta_i(t)\Theta_i^{-m_i}(t)\phi_i^{\check{\sigma}}(t)^{\alpha_i}\right]\Delta t$$

$$\leq \left(\prod_{j=1}^n C_j^{-\alpha_j}\right)\sum_{i=1}^n K_i \int_0^\infty \vartheta_i^{1-\frac{\alpha_i}{\beta_i}}(t)\Theta_i^{\frac{\alpha_i - m_i}{\beta_i}}(t)\left(h_i^{\check{\sigma}}(t)f_i(t)g_i(t)\right)^{\frac{\alpha_i}{\beta_i}}\Delta t,$$

where

$$K_i = \beta_i C_i^{\frac{\alpha_i}{\beta_i}}\left(\frac{p_i\kappa_i}{m_i - \beta_i}\right)^{\frac{\alpha_i}{\beta_i}} > 0 \quad \text{and} \quad C_i > 0, \quad \forall i = 1,\dots,n.$$

There are many special cases that can be derived from Theorems 4 and 5. For instance, we can deduce inequality (9) from Theorem 5 by taking the time scale equals $\mathbb{R}$, $\vartheta_i(t) = 1$ (this leads to $\Theta_i(x) = x$), $h_i(x) = 1/f_i(x)$, and replacing $g_i(x)$ by $g_i(x)/x$.

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