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RESEARCH ARTICLE

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A fractional order co-infection model between malaria and filariasis epidemic

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ABSTRACT

In this article, we investigate a mathematical malaria-filariasis co-infection model with the assistance of the non-integer order operator. Using the fractal-fractional operator in the Caputo-Fabrizio (CF) sense, it has been possible to understand the dynamical behaviour and complicatedness of the malaria-filariasis model. An investigation of the existence and uniqueness of the solution employs fixed-point theory. Ulam-Hyers stability helps examine the stability analysis of the proposed co-infection model. The malaria-filariasis model has been investigated using the Toufik-Atanagana (TA), a sophisticated numerical method for these biological co-infection models. With the help of numerical procedures, we provide the approximate solutions for the proposed model. A variety of fractal dimension and fractional order options are utilized for the presentation of the results. When we adjust sensitive parameters like τ and γ, the graphical representation illustrates the system's behaviour and identifies suitable parameter ranges for solutions. In addition, we evaluate the model along with the regarded operators and various β_1 values using an exceptional graphical representation.

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Existence and uniqueness; fractal-fractional derivative; Malaria-filariasis model; numerical scheme; Ulam-Hyers stability

1. Introduction

Fractional calculus is an effective tool used to solve mathematical problems in the real world. Fractional-order derivatives find numerous applications in various fields such as numerical analysis, physics, and biomathematics. This field is gaining popularity among researchers due to its potential to provide plausible results. Fractional-order derivatives are essential in evaluating biological models (Miller & Ross, 1993; Sabatier, Agrawal, & Machado, 2007). Some fractional derivatives have singular kernels, while others have non-singular kernels. The fractal-fractional operator is a more accurate and efficient operator commonly used in biological models. The Caputo, Caputo-Fabrizio (CF), and Atangana-Baleanu (AB) types of fractal fractional derivatives are powerful tools for analyzing biological systems and other models. Memory effects are often observed in biological systems, making the usefulness of fractional-order operators even more essential in these cases.

Today, Malaria – filariasis coinfection is becoming a hazardous disease for the world. This disease is an old parasitic disease for human beings. This disease is arising as a critical situation created for the world population. This illness is a medical challenge at this time for researchers and doctors. We see a high effect of this disease in Africa and Asia; These continents, in this illness, have an enormous death rate. The World Health Organization says that in 2018 approximately 438,000 people died from the sickness of malaria; 90 per cent of deaths occurred only in Africa (World Health Organization, n.d.). Malaria is a disease spread by a mosquito's bite; when infected mosquitoes bite a human, the virus of this disease enters the human body (Nzeako, Okunnuga, Nduka, & Ezenwaka, 2016). The female anopheles mosquito plays a central role in the dispersion of malaria (Amoah-Mensah, Dontwi, & Bonyah, 2018; Mutua, Wang, & Vaidya, 2015; Okosun & Makinde, 2014). When we are infected with malaria, the symptoms of this illness like high fever, pain in the head, feeling the cold, and muscle aches. In some cases, patients have also suffered from vomiting and diarrhoea. Some of the patients infected with malaria have also been infected with anaemia and jaundice because of this disease's effect on the red blood cells of the human body. Every year, approximately 290 million people suffer from malaria. The death rate is more

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than we think, and about 400,000 people die from this disease each year.

Filariasis is a disease that can spread through mosquito bites. It is caused by three types of filaria parasites, which are known as Wuchereria bancrofti, Brugia timori, and Brugia malayi. According to a 2018 report by the World Health Organization, around 856 million people in 52 countries are at risk of contracting filariasis. To prevent the spread of this parasitic disease, it is important to take precautions and use preventive chemotherapy (Abdullahi, Alaku, & Hudu, 2015; Adegnika et al., 2010; Chandrakala & Zulfeen, 2016). When a mosquito bites a person who has lymphatic filariasis, the mosquito can also become infected. This is because the person's blood contains microscopic worms that enter the mosquito's body. When the infected mosquito bites another person, the microscopic worms can enter their skin and blood, causing the infection to spread. The worms can live in the lymphatic vessels for up to 8 years and produce many microfilariae (Bhunu & Mushayabasa, 2012). Symptoms of lymphatic filariasis include swelling in the legs, arms, breasts, and genitals.

Fractional calculus has become a widely studied branch of mathematics in recent years, and its applications have greatly enhanced the field of mathematical analysis (Mainardi, 2012). In particular, fractional calculus has proven to be a valuable tool for modelling biological processes that involve memory effects. For example, when developing a mathematical model of an infectious disease in the natural sciences, the use of fractional-order derivatives is crucial for finding numerical solutions (Kumar, Kumar, Samet, & Dutta, 2021; Kumar, Kumar, Samet, Gómez-Aguilar, & Osman, 2020; Kumar & Kumar, 2022). These derivatives are more efficient and effective than integer-order derivatives when dealing with biological systems (Caputo, 1969). A significant concept in the study of derivatives in fractional calculus is the Caputo derivative, which has a singular kernel, but there are also other operators like the CF operator and AB operator that have non-singular kernels (Caputo & Fabrizio, 2015; Kiryakova, 1993). In the past few years, several biological models have been solved with the help of fractional derivatives (Losada & Nieto, 2015; Uçar, Uçar, Özdemir, & Hammouch, 2019). However, dealing with fractional-order biological systems can be challenging due to non-linearity. The SVIR epidemic model is thoroughly examined by the authors, who utilize Lyapunov functions to provide methods for preventing the spread of disease (Alkhazzan, Wang, Nie, Khan, & Alzabut, 2023). This paper (Kumar, Kumar, & Jleli, 2022),, provides insights on how to investigate the dynamics and complexity of food chain models. Some of the notable derivatives include the AB derivative, which has become a pillar of fractional calculus in recent decades (Alkahtani & Atangana, 2016). The authors utilize a fractional differential operator to conduct a thorough examination of the tumour growth model that incorporates nonlinearity (Alzabut, Dhineshbabu, Selvam, Gómez-Aguilar, & Khan, 2023). Fractal-fractional operators have three different types of kernel, including power law, exponential decay, and Mittag-Leffler. These operators are considered reliable for biological models and are significant from a researcher's perspective. It has been noted that the analytical and numerical computations for various fractional-order and fractal dimensions support the dynamics converging effects more strongly than they do for an integer order. The authors analyze the impact of waterborne diseases and COVID-19 using the fractal-fractional operator on human health (Khan, Alzabut, Shah, et al., 2023; Khan, Alzabut, Tunç, et al., 2023). In this paper, we analyze a nonlinear model and propose a practical and efficient solution plan.

1.1. A Summary of the paper

The following parts provide an organized breakdown of the entire work: Section 2 provides fundamental definitions for fractional calculus. The fractal-fractional model of malaria and filariasis is discussed in section 3 of this article. In section 4, we discuss the uniqueness, and existence of the model solution, non-negativity, and as well as its stability. Section 4 is further broken into sections 4, 4.1 and 4.2. In Section 5, numerical techniques for a model of fractional order malaria and filariasis are described. In Section 6, there is a numerical simulation and comments. Section 7 provides the conclusion.

2. Preliminaries

In this section, we look at a few fractional operator definitions and theorems in detail.

Definition 2.1 (Atangana & Qureshi, 2019; Li, Liu, & Khan, 2020). *Let $\mathscr{F}(t)$ should be a continuous function in an open interval (a, b) and along with fractional order $0 < \beta_1 \leq 1$ and fractal dimension $0 < \beta_2 \leq 1$, in the Riemann-Liouville (RL) derivative with power law kernel is defined as follows:*

$$ {}^{FFP}_{0}D^{\beta_1,\beta_2}_{0,t}(\mathscr{F}(t)) = \frac{1}{\Gamma(k-\beta_1)}\frac{d}{dt^{\beta_2}}\int_0^t (t-u)^{k-\beta_1-1}\mathscr{F}(u)du, \tag{1}$$

with $k-1 < \beta_1, \beta_2 \leq k \in \mathbb{N}$ and $\frac{d\mathscr{F}(u)}{du^{\beta_2}} = \lim_{t\to u}\frac{\mathscr{F}(t)-\mathscr{F}(u)}{t^{\beta_2}-u^{\beta_2}}$.

Definition 2.2 (Atangana & Qureshi, 2019; Li et al., 2020). *Let $\mathscr{F}(t)$ should be a continuous function in an* *open interval (a, b) and along with fractional order $0 < \beta_1 \le 1$ and fractal dimension $0 < \beta_2 \le 1$, in the Riemann-Liouville (RL) derivative with exponentially decaying kernel is defined as follows:*

$$^{FFE}{}_0D_{0,t}^{\beta_1,\beta_2}(\mathscr{F}(t)) = \frac{B(\beta_1)}{1-\beta_1}\frac{d}{dt^{\beta_2}}\int_0^t exp\left(\frac{-\beta_1}{1-\beta_1}(t-u)\right)\mathscr{F}(u)du, \quad (2)$$

the normalized constant is defined as follows: $B(0) = 1,\ B(1) = 1.$

Definition 2.3 *(*Atangana & Qureshi, 2019*;* Li et al., 2020*). Let $\mathscr{F}(t)$ should be a continuous function in an open interval (a, b) and along with fractional order $0 < \beta_1 \le 1$ and fractal dimension $0 < \beta_2 \le 1$, with power law kernel is defined as follows:*

$$^{FFP}{}_0I_{0,t}^{\beta_1,\beta_2}(\mathscr{F}(t)) = \frac{\beta_2}{\Gamma(\beta_1)}\int_0^t (t-u)^{\beta_1-1}u^{\beta_2-1}\mathscr{F}(u)du. \quad (3)$$

Definition 2.4 *(*Atangana & Qureshi, 2019*;* Li et al., 2020*). Let $\mathscr{F}(t)$ should be a continuous function in an open interval (a, b) and along with fractional order $0 < \beta_1 \le 1$ and fractal dimension $0 < \beta_2 \le 1$, with exponential decay kernel is defined as follows:*

$$^{FFE}{}_0I_{0,t}^{\beta_1,\beta_2}(\mathscr{F}(t)) = \frac{\beta_2(1-\beta_1)t^{\beta_2-1}\mathscr{F}(t)}{B(\beta_1)} + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathscr{F}(u)du. \quad (4)$$

Theorem 2.1 *(*Ali, Shah, Zada, & Kumam, 2020*;* Granas & Dugundji, 2003*). Let $\Theta : \Phi \to \Phi$ be an operator and completely continuous. Let*

$$\mathfrak{S}(\Theta) = \{S_h \in \Phi : S_h = \varsigma\Theta(S_h), \varsigma \in [0,1]\}.$$

The operator Θ has at least one fixed point or the set $\mathfrak{S}(\Phi)$ is not bounded.

Lemma 2.1 *(*Xu, Saifullah, Ali, & Adnan, 2022*). This problem has a solution*

$$^{FFE}{}_0D_{0,t}^{\beta_1,\beta_2}\mathscr{H}(t) = \mathbb{F}(\mathscr{H}(t),t) + \varphi(t)$$
$$\mathscr{H}(0) = \mathscr{H}_0,$$

if the following condition is true,

$$\left|\mathscr{H}(t) - \left\{\mathscr{H}(0) + (\mathbb{F}(\mathscr{H}(t),t) - \varphi_0(t))\frac{\beta_2(1-\beta_1)}{B(\beta_1)}t^{\beta_2-1} + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_1-1}\mathbb{F}(\mathscr{H}(u),u)du\right\}\right| \le \left(\frac{\beta_2(1-\beta_1)\mathbb{T}^{\beta_2-1}}{B(\beta_1)} + \frac{\beta_2\mathbb{T}^{\beta_1}}{B(\beta_1)}\right)\aleph_{\mathbb{F}}. \quad (5)$$

3. The mathematical formulation of the malaria-filariasis model

In this part, we convey fractional malaria-filariasis behavior and depict the proposed model. We have nine equations in our malaria-filariasis system. In this model, the complete population of pregnant women at the time "t" is divided into separate subpopulations. S_h represents the individual pregnant women who adhered to a medical prenatal program. Individually susceptible women who do not follow the antenatal medical regimen are classified as S_2. I_m represents those pregnant women only infected with malaria; I_f represents those pregnant women only infected with filariasis. I_{mf} represents those pregnant women infected with both malaria and filariasis. T represents those pregnant women treated for malaria-filariasis. That's why $N_h = T + I_{mf} + I_f + I_m + S_2 + S_h$. Because the same vector that spreads malaria also spreads filariasis (Anopheles mosquito), this vector population is represented by N_υ. The S_υ stands for the susceptible mosquito; the E_υ for the exposed mosquito, and the I_υ for the infected mosquito, therefore $N_\upsilon = I_\upsilon + E_\upsilon + S_\upsilon$. The host population is recruited from the population of susceptible pregnant women who adhere to the medical prenatal program at a steady per capita rate of A_h. The fraction of pregnant women who are complying with antenatal medical programs is represented by the π, and τ is represented by the rate of loss of immunity to both malaria and filariasis. The natural death rate was represented by μ_h, while the rate of progression from S_h to S_2 was represented by δ. $K_{ma}S_h$ and K_fS_h have been chosen to depict the force of infection. That's why $K_{ma} = \frac{\alpha a I_\upsilon}{N_h}$, and $K_f = \frac{\beta(I_m+I_{mf}+kI_f)}{N_h}$. μ_h is the rate at which pregnant women die naturally, and the disease-induced death rate is denoted by the letter γ. σ_h is the percentage of susceptible pregnant women who do not comply with the antenatal medical program that develops malaria symptoms. ϕ_1 stands for the rate at which malaria is treated, and ψ stands for the pace at which malaria progresses to the point when filariasis symptoms appear. The treatment rate for filariasis is denoted by ϕ_2, whereas the treatment rate for malaria-filariasis is denoted by ϕ_3. The recruitment of mosquitoes has determined the rate A_ν of change in the susceptible mosquito population. The force of infection that created a rate of change in the exposed mosquito population has been denoted as the K_ν. μ_υ represents the natural mortality rate of mosquitoes, and σ_υ denotes a decrease in the pace of mosquito progression. α represents the rate of mosquito bites. The chance of malaria transmission in pregnant women per bite was represented by a, whereas the transmission rate between infectious and susceptible mosquitoes was represented by K_υ. b is the likelihood that any pregnant woman with the infection will cause mosquitoes to transmit the disease. β represents the actual contact rate of infection between infectious groups. Other modification parameters, such as ϵ, θ, ρ, and ν, are also present, as we know.

The following assumptions are included in the model's construction:

- In the tropics, we solely consider the population of pregnant women who are considered lonely.
- Birth and death rates exist.
- The mother and the kids have no vertical transmission.
- Malaria and filariasis disease-contaminated pregnant ladies have treatment also available.
- When pregnant women are infected with this disease, then after the treatment affects immunity; therefore, the possibility of again being infected with this disease.

This paper presents a realistic and well-posed domain for an epidemic model. However, due to state variables and parameter variations, medical intervention strategies such as treatment are required to control the spread of co-infectious diseases. Therefore the proposed co-infection model (Ogunmiloro, 2019) is written as:

$$\begin{aligned}
\frac{dS_h}{dt} &= \pi A_h - \mu_h S_h - \delta S_h - K_f S_h - K_{ma} S_h + \tau T,\\
\frac{dS_2}{dt} &= (1-\pi)A_h + K_f S_h + K_{ma} S_h + \delta S_h - \epsilon\sigma_h S_2 - \gamma S_2 - \mu_h S_2,\\
\frac{dI_m}{dt} &= \epsilon\sigma_h S_2 - \psi I_m - \phi_1 I_m - \mu_h I_m - \theta K_f I_m,\\
\frac{dI_f}{dt} &= \psi I_m - \gamma I_f - \phi_2 I_f - \mu_h I_f - \rho K_{ma} I_f,\\
\frac{dI_{mf}}{dt} &= \theta K_f I_m + \rho K_{ma} I_f - \phi_3 I_{mf} - \upsilon\phi I_{mf} - \eta\gamma I_{mf} - \mu_h I_{mf},\\
\frac{dT}{dt} &= \phi_1 I_m + \phi_2 I_f + \upsilon\phi I_{mf} + \phi_3 I_{mf} - \mu_h T - \tau T,\\
\frac{dS_\upsilon}{dt} &= A_\upsilon - K_\upsilon S_\upsilon - \mu_\upsilon S_\upsilon,\\
\frac{dE_\upsilon}{dt} &= K_\upsilon S_\upsilon - \sigma_\upsilon E_\upsilon - \mu_\upsilon E_\upsilon,\\
\frac{dI_\upsilon}{dt} &= \sigma_\upsilon E_\upsilon - \mu_\upsilon I_\upsilon,
\end{aligned} \tag{6}$$

with the initial conditions $S_h(0) = S_{h,0}$, $S_2(0) = S_{2,0}$, $I_m(0) = I_{m,0}$, $I_f(0) = I_{f,0}$, $I_{mf}(0) = I_{mf,0}$, $T(0) = T_0$, $S_\upsilon(0) = S_{\upsilon,0}$, $E_\upsilon(0) = E_{\upsilon,0}$, $I_\upsilon(0) = I_{\upsilon,0}$.

We operate the fractal fractional-order derivative in system (6), then we get

$$\begin{aligned}
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} S_h &= \pi A_h - \mu_h S_h - \delta S_h - K_f S_h - K_{ma} S_h + \tau T,\\
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} S_2 &= (1-\pi)A_h + K_f S_h + K_{ma} S_h + \delta S_h - \epsilon\sigma_h S_2 - \gamma S_2 - \mu_h S_2,\\
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} I_m &= \epsilon\sigma_h S_2 - \psi I_m - \phi_1 I_m - \mu_h I_m - \theta K_f I_m,\\
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} I_f &= \psi I_m - \gamma I_f - \phi_2 I_f - \mu_h I_f - \rho K_{ma} I_f,\\
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} I_{mf} &= \theta K_f I_m + \rho K_{ma} I_f - \phi_3 I_{mf} - \upsilon\phi I_{mf} - \eta\gamma I_{mf} - \mu_h I_{mf},\\
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} T &= \phi_1 I_m + \phi_2 I_f + \upsilon\phi I_{mf} + \phi_3 I_{mf} - \mu_h T - \tau T,\\
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} S_\upsilon &= A_\upsilon - K_\upsilon S_\upsilon - \mu_\upsilon S_\upsilon,\\
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} E_\upsilon &= K_\upsilon S_\upsilon - \sigma_\upsilon E_\upsilon - \mu_\upsilon E_\upsilon,\\
{}^{FFE}_{\ 0}D^{\beta_1,\beta_2}_{0,t} I_\upsilon &= \sigma_\upsilon E_\upsilon - \mu_\upsilon I_\upsilon,
\end{aligned} \tag{7}$$

with the initial conditions $S_h(0) = S_{h,0}$, $S_2(0) = S_{2,0}$, $I_m(0) = I_{m,0}$, $I_f(0) = I_{f,0}$, $I_{mf}(0) = I_{mf,0}$, $T(0) = T_0$, $S_\upsilon(0) = S_{\upsilon,0}$, $E_\upsilon(0) = E_{\upsilon,0}$, $I_\upsilon(0) = I_{\upsilon,0}$.

4. Existence and uniqueness of the solution

In this part, we build the existence theory for the proposed model. Now, we rewrite the system (7) with this structure as follows:

$$\begin{aligned}
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} S_h(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_1(S_h(t),t)),\\
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} S_2(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_2(S_2(t),t)),\\
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} I_m(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_3(I_m(t),t)),\\
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} I_f(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_4(I_f(t),t)),\\
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} I_{mf}(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_5(I_{mf}(t),t)),\\
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} T(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_6(T(t),t)),\\
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} S_\upsilon(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_7(S_\upsilon(t),t)),\\
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} E_\upsilon(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_8(E_\upsilon(t),t)),\\
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} I_\upsilon(t) &= \beta_2 t^{\beta_2-1}(\mathbb{F}_9(I_\upsilon(t),t)),
\end{aligned} \tag{8}$$

where

$$\begin{cases}
\mathbb{F}_1(S_h(t),t) = \pi A_h - \mu_h S_h - \delta S_h - K_f S_h - K_{ma} S_h + \tau T,\\
\mathbb{F}_2(S_2(t),t) = (1-\pi)A_h + K_f S_h + K_{ma} S_h + \delta S_h - \epsilon\sigma_h S_2 - \gamma S_2 - \mu_h S_2,\\
\mathbb{F}_3(I_m(t),t) = \epsilon\sigma_h S_2 - \psi I_m - \phi_1 I_m - \mu_h I_m - \theta K_f I_m,\\
\mathbb{F}_4(I_f(t),t) = \psi I_m - \gamma I_f - \phi_2 I_f - \mu_h I_f - \rho K_{ma} I_f,\\
\mathbb{F}_5(I_{mf}(t),t) = \theta K_f I_m + \rho K_{ma} I_f - \phi_3 I_{mf} - \upsilon\phi I_{mf} - \eta\gamma I_{mf} - \mu_h I_{mf},\\
\mathbb{F}_6(T(t),t) = \phi_1 I_m + \phi_2 I_f + \upsilon\phi I_{mf} + \phi_3 I_{mf} - \mu_h T - \tau T,\\
\mathbb{F}_7(S_\upsilon(t),t) = A_\upsilon - K_\upsilon S_\upsilon - \mu_\upsilon S_\upsilon,\\
\mathbb{F}_8(E_\upsilon(t),t) = K_\upsilon S_\upsilon - \sigma_\upsilon E_\upsilon - \mu_\upsilon E_\upsilon,\\
\mathbb{F}_9(I_\upsilon(t),t) = \sigma_\upsilon E_\upsilon - \mu_\upsilon I_\upsilon.
\end{cases}$$

System (8) can be expressed as follows:

$$\begin{aligned}
{}^{CF}_{\ 0}D^{\beta_1}_{0,t} \mathscr{H}(t) &= \beta_2 t^{\beta_2-1}\mathbb{F}(\mathscr{H}(t),t),\\
\mathscr{H}(0) &= \mathscr{H}_0.
\end{aligned} \tag{9}$$

We operate the fractional integral, we have

$$\mathscr{H}(t) = \mathscr{H}(0) + \frac{\beta_2 t^{\beta_2-1}(\beta_1-1)}{B(\beta_1)}\mathbb{F}(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}(\mathscr{H}(u),u)du,$$

where

$$\mathscr{H}(t) = \begin{cases} S_h(t)\\ S_2(t)\\ I_m(t)\\ I_f(t)\\ I_{mf}(t),\\ T(t)\\ S_\upsilon(t)\\ E_\upsilon(t)\\ I_\upsilon(t)\end{cases} \quad \mathbb{F}(\mathscr{H}(t),t) = \begin{cases} \mathbb{F}_1(S_h(t),t)\\ \mathbb{F}_2(S_2(t),t)\\ \mathbb{F}_3(I_m(t),t)\\ \mathbb{F}_4(I_f(t),t)\\ \mathbb{F}_5(I_{mf}(t),t)\\ \mathbb{F}_6(T(t),t)\\ \mathbb{F}_7(S_\upsilon(t),t)\\ \mathbb{F}_8(E_\upsilon(t),t)\\ \mathbb{F}_9(I_\upsilon(t),t)\end{cases}$$

We operate the fractal-fractional integral in the CF sense in the system (8), then we get

$$
\begin{aligned}
S_h(t) &= S_h(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_1(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_1(\mathscr{H}(u),u)du,\\
S_2(t) &= S_2(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_2(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_2(\mathscr{H}(u),u)du,\\
I_m(t) &= I_m(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_3(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_3(\mathscr{H}(u),u)du,\\
I_f(t) &= I_f(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_4(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_4(\mathscr{H}(u),u)du,\\
I_{mf}(t) &= I_{mf}(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_5(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_5(\mathscr{H}(u),u)du,\\
T(t) &= T(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_6(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_6(\mathscr{H}(u),u)du,\\
S_\upsilon(t) &= S_\upsilon(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_7(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_7(\mathscr{H}(u),u)du,\\
E_\upsilon(t) &= E_\upsilon(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_8(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_8(\mathscr{H}(u),u)du,\\
I_\upsilon(t) &= I_\upsilon(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_9(\mathscr{H}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_9(\mathscr{H}(u),u)du.
\end{aligned} \tag{10}
$$

In this part, we analyze the existence and uniqueness of the solution in our proposed model with the help of the fixed point theory. A Banach space can be defined as follows to demonstrate the existence and unique solution for the proposed model. We defined as $\Phi_n = \mathbb{G}[0,\mathbb{T}]$ is space of all functions $S_h, S_2, \dots, I_\upsilon$ respectively to $n = 1,2,\dots,9$. Φ_n generates a Banach space when provided with the norm $\|S_h\| = \max_{t\in[0,\mathbb{T}]}|S_h|$, $\|S_2\| = \max_{t\in[0,\mathbb{T}]}|S_2|$, …, $\|I_\upsilon\| = \max_{t\in[0,\mathbb{T}]}|I_\upsilon|$ respectively to $n = 1,2,\dots,9$. Therefore the norm is established in the product space as

$$
\begin{aligned}
\|\mathbb{U}\| &= \|(S_h, S_2, I_m, I_f, I_{mf}, T, S_\upsilon, E_\upsilon, I_\upsilon)\| = \|(S_h, S_2, \dots, I_\upsilon)\|,\\
&= \|S_h\| + \|S_2\| + \dots + \|I_\upsilon\|.
\end{aligned}
$$

A Banach space is established as

$$\Phi = (\Phi_1 \times \Phi_2 \times \Phi_3 \times \Phi_4 \times \Phi_5 \times \Phi_6 \times \Phi_7 \times \Phi_8 \times \Phi_9, \|\mathbb{U}\|).$$

Let the operator $\Theta : \Phi \to \Phi$ is defined in system (10), then we have

$$
\Theta(\mathscr{H})(t) = \begin{pmatrix} \Theta_1(\mathbb{U})(t)\\ \Theta_2(\mathbb{U})(t)\\ \Theta_3(\mathbb{U})(t)\\ \Theta_4(\mathbb{U})(t)\\ \Theta_5(\mathbb{U})(t)\\ \Theta_6(\mathbb{U})(t)\\ \Theta_7(\mathbb{U})(t)\\ \Theta_8(\mathbb{U})(t)\\ \Theta_9(\mathbb{U})(t) \end{pmatrix} \tag{11}
$$

where

$$
\begin{aligned}
\Theta_1(\mathscr{H})(t) &= S_h(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_1(S_h(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_1(S_h(u),u)du,\\
\Theta_2(\mathscr{H})(t) &= S_2(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_2(S_2(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_2(S_2(u),u)du,\\
\Theta_3(\mathscr{H})(t) &= I_m(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_3(I_m(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_3(I_m(u),u)du,\\
\Theta_4(\mathscr{H})(t) &= I_f(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_4(I_f(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_4(I_f(u),u)du,\\
\Theta_5(\mathscr{H})(t) &= I_{mf}(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_5(I_{mf}(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_5(I_{mf}(u),u)du,\\
\Theta_6(\mathscr{H})(t) &= T(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_6(T(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_6(T(u),u)du,\\
\Theta_7(\mathscr{H})(t) &= S_\upsilon(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_7(S_\upsilon(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_7(S_\upsilon(u),u)du,\\
\Theta_8(\mathscr{H})(t) &= E_\upsilon(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_8(E_\upsilon(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_8(E_\upsilon(u),u)du,\\
\Theta_9(\mathscr{H})(t) &= I_\upsilon(0) + \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1}\mathbb{F}_9(I_\upsilon(t),t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}\mathbb{F}_9(I_\upsilon(u),u)du.
\end{aligned} \tag{12}
$$

Theorem 4.1. *Let $\mathbb{F}_n : \mathscr{I} \times \mathscr{R}^9 \to \mathscr{R}$ are continuous functions and these constants are $\mathscr{L}_{1,\mathbb{F}_n}$, $\mathscr{L}_{2,\mathbb{F}_n}, \ldots,$ and $\mathscr{L}_{9,\mathbb{F}_n} > 0$, such that $\forall$ S_h, $\widehat{S_h}$, S_2, $\widehat{S_2}$, $I_m, \widehat{I_m}$, $I_f, \widehat{I_f}$, $I_{mf}, \widehat{I_{mf}}$, $T, \hat{T}$, $S_\upsilon, \hat{S_\upsilon}$, $E_\upsilon, \hat{E_\upsilon}$, $I_\upsilon, \hat{I_\upsilon}$ $\in \Phi$, where $\mathscr{I} = [0, \mathbb{T}]$, $n = 1, 2, \ldots, 9$, then we get*

$$\|\mathbb{F}_n(\mathbb{U}, t) - \mathbb{F}_n(\hat{\mathbb{U}}, t)\| \leq \mathscr{L}_{1,\mathbb{F}_n}\|S_h - \widehat{S_h}\| + \mathscr{L}_{2,\mathbb{F}_n}\| S_2 - \widehat{S_2}\| + \ldots + \mathscr{L}_{9,\mathbb{F}_n}\|I_\upsilon - \hat{I_\upsilon}\|.$$

In addition, suppose that if the condition $\Xi_\Theta \nabla_{\mathbb{F}_1} + \Xi_\Theta \nabla_{\mathbb{F}_2} + \ldots + \Xi_\Theta \nabla_{\mathbb{F}_9} < 1$ are fulfilled, then system (7) has a unique solution, where

$$\Xi_\Theta = \left\{ \frac{\beta_2(1-\beta_1)}{B(\beta_1)} \mathbb{T}^{\beta_2 - 1} + \frac{\beta_1 \mathbb{T}^{\beta_2}}{B(\beta_1)} \right\},$$

$$\begin{aligned}
\nabla_{\mathbb{F}_1} &= \mathscr{L}_{1,\mathbb{F}_1} + \mathscr{L}_{2,\mathbb{F}_1} + \mathscr{L}_{3,\mathbb{F}_1} + \mathscr{L}_{4,\mathbb{F}_1} + \mathscr{L}_{5,\mathbb{F}_1} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_1} + \mathscr{L}_{7,\mathbb{F}_1} + \mathscr{L}_{8,\mathbb{F}_1} + \mathscr{L}_{9,\mathbb{F}_1}, \\
\nabla_{\mathbb{F}_2} &= \mathscr{L}_{1,\mathbb{F}_2} + \mathscr{L}_{2,\mathbb{F}_2} + \mathscr{L}_{3,\mathbb{F}_2} + \mathscr{L}_{4,\mathbb{F}_2} + \mathscr{L}_{5,\mathbb{F}_2} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_2} + \mathscr{L}_{7,\mathbb{F}_2} + \mathscr{L}_{8,\mathbb{F}_2} + \mathscr{L}_{9,\mathbb{F}_2}, \\
\nabla_{\mathbb{F}_3} &= \mathscr{L}_{1,\mathbb{F}_3} + \mathscr{L}_{2,\mathbb{F}_3} + \mathscr{L}_{3,\mathbb{F}_3} + \mathscr{L}_{4,\mathbb{F}_3} + \mathscr{L}_{5,\mathbb{F}_3} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_3} + \mathscr{L}_{7,\mathbb{F}_3} + \mathscr{L}_{8,\mathbb{F}_3} + \mathscr{L}_{9,\mathbb{F}_3}, \\
\nabla_{\mathbb{F}_4} &= \mathscr{L}_{1,\mathbb{F}_4} + \mathscr{L}_{2,\mathbb{F}_4} + \mathscr{L}_{3,\mathbb{F}_4} + \mathscr{L}_{4,\mathbb{F}_4} + \mathscr{L}_{5,\mathbb{F}_4} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_4} + \mathscr{L}_{7,\mathbb{F}_4} + \mathscr{L}_{8,\mathbb{F}_4} + \mathscr{L}_{9,\mathbb{F}_4}, \\
\nabla_{\mathbb{F}_5} &= \mathscr{L}_{1,\mathbb{F}_5} + \mathscr{L}_{2,\mathbb{F}_5} + \mathscr{L}_{3,\mathbb{F}_5} + \mathscr{L}_{4,\mathbb{F}_5} + \mathscr{L}_{5,\mathbb{F}_5} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_5} + \mathscr{L}_{7,\mathbb{F}_5} + \mathscr{L}_{8,\mathbb{F}_5} + \mathscr{L}_{9,\mathbb{F}_5}, \\
\nabla_{\mathbb{F}_6} &= \mathscr{L}_{1,\mathbb{F}_6} + \mathscr{L}_{2,\mathbb{F}_6} + \mathscr{L}_{3,\mathbb{F}_6} + \mathscr{L}_{4,\mathbb{F}_6} + \mathscr{L}_{5,\mathbb{F}_6} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_6} + \mathscr{L}_{7,\mathbb{F}_6} + \mathscr{L}_{8,\mathbb{F}_6} + \mathscr{L}_{9,\mathbb{F}_6}, \\
\nabla_{\mathbb{F}_7} &= \mathscr{L}_{1,\mathbb{F}_7} + \mathscr{L}_{2,\mathbb{F}_7} + \mathscr{L}_{3,\mathbb{F}_7} + \mathscr{L}_{4,\mathbb{F}_7} + \mathscr{L}_{5,\mathbb{F}_7} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_7} + \mathscr{L}_{7,\mathbb{F}_7} + \mathscr{L}_{8,\mathbb{F}_7} + \mathscr{L}_{9,\mathbb{F}_7}, \\
\nabla_{\mathbb{F}_8} &= \mathscr{L}_{1,\mathbb{F}_8} + \mathscr{L}_{2,\mathbb{F}_8} + \mathscr{L}_{3,\mathbb{F}_8} + \mathscr{L}_{4,\mathbb{F}_8} + \mathscr{L}_{5,\mathbb{F}_8} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_8} + \mathscr{L}_{7,\mathbb{F}_8} + \mathscr{L}_{8,\mathbb{F}_8} + \mathscr{L}_{9,\mathbb{F}_8}, \\
\nabla_{\mathbb{F}_9} &= \mathscr{L}_{1,\mathbb{F}_9} + \mathscr{L}_{2,\mathbb{F}_9} + \mathscr{L}_{3,\mathbb{F}_9} + \mathscr{L}_{4,\mathbb{F}_9} + \mathscr{L}_{5,\mathbb{F}_9} \\
&\quad + \mathscr{L}_{6,\mathbb{F}_9} + \mathscr{L}_{7,\mathbb{F}_9} + \mathscr{L}_{8,\mathbb{F}_9} + \mathscr{L}_{9,\mathbb{F}_9}.
\end{aligned}$$

Proof. Let us define $\sup_{t\in\mathscr{I}} \mathbb{F}_1(0,0,0,0,0,0,0,0,0,t) = \Upsilon_{\mathbb{F}_1} < \infty$, $\sup_{t\in\mathscr{I}} \mathbb{F}_2(0,0,0,0,0,0,0,0,0,t) = \Upsilon_{\mathbb{F}_2} < \infty$, ..., $\sup_{t\in\mathscr{I}} \mathbb{F}_9(0,0,0,0,0,0,0,0,0,t) = \Upsilon_{\mathbb{F}_9} < \infty$. First, we prove that $\Theta(\mathfrak{S}_\varkappa) \subset \mathfrak{S}_\varkappa$ and let $\mathfrak{S}_\varkappa$ be closed convex ball (i.e $\mathfrak{S}_\varkappa = \{\mathbb{U} \in \Theta : \|\mathbb{U}\| \leq \varkappa\}$). Let $\mathfrak{S}_h, S_2, \ldots, I_\upsilon \in S_\varkappa$, we have

$$\begin{aligned}
\|\Theta_1(\mathbb{U})\| &\leq \frac{\beta_2(1-\beta_1)}{B(\beta_1)} \mathbb{T}^{\beta_2-1} \max_{t\in\mathscr{I}}(|\mathbb{F}_1(S_h(t), t) \\
&\quad - \mathbb{F}_1(0,0,0,0,0,0,0,0,0,t)| \\
&\quad + |\mathbb{F}_1(0,0,0,0,0,0,0,0,0,t)|) \\
&\quad + \frac{\beta_1\beta_2}{B(\beta_1)} \max_{t\in\mathscr{I}} \int_0^t u^{\beta_2-1}(|\mathbb{F}_1(S_h(u), u) \\
&\quad - \mathbb{F}_1(0,0,0,0,0,0,0,0,0,u)| \\
&\quad + |\mathbb{F}_1(0,0,0,0,0,0,0,0,0,u)|) du, \\
&\leq \frac{\beta_2(1-\beta_1)}{B(\beta_1)} \mathbb{T}^{\beta_2-1}(\mathscr{L}_{1,\mathbb{F}_1}\|S_h\| + \mathscr{L}_{2,\mathbb{F}_1}\|S_2\| \\
&\quad + \ldots + \mathscr{L}_{9,\mathbb{F}_1}\|I_\upsilon\| + \Upsilon_{\mathbb{F}_1}) \\
&\quad + \frac{\beta_1}{B(\beta)} \mathbb{T}^{\beta_2}(\mathscr{L}_{1,\mathbb{F}_1}\|S_h\| + \mathscr{L}_{2,\mathbb{F}_1}\|S_2\| \\
&\quad + \ldots + \mathscr{L}_{9,\mathbb{F}_1}\|I_\upsilon\| + \Upsilon_{\mathbb{F}_1}) \\
&\leq \Xi_\Theta \nabla_{\mathbb{F}_1}\left(\frac{\varkappa}{9} + \Upsilon_{\mathbb{F}_1}\right), \\
&\leq \frac{\varkappa}{9}.
\end{aligned} \tag{13}$$

Likewise, we get

$$\begin{aligned}
\|\Theta_2(\mathbb{U})\| &\leq \frac{\varkappa}{9}, \\
\|\Theta_3(\mathbb{U})\| &\leq \frac{\varkappa}{9}, \\
\|\Theta_4(\mathbb{U})\| &\leq \frac{\varkappa}{9}, \\
\|\Theta_5(\mathbb{U})\| &\leq \frac{\varkappa}{9}, \\
\|\Theta_6(\mathbb{U})\| &\leq \frac{\varkappa}{9}, \\
\|\Theta_7(\mathbb{U})\| &\leq \frac{\varkappa}{9}, \\
\|\Theta_8(\mathbb{U})\| &\leq \frac{\varkappa}{9}, \\
\|\Theta_9(\mathbb{U})\| &\leq \frac{\varkappa}{9},
\end{aligned} \tag{14}$$

with the use of the Φ definition, with the help of equations (13) and (14), we get

$$\|\Theta(\mathbb{U})\| = \|\Theta(S_h, S_2, \ldots, I_\upsilon)\| \leq \varkappa. \tag{15}$$

When $\mathbb{U}, \hat{\mathbb{U}} \in \Phi$, for each $t \in \mathscr{I}$, we get

$$\begin{aligned}
&\|\Theta_1(\mathbb{U}) - \Theta_1(\hat{\mathbb{U}})\| \\
&\leq \frac{\beta_2(1-\beta_1)}{B(\beta_1)} \mathbb{T}^{\beta_2-1} \max_{t\in\mathscr{I}} \left(\left| \mathbb{F}_1(S_h(t), t) - \mathbb{F}_1(\widehat{S_h}(t), t) \right| \right) \\
&\quad + \frac{\beta_1\beta_2}{B(\beta_1)} \max_{t\in\mathscr{I}} \int_0^t u^{\beta_2-1} \left(\left| \mathbb{F}_1(S_h(u), u) - \mathbb{F}_1(\widehat{S_h}(u), u) \right| \right) du, \\
&\leq \Xi_\Theta \nabla_{\mathbb{F}_1} \|\mathbb{U} - \hat{\mathbb{U}}\|.
\end{aligned} \tag{16}$$

Likewise, we get

$$\begin{aligned}
\|\Theta_2(\mathbb{U}) - \Theta_2(\hat{\mathbb{U}})\| &\leq \Xi_\Theta \nabla_{\mathbb{F}_2} \|\mathbb{U} - \hat{\mathbb{U}}\|, \\
\|\Theta_3(\mathbb{U}) - \Theta_3(\hat{\mathbb{U}})\| &\leq \Xi_\Theta \nabla_{\mathbb{F}_3} \|\mathbb{U} - \hat{\mathbb{U}}\|, \\
\|\Theta_4(\mathbb{U}) - \Theta_4(\hat{\mathbb{U}})\| &\leq \Xi_\Theta \nabla_{\mathbb{F}_4} \|\mathbb{U} - \hat{\mathbb{U}}\|, \\
\|\Theta_5(\mathbb{U}) - \Theta_5(\hat{\mathbb{U}})\| &\leq \Xi_\Theta \nabla_{\mathbb{F}_5} \|\mathbb{U} - \hat{\mathbb{U}}\|, \\
\|\Theta_6(\mathbb{U}) - \Theta_6(\hat{\mathbb{U}})\| &\leq \Xi_\Theta \nabla_{\mathbb{F}_6} \|\mathbb{U} - \hat{\mathbb{U}}\|, \\
\|\Theta_7(\mathbb{U}) - \Theta_7(\hat{\mathbb{U}})\| &\leq \Xi_\Theta \nabla_{\mathbb{F}_7} \|\mathbb{U} - \hat{\mathbb{U}}\|, \\
\|\Theta_8(\mathbb{U}) - \Theta_8(\hat{\mathbb{U}})\| &\leq \Xi_\Theta \nabla_{\mathbb{F}_8} \|\mathbb{U} - \hat{\mathbb{U}}\|, \\
\|\Theta_9(\mathbb{U}) - \Theta_9(\hat{\mathbb{U}})\| &\leq \Xi_\Theta \nabla_{\mathbb{F}_9} \|\mathbb{U} - \hat{\mathbb{U}}\|.
\end{aligned} \tag{17}$$

With the help of equations (16) and (17), then we get

$$\begin{aligned}
&\|\Theta(\mathbb{U}) - \Theta(\hat{\mathbb{U}})\| \\
&\leq (\Xi_\Theta \nabla_{\mathbb{F}_1} + \Xi_\Theta \nabla_{\mathbb{F}_2} + \ldots + \Xi_\Theta \nabla_{\mathbb{F}_9}) \|\mathbb{U} - \bar{\mathbb{U}}\|.
\end{aligned} \tag{18}$$

Since $\Xi_\Theta \nabla_{\mathbb{F}_1} + \Xi_\Theta \nabla_{\mathbb{F}_2} + \ldots + \Xi_\Theta \nabla_{\mathbb{F}_9} < 1$. So therefore, $\Theta(\mathbb{U})$ is a contraction operator. With the use of the

Banach contraction theorem and $\Theta(\mathbb{U})$ has a unique fixed point. Hence the proposed model (7) has a unique solution. □

Theorem 4.2. *Let $\zeta_{1,\mathbb{F}_n}, \zeta_{2,\mathbb{F}_n}, \zeta_{3,\mathbb{F}_n}, \zeta_{4,\mathbb{F}_n}, \zeta_{5,\mathbb{F}_n}, \zeta_{6,\mathbb{F}_n}, \zeta_{7,\mathbb{F}_n}, \zeta_{8,\mathbb{F}_n}, \zeta_{9,\mathbb{F}_n}, \zeta_{10,\mathbb{F}_n}, (n = 1, 2, ..., 9) : \mathscr{I} \to \mathscr{R}^+$ such that $\forall\ S_h, S_2, ..., l_\upsilon \in \Phi$, we have*

$$|\mathbb{F}_n(\mathbb{U}(t), t)| \leq \zeta_{1,\mathbb{F}_n}(t) + \zeta_{2,\mathbb{F}_n}(t)|S_h(t)| + \zeta_{3,\mathbb{F}_n}(t)|S_2(t)| + ... + \zeta_{10,\mathbb{F}_n}(t)|l_\upsilon(t)|,$$

with $\sup_{t\in\mathscr{I}} \zeta_{1,\mathbb{F}_n}(t) = \widetilde{\zeta_{1,\mathbb{F}_n}}$, $\sup_{t\in\mathscr{I}} \zeta_{2,\mathbb{F}_n}(t) = \widetilde{\zeta_{2,\mathbb{F}_n}}$, ..., $\sup_{t\in\mathscr{I}} \zeta_{10,\mathbb{F}_n}(t) = \widetilde{\zeta_{10,\mathbb{F}_n}}$ and $\widetilde{\zeta_{1,\mathbb{F}_1}}, \widetilde{\zeta_{1,\mathbb{F}_2}}, ..., \widetilde{\zeta_{1,\mathbb{F}_9}} > 0$.

Other assumptions are

$$\Xi_\Theta\Big(\widetilde{\zeta_{q,\mathbb{F}_1}} + \widetilde{\zeta_{q,\mathbb{F}_2}} + \widetilde{\zeta_{q,\mathbb{F}_3}} + \widetilde{\zeta_{q,\mathbb{F}_4}} + \widetilde{\zeta_{q,\mathbb{F}_5}} + \widetilde{\zeta_{q,\mathbb{F}_6}} + \widetilde{\zeta_{q,\mathbb{F}_7}} + \widetilde{\zeta_{q,\mathbb{F}_8}} + \widetilde{\zeta_{q,\mathbb{F}_9}}\Big) < 1$$

where $q = 2, 3, ..., 10$, and

$$\Xi_0 = \min\Big\{1 - \Xi_\Theta\Big(\widetilde{\zeta_{2,\mathbb{F}_1}} + \widetilde{\zeta_{2,\mathbb{F}_2}} + ... + \widetilde{\zeta_{2,\mathbb{F}_9}}\Big), 1 - \Xi_\Theta\Big(\widetilde{\zeta_{3,\mathbb{F}_1}} + \widetilde{\zeta_{3,\mathbb{F}_2}} + ... + \widetilde{\zeta_{3,\mathbb{F}_9}}\Big), ..., 1 - \Xi_\Theta\Big(\widetilde{\zeta_{10,\mathbb{F}_1}} + \widetilde{\zeta_{10,\mathbb{F}_2}} + ... + \widetilde{\zeta_{10,\mathbb{F}_9}}\Big)\Big\},$$

then the system (7) has at least one solution.

Proof. Let $\Theta : \Phi \to \Phi$ be an operator and completely continuous. We can say that Θ operator is continuous because $\mathbb{F}_n, (n = 1, 2, ..., 9)$ is continuous. Let $B \subseteq \Phi$ be a bounded set and there $\exists$ constants $k_{\mathbb{F}_n} > 0$, such that, $\max_{t\in\mathscr{I}}|\mathbb{F}_n(\mathbb{U}(t), t)| \leq k_{\mathbb{F}_n}, \forall\ \mathbb{U} \in B$. We have

$$\begin{aligned} \|\Theta_1(\mathbb{U})\| &\leq \frac{\beta_2(1-\beta_1)}{B(\beta_1)}\mathbb{T}^{\beta_2-1}\max_{t\in\mathscr{I}}|\mathbb{F}_1(S_h(t), t)| \\ &\quad + \frac{\beta_1\beta_2}{B(\beta_1)}\max_{t\in\mathscr{I}}\int_0^t u^{\beta_2-1}|\mathbb{F}_1(S_h(u), u)|du, \\ &\leq \Xi_\Theta k_{\mathbb{F}_1}. \end{aligned} \tag{19}$$

Likewise, we get

$$\begin{aligned} \|\Theta_2(\mathbb{U})\| &\leq \Xi_\Theta k_{\mathbb{F}_2}, \\ \|\Theta_3(\mathbb{U})\| &\leq \Xi_\Theta k_{\mathbb{F}_3}, \\ \|\Theta_4(\mathbb{U})\| &\leq \Xi_\Theta k_{\mathbb{F}_4}, \\ \|\Theta_5(\mathbb{U})\| &\leq \Xi_\Theta k_{\mathbb{F}_5}, \\ \|\Theta_6(\mathbb{U})\| &\leq \Xi_\Theta k_{\mathbb{F}_6}, \\ \|\Theta_7(\mathbb{U})\| &\leq \Xi_\Theta k_{\mathbb{F}_7}, \\ \|\Theta_8(\mathbb{U})\| &\leq \Xi_\Theta k_{\mathbb{F}_8}, \\ \|\Theta_9(\mathbb{U})\| &\leq \Xi_\Theta k_{\mathbb{F}_9}. \end{aligned} \tag{20}$$

We proved that $\Theta(S_h, S_2, ..., l_\upsilon)$ is uniformly bounded with the use of equations (19) and (20).

Now, we show that Θ is equicontinuous. Let $0 \leq t_1 \leq t_2 \leq \mathbb{T}$, therefore

$$\begin{aligned} \|\Theta_1(\mathbb{U}(t_2)) - \Theta_1(\mathbb{U}(t_1))\| \leq \Bigg[\Bigg|&\frac{\beta_2(1-\beta_1)}{B(\beta_1)}\big({t_2}^{\beta_2-1} - {t_1}^{\beta_2-1}\big)\{\mathbb{F}_1(S_h(t_2), ..., l_\upsilon(t_2), t_2) \\ &- \mathbb{F}_1(S_h(t_1), ..., l_\upsilon(t_1), t_1)\}\Bigg|\Bigg|\frac{\beta_1 k_{\mathbb{F}_1}}{B(\beta_1)}\big({t_2}^{\beta_2-1} - {t_1}^{\beta_2-1}\big)\Bigg|\Bigg] \\ &\to 0 \quad \text{when} \quad t_2 \to t_1. \end{aligned} \tag{21}$$

Likewise, we get

$$\begin{aligned} \|\Theta_2(\mathbb{U}(t_2)) - \Theta_2(\mathbb{U}(t_1))\| \leq \Bigg[\Bigg|&\frac{\beta_2(1-\beta_1)}{B(\beta_1)}\big({t_2}^{\beta_2-1} - {t_1}^{\beta_2-1}\big)\{\mathbb{F}_2(S_h(t_2), ..., l_\upsilon(t_2), t_2) \\ &- \mathbb{F}_2(S_h(t_1), ..., l_\upsilon(t_1), t_1)\}\Bigg|\Bigg|\frac{\beta_1 k_{\mathbb{F}_1}}{B(\beta_1)}\big({t_2}^{\beta_2-1} - {t_1}^{\beta_2-1}\big)\Bigg|\Bigg] \\ &\to 0 \quad \text{when} \quad t_2 \to t_1, \\ &\quad\vdots \\ \|\Theta_9(\mathbb{U}(t_2)) - \Theta_9(\mathbb{U}(t_1))\| \leq \Bigg[\Bigg|&\frac{\beta_2(1-\beta_1)}{B(\beta_1)}\big({t_2}^{\beta_2-1} - {t_1}^{\beta_2-1}\big)\{\mathbb{F}_9(S_h(t_2), ..., l_\upsilon(t_2), t_2) \\ &- \mathbb{F}_9(S_h(t_1), ..., l_\upsilon(t_1), t_1)\}\Bigg|\Bigg|\frac{\beta_1 k_{\mathbb{F}_1}}{B(\beta_1)}\big({t_2}^{\beta_2-1} - {t_1}^{\beta_2-1}\big)\Bigg|\Bigg] \\ &\to 0 \quad \text{when} \quad t_2 \to t_1. \end{aligned} \tag{22}$$

Thus, $\Theta(\mathbb{U}) = \Theta(S_h, S_2, ..., l_\upsilon)$ is equicontinuous.

So therefore, $\Theta(S_h, S_2, ..., I_\upsilon)$ is completely continuous.

Now, we prove that $\mathfrak{S} = \{\mathbb{U} \in \Phi : (S_h, S_2, ..., I_\upsilon) = \varsigma\Theta(\mathbb{U}), \varsigma \in [0,1]\}$ is a bounded. Let $\mathbb{U} \in \mathfrak{S}$, then $\mathbb{U} = \varsigma\Theta(\mathbb{U})$. When $t \in \mathscr{I}$, then $S_h(t) = \varsigma\Theta_1(\mathbb{U})(t)$, $S_2(t)=\varsigma\Theta_2(\mathbb{U})(t)$, $I_m(t)=\varsigma\Theta_3(\mathbb{U})(t)$, $I_f(t)=\varsigma\Theta_4(\mathbb{U})(t)$, $I_{mf}(t)=\varsigma\Theta_5(\mathbb{U})(t)$, $T(t)=\varsigma\Theta_6(\mathbb{U})(t)$, $S_\upsilon(t)=\varsigma\Theta_7(\mathbb{U})(t)$, $E_\upsilon(t)=\varsigma\Theta_8(\mathbb{U})(t)$ and $I_\upsilon(t)=\varsigma\Theta_9(\mathbb{U})(t)$. Then

$$|S_h(t)| \le \left[\frac{\beta_2(1-\beta_1)}{B(\beta_1)}\mathbb{T}^{\beta_2-1} + \frac{\beta_1\mathbb{T}^{\beta_2}}{B(\beta_1)}\right] \left(\zeta_{1,\mathbb{F}_1}(t) + \zeta_{2,\mathbb{F}_1}(t)|S_h(t)| + ... + \zeta_{10,\mathbb{F}_1}(t)|I_\upsilon(t)|\right), \tag{23}$$

we simplifying equation (23), we have

$$\|S_h(t)\| \le \Xi_\Theta\left(\widetilde{\zeta_{1,\mathbb{F}_1}(t)} + \widetilde{\zeta_{2,\mathbb{F}_1}(t)}|S_h(t)| + ... + \widetilde{\zeta_{10,\mathbb{F}_1}(t)}|I_\upsilon(t)|\right). \tag{24}$$

We apply a similar process then we get

$$\|S_2(t)\| \le \Xi_\Theta\left(\widetilde{\zeta_{1,\mathbb{F}_2}(t)} + \widetilde{\zeta_{2,\mathbb{F}_2}(t)}|S_h(t)| + ... + \widetilde{\zeta_{10,\mathbb{F}_2}(t)}|I_\upsilon(t)|\right),$$
$$\vdots$$
$$\|I_\upsilon(t)\| \le \Xi_\Theta\left(\widetilde{\zeta_{1,\mathbb{F}_9}(t)} + \widetilde{\zeta_{2,\mathbb{F}_9}(t)}|S_h(t)| + ... + \widetilde{\zeta_{10,\mathbb{F}_9}(t)}|I_\upsilon(t)|\right). \tag{25}$$

Now, we add the equations (24) and (25), then we get

$$\begin{aligned}&\|S_h(t)\| + \|S_2(t)\| + ... + \|I_\upsilon(t)\| \\ &\le \Xi_\Theta\left(\widetilde{\zeta_{1,\mathbb{F}_1}(t)} + \widetilde{\zeta_{1,\mathbb{F}_2}(t)} + ... + \widetilde{\zeta_{1,\mathbb{F}_9}(t)}\right) \\ &+\Xi_\Theta\left(\widetilde{\zeta_{2,\mathbb{F}_1}(t)} + \widetilde{\zeta_{2,\mathbb{F}_2}(t)} + ... + \widetilde{\zeta_{2,\mathbb{F}_9}(t)}\right)\|S_h(t)\| \\ &\vdots \\ &+\Xi_\Theta\left(\widetilde{\zeta_{10,\mathbb{F}_1}(t)} + \widetilde{\zeta_{10,\mathbb{F}_2}(t)} + ... + \widetilde{\zeta_{10,\mathbb{F}_9}(t)}\right)\|I_\upsilon(t)\|.\end{aligned} \tag{26}$$

Consequently, we obtain

$$\|S_h(t), S_2(t), ..., I_\upsilon(t)\| \le \frac{\Xi_\Theta\left(\widetilde{\zeta_{1,\mathbb{F}_1}(t)} + \widetilde{\zeta_{1,\mathbb{F}_2}(t)} + ... + \widetilde{\zeta_{1,\mathbb{F}_9}(t)}\right)}{\Xi_0}.$$

Thus, using the Leray-Schauder fixed point theorem, the proposed model (7) has at least one solution. □

4.1. Non-negativity condition

In this section, we prove that the system classes are non-negative for all t. This implies that the co-infection model has non-negative solutions for non-negative initial values for all $t > 0$.

Lemma 4.1 (Xu et al., 2022). *Let be suppose that initial condition $\mathbb{W}(t) \ge 0$, where*

$$\mathbb{W}(t) = (S_h, S_2, I_m, I_f, I_{mf}, T, S_\upsilon, E_\upsilon, I_\upsilon),$$

for every $t > 0$ the solution of the system (6) are non-negative. Now, $\lim_{t\to\infty}\mathbb{Z}(t) \le \frac{\beta_2}{\mu}$, where $\mu = \mu_h + \mu_\nu$ and $\mathbb{Z}(t) = S_h(t) + S_2(t) + I_m(t) + I_f(t) + I_{mf}(t) + T(t) + S_\upsilon(t) + E_\upsilon(t) + I_\upsilon(t)$.

Proof. Let $t_1 = \sup\{t > 0 : \mathbb{W}(t) > 0 \text{ and } \mathbb{W}(t) \in [0,1]\}$ and $t_1 > 0$. We take the first equation of the system (6), then we get

$$\frac{dS_h}{dt} = \pi A_h + \tau T - (\mu_h + \delta + K_f + K_{ma})S_h. \tag{27}$$

We solve the equation (27), and we get

$$\frac{d}{dt}[S_h \exp((\mu_h + \delta + K_f + K_{ma})t)] = \beta_2 \exp((\mu_h + \delta + K_f + K_{ma})t). \tag{28}$$

We simplified the equation (28), and we get

$$S_h(t_1) = S_h(0)\exp((\mu_h + \delta + K_f + K_{ma})t_1) + \beta_2 > 0. \tag{29}$$

The previous similar process applies to other equations in system (6), then we get $\mathbb{W}(t) > 0\ \forall\ t > 0$. Consider that $0 < S_h, S_2, I_m, I_f, I_{mf}, T, S_\upsilon, E_\upsilon, I_\upsilon \le \mathbb{Z}(t)$. The resulting value is obtained after summing up the state variables in the system (6), we have

$$\frac{d}{dt}\mathbb{Z}(t) = \beta_2 - \mu\mathbb{Z}(t).$$

Hence,

$$\lim_{t\to\infty}\mathbb{Z}(t) \le \frac{\beta_2}{\mu}.$$

Which indeed completes the proof of the lemma.

Lemma 4.2. *We prove that system (6) is bounded in the feasible region as:*

$$\mathbb{R} = \left\{(S_h, S_2, ..., I_\upsilon) \in \mathscr{R}^9_+ : 0 \le \mathbb{Z}(t) \le \frac{\beta_2}{\mu}\right\}.$$

Proof. We have added up all the compartments of the system (6), and we get

$$\begin{aligned}\frac{d}{dt}\mathbb{Z}(t) &= \beta_2 - \mu\mathbb{Z}(t), \\ \frac{d}{dt}\mathbb{Z}(t) + \mu\mathbb{Z}(t) &= \beta_2.\end{aligned} \tag{30}$$

We solve the equation (30), we get

$$\mathbb{Z}(t) \le ce^{-\mu t} + \frac{\beta_2}{\mu}. \tag{31}$$

When $t \to \infty$ in equation (31), then we get $\mathbb{Z}(t) \le \frac{\beta_2}{\mu}$.

Here, we will explore the positivity of the system (7). To do so, we will follow the previous steps.

$${}^{FFE}_{0}D^{\beta_1,\beta_2}_{0,t}\mathbb{Z}(t) = \beta_2 - \mu\mathbb{Z}(t),$$

With the help of the previous equation, we get

$$\lim_{t\to\infty} \mathbb{Z}(t) \leq \frac{\beta_2}{\mu}.$$

So,

$$\mathbb{R} = \left\{ (S_h, S_2, ..., I_\upsilon) \in \mathscr{R}_+^9 : 0 \leq \mathbb{Z}(t) \leq \frac{\beta_2}{\mu} \right\}.$$

Hence, the proof of the lemma is complete.

4.2. Ulam-Hyer stability condtion

In this section, we establish some terms and conditions of stability for proposed model. Let $\varphi(t)$ is a perturbed parameter.

(*i*) $|\varphi(t)| \leq \epsilon$, for $\epsilon > 0$,

(*ii*) ${}^{FFE}{}_0D_{0,t}^{\beta_1,\beta_2} \mathscr{H}(t) = \mathbb{F}(\mathscr{H}(t), t) + \varphi(t)$.

Theorem 4.3 (Xu et al., 2022). *If the following condition hold* $\wp < 1$, *where* $\wp = (\frac{\beta_2(1-\beta_1)\mathbb{T}^{\beta_2-1}}{B(\beta_1)} + \frac{\beta_2\mathbb{T}^{\beta_1}}{B(\beta_1)})\aleph_\mathbb{F}$ *and with use of systems* (8), (9) *and Lemma* (2.1), *then Ulam-Hyers stability exists for the solution of the proposed model.*

Proof. The proposed model has a unique solution, as we have demonstrated, let $\mathscr{H} \in \Phi$ be solution and $\bar{\mathscr{H}} \in \Phi$ be a unique solution of the system (7), we have

$$\begin{aligned}
&|\mathscr{H}(t) - \bar{\mathscr{H}}(t)| \\
&= \left| \mathscr{H}(t) - \left\{ \overline{\mathscr{H}(0)} + (\mathbb{F}(\bar{\mathscr{H}}(t), t) - \mathbb{F}_0(t)) \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1} \right.\right. \\
&\quad \left.\left. + \frac{\beta_1\beta_2}{B(\beta_1)} \int_0^t u^{\beta_1-1} \mathbb{F}(\bar{\mathscr{H}}(u), u) du \right\} \right|, \\
&\leq \left| \mathscr{H}(t) - \left\{ \mathscr{H}(0) + (\mathbb{F}(\mathscr{H}(t), t) - \mathbb{F}_0(t)) \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1} \right.\right. \\
&\quad \left.\left. + \frac{\beta_1\beta_2}{B(\beta_1)} \int_0^t u^{\beta_1-1} \mathbb{F}(\mathscr{H}(u), u) du \right\} \right| \\
&\quad + \left| \left\{ \mathscr{H}(0) + (\mathbb{F}(\mathscr{H}(t), t) - \mathbb{F}_0(t)) \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1} \right.\right. \\
&\quad \left.\left. + \frac{\beta_1\beta_2}{B(\beta_1)} \int_0^t u^{\beta_1-1} \mathbb{F}(\mathscr{H}(u), u) du \right\} \right| \\
&\quad - \left| \left\{ \bar{\mathscr{H}}(0) + (\mathbb{F}(\bar{\mathscr{H}}(t), t) - \mathbb{F}_0(t)) \frac{\beta_2(1-\beta_1)}{B(\beta_1)} t^{\beta_2-1} \right.\right. \\
&\quad \left.\left. + \frac{\beta_1\beta_2}{B(\beta_1)} \int_0^t u^{\beta_1-1} \mathbb{F}(\bar{\mathscr{H}}(u), u) du \right\} \right|, \\
&\leq \beth_{\beta_1,\beta_2} + \frac{\beta_2(1-\beta_1)\mathbb{T}^{\beta_2-1}\aleph_\mathbb{F}}{B(\beta_1)} \|\mathscr{H} - \bar{\mathscr{H}}\| + \frac{\beta_2\mathbb{T}^{\beta_1}\aleph_\mathbb{F}}{B(\beta_1)} \|\mathscr{H} - \bar{\mathscr{H}}\|, \\
&\leq \beth_{\beta_1,\beta_2} + \wp \|\mathscr{H} - \bar{\mathscr{H}}\|.
\end{aligned}$$

Based on the result mentioned above, we have

$$\|\mathscr{H} - \bar{\mathscr{H}}\| \leq \frac{\beth_{\beta_1,\beta_2}}{1-\wp} \|\mathscr{H} - \bar{\mathscr{H}}\|.$$

We can conclude that the system's solution is stable. So, the proof is finished. □

5. A Numerical technique for malaria – filariasis model

It is not a simple task to deal with nonlinearity when using fractional derivatives in a biological model. Working with non-linearity while using a fractional model is a complex task. We've used a few new numerical algorithms to solve biological models in recent years. These numerical techniques play a significant role in determining our system's approximate solution. We create a fractional system to find an approximate solution in the first stage.

5.1. Numerical scheme with fractal fractional in Caputo sense

We use the model to create a numerical scheme by starting with a power-law scenario. We write the proposed model in terms of Volterra representation in the RL sense before the beginning of the scheme (Khan, Atangana, Muhammad, & Alzahrani, 2021).

$$ {}^{FFP}{}_0D_{0,t}^{\beta_1,\beta_2} \mathfrak{F}(t) = \frac{1}{\Gamma(1-\beta_1)} \frac{d}{dt} \int_0^t \frac{(t-u)^{-\beta_1}}{\beta_2 t^{\beta_2-1}} \mathfrak{F}(u) du, \tag{32}$$

We are considering the fractional differential equation, outcomes displayed below:

$$\begin{aligned}
{}^{RL}_0D_{0,t}^{\beta_1} S_h(t) &= \beta_2 t^{\beta_2-1} k_1(S_h, t), \\
{}^{RL}_0D_{0,t}^{\beta_1} S_2(t) &= \beta_2 t^{\beta_2-1} k_2(S_2, t), \\
{}^{RL}_0D_{0,t}^{\beta_1} I_m(t) &= \beta_2 t^{\beta_2-1} k_3(I_m, t), \\
{}^{RL}_0D_{0,t}^{\beta_1} I_f(t) &= \beta_2 t^{\beta_2-1} k_4(I_f, t), \\
{}^{RL}_0D_{0,t}^{\beta_1} I_{mf}(t) &= \beta_2 t^{\beta_2-1} k_5(I_{mf}, t), \\
{}^{RL}_0D_{0,t}^{\beta_1} T(t) &= \beta_2 t^{\beta_2-1} k_6(T, t), \\
{}^{RL}_0D_{0,t}^{\beta_1} S_\upsilon(t) &= \beta_2 t^{\beta_2-1} k_7(S_\upsilon, t), \\
{}^{RL}_0D_{0,t}^{\beta_1} E_\upsilon(t) &= \beta_2 t^{\beta_2-1} k_8(E_\upsilon, t), \\
{}^{RL}_0D_{0,t}^{\beta_1} I_\upsilon(t) &= \beta_2 t^{\beta_2-1} k_9(I_\upsilon, t),
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
k_1(S_h, t) &= \pi A_h - \mu_h S_h - \delta S_h - K_f S_h - K_{ma} S_h + \tau T, \\
k_2(S_2, t) &= (1-\pi)A_h + K_f S_h + K_{ma} S_h + \delta S_h \\
&\quad - \epsilon\sigma_h S_2 - \gamma S_2 - \mu_h S_2, \\
k_3(I_m, t) &= \epsilon\sigma_h S_2 - \psi I_m - \phi_1 I_m - \mu_h I_m - \theta K_f I_m, \\
k_4(I_f, t) &= \psi I_m - \gamma I_f - \phi_2 I_f - \mu_h I_f - \rho K_{ma} I_f, \\
k_5(I_{mf}, t) &= \theta K_f I_m + \rho K_{ma} I_f - \phi_3 I_{mf} - \upsilon\phi I_{mf} - \eta\gamma I_{mf} - \mu_h I_{mf}, \\
k_6(T, t) &= \phi_1 I_m + \phi_2 I_f + \upsilon\phi I_{mf} + \phi_3 I_{mf} - \mu_h T - \tau T, \\
k_7(S_\upsilon, t) &= A_\upsilon - K_\upsilon S_\upsilon - \mu_\upsilon S_\upsilon, \\
k_8(E_\upsilon, t) &= K_\upsilon S_\upsilon - \sigma_\upsilon E_\upsilon - \mu_\upsilon E_\upsilon, \\
k_9(I_\upsilon, t) &= \sigma_\upsilon E_\upsilon - \mu_\upsilon I_\upsilon.
\end{aligned} \tag{34}$$

To develop a numerical scheme for the fractal fractional malaria-filariasis model, we apply the RL fractional integral to the system (33), then we get

$$
\begin{aligned}
S_h(t)-S_h(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_1(S_h,u)du,\\
S_2(t)-S_2(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_2(S_2,u)du,\\
I_m(t)-I_m(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_3(I_m,u)du,\\
I_f(t)-I_f(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_4(I_f,u)du,\\
I_{mf}(t)-I_{mf}(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_5(I_{mf},u)du,\\
T(t)-T(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_6(T,u)du,\\
S_\upsilon(t)-S_\upsilon(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_7(S_\upsilon,u)du,\\
E_\upsilon(t)-E_\upsilon(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_8(E_\upsilon,u)du,\\
I_\upsilon(t)-I_\upsilon(0)&=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t-u)^{\beta_1-1}k_9(I_\upsilon,u)du.
\end{aligned}
\tag{35}
$$

Initially, we merely resolve the system's (35) first equation. Other equations obtain solutions that are analogous to those of the first equation.

$$
S_h(t)-S_h(0)=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^t u^{\beta_2-1}(t_{n+1}-u)^{\beta_1-1}k_1(S_h,u)du,
\tag{36}
$$

put $t=t_{n+1}$, in equation (36) then we get

$$
S_h(t_{n+1})-S_h(0)=\frac{\beta_2}{\Gamma(\beta_1)}\int_0^{t_{n+1}} u^{\beta_2-1}(t_{n+1}-u)^{\beta_1-1}k_1(S_h,u)du.
\tag{37}
$$

We simplify the equation (37), we have

$$
S_h(t_{n+1}) = S_h(0)+\frac{\beta_2}{\Gamma(\beta_1)}\sum_{\mathfrak{q}=0}^{n}\int_{t_\mathfrak{q}}^{t_{\mathfrak{q}+1}} u^{\beta_2-1}(t_{n+1}-u)^{\beta_1-1}k_1(S_h,u)du,
\tag{38}
$$

with the help of the Lagrangian interpolation technique for finding the approximate function $u^{\beta_2-1}k_1(S_h,u)$ in the interval $[t_\mathfrak{q},t_{\mathfrak{q}+1}]$ into equation (38), then we get

$$
\mathscr{B}^1_\mathfrak{q}(u)=\frac{u-t_{\mathfrak{q}-1}}{t_\mathfrak{q}-t_{\mathfrak{q}-1}}t_\mathfrak{q}^{\beta_2-1}k_1(S_{h,\mathfrak{q}},t_\mathfrak{q}) -\frac{u-t_\mathfrak{q}}{t_\mathfrak{q}-t_{\mathfrak{q}-1}}t_{\mathfrak{q}-1}^{\beta_2-1}k_1(S_{h,\mathfrak{q}-1},t_{\mathfrak{q}-1}).
\tag{39}
$$

We apply equation (39) to equation (38), then we get

$$
S_h(t_{n+1})=S_h(0) +\frac{\beta_2}{\Gamma(\beta_1)}\sum_{\mathfrak{q}=0}^{n}\int_{t_\mathfrak{q}}^{t_{\mathfrak{q}+1}} u^{\beta_2-1}(t_{n+1}-u)^{\beta_1-1}\mathscr{B}^1_\mathfrak{q}(u)du,
\tag{40}
$$

Equation (40) can be solved further to produce the following results,

$$
S_h(t_{n+1})=S_h(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_1(S_{h,\mathfrak{q}},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_1(S_{h,\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],
\tag{41}
$$

where

$$
\begin{aligned}
\mathfrak{e}_{n,\mathfrak{q}}&=\Big[(n-\mathfrak{q}+1)^{\beta_1}(n-\mathfrak{q}+\beta_1+2) -(n-\mathfrak{q})^{\beta_1}(n-\mathfrak{q}+2\beta_1+2)\Big],\\
\mathfrak{f}_{n,\mathfrak{q}}&=\Big[(n-\mathfrak{q}+1)^{\beta_1+1}-(n-\mathfrak{q})^{\beta_1}(n-\mathfrak{q}+\beta_1+1)\Big],
\end{aligned}
$$

$n=0,1,2,\dots,N$ and $\mathfrak{q}=1,2,3,\dots,n$.

The previous similar process applies to other equations, then we get

$$
\begin{aligned}
S_2(t_{n+1})&=S_2(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_2(S_{2,\mathfrak{q}},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_2(S_{2,\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],\\
I_m(t_{n+1})&=I_m(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_3(I_{m,\mathfrak{q}},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_3(I_{m,\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],\\
I_f(t_{n+1})&=I_f(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_4(I_{f,\mathfrak{q}},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_4(I_{f,\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],\\
I_{mf}(t_{n+1})&=I_{mf}(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_5(I_{mf,\mathfrak{q}},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_5(I_{mf,\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],\\
T(t_{n+1})&=T(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_6(T_\mathfrak{q},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_6(T_{\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],\\
S_\upsilon(t_{n+1})&=S_\upsilon(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_7(S_{\upsilon,\mathfrak{q}},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_7(S_{\upsilon,\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],\\
E_\upsilon(t_{n+1})&=E_\upsilon(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_8(E_{\upsilon,\mathfrak{q}},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_8(E_{\upsilon,\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],\\
I_\upsilon(t_{n+1})&=I_\upsilon(0)+\frac{h^{\beta_1}\beta_2}{\Gamma(\beta_1+2)}\sum_{\mathfrak{q}=1}^{n}\Big[t_\mathfrak{q}^{\beta_2-1}k_9(I_{\upsilon,\mathfrak{q}},t_\mathfrak{q})\mathfrak{e}_{n,\mathfrak{q}} - t_{\mathfrak{q}-1}^{\beta_2-1}k_9(I_{\upsilon,\mathfrak{q}-1},t_{\mathfrak{q}-1})\mathfrak{f}_{n,\mathfrak{q}}\Big],
\end{aligned}
\tag{42}
$$

where

$$\mathfrak{e}_{n,\mathfrak{q}} = \Big[(n-\mathfrak{q}+1)^{\beta_1}(n-\mathfrak{q}+\beta_1+2) - (n-\mathfrak{q})^{\beta_1}(n-\mathfrak{q}+2\beta_1+2)\Big],$$
$$\mathfrak{f}_{n,\mathfrak{q}} = \Big[(n-\mathfrak{q}+1)^{\beta_1+1} - (n-\mathfrak{q})^{\beta_1}(n-\mathfrak{q}+\beta_1+1)\Big],$$

$n = 0, 1, 2, \ldots, N$ and $\mathfrak{q} = 1, 2, 3, \ldots, n$.

5.2. Numerical scheme with fractal fractional in CF sense

We now converted the proposed model to the fractal-fractional in the CF sense. Therefore we are developing the numerical approach in the CF sense, the structure as follows:

$$\begin{aligned}
{}^{CF}_{0}D^{\beta_1}_{0,t}S_h(t) &= \beta_2 t^{\beta_2-1}k_1(S_h,t),\\
{}^{CF}_{0}D^{\beta_1}_{0,t}S_2(t) &= \beta_2 t^{\beta_2-1}k_2(S_2,t),\\
{}^{CF}_{0}D^{\beta_1}_{0,t}I_m(t) &= \beta_2 t^{\beta_2-1}k_3(I_m,t),\\
{}^{CF}_{0}D^{\beta_1}_{0,t}I_f(t) &= \beta_2 t^{\beta_2-1}k_4(I_f,t),\\
{}^{CF}_{0}D^{\beta_1}_{0,t}I_{mf}(t) &= \beta_2 t^{\beta_2-1}k_5(I_{mf},t),\\
{}^{CF}_{0}D^{\beta_1}_{0,t}T(t) &= \beta_2 t^{\beta_2-1}k_6(T,t),\\
{}^{CF}_{0}D^{\beta_1}_{0,t}S_\upsilon(t) &= \beta_2 t^{\beta_2-1}k_7(S_\upsilon,t),\\
{}^{CF}_{0}D^{\beta_1}_{0,t}E_\upsilon(t) &= \beta_2 t^{\beta_2-1}k_8(E_\upsilon,t),\\
{}^{CF}_{0}D^{\beta_1}_{0,t}I_\upsilon(t) &= \beta_2 t^{\beta_2-1}k_9(I_\upsilon,t).
\end{aligned} \tag{43}$$

We apply the CF integral in equation (43), then we get

$$\begin{aligned}
S_h(t)-S_h(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_1(S_h,t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_1(S_h,u)du,\\
S_2(t)-S_2(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_2(S_2,t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_2(S_2,u)du,\\
I_m(t)-I_m(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_3(I_m,t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_3(I_m,u)du,\\
I_f(t)-I_f(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_4(I_f,t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_4(I_f,u)du,\\
I_{mf}(t)-I_{mf}(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_5(I_{mf},t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_5(I_{mf},u)du,\\
T(t)-T(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_6(T,t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_6(T,u)du,\\
S_\upsilon(t)-S_\upsilon(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_7(S_\upsilon,t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_7(S_\upsilon,u)du,\\
E_\upsilon(t)-E_\upsilon(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_8(E_\upsilon,t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_8(E_\upsilon,u)du,\\
I_\upsilon(t)-I_\upsilon(0) &= \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_9(I_\upsilon,t)+\frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_9(I_\upsilon,u)du.
\end{aligned} \tag{44}$$

Initially, we merely resolve the system's (44) first equation. Other equations obtain solutions that are analogous to those of the first equation.

$$S_h(t)-S_h(0) = \frac{\beta_2 t^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_1(S_h,t) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^t u^{\beta_2-1}k_1(S_h,u)du, \tag{45}$$

we put $t = t_{n+1}$ in equation (45), then we get

$$S_h(t_{n+1})-S_h(0) = \frac{\beta_2 t_n^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_1(S_h^n,t_n) + \frac{\beta_1\beta_2}{B(\beta_1)}\int_0^{t_{n+1}} u^{\beta_2-1}k_1(S_h,u)du. \tag{46}$$

Now, we simplified equation (46), and we get

$$\begin{aligned}
S_h(t_{n+1}) = S_h(0) &+ \frac{\beta_2 t_n^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_1(S_h^n,t_n)\\
&- \frac{\beta_2 t_{n-1}^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_1(S_h^{n-1},t_{n-1})\\
&+ \frac{\beta_1\beta_2}{B(\beta_1)}\int_{t_n}^{t_{n+1}} u^{\beta_2-1}k_1(S_h,u)du.
\end{aligned}$$

The following outcome is obtained using the Lagrange polynomial concept:

$$\begin{aligned}
S_h(t_{n+1}) = S_h(0) &+ \frac{\beta_2 t_n^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_1(S_h^n,t_n)\\
&- \frac{\beta_2 t_{n-1}^{\beta_2-1}(1-\beta_1)}{B(\beta_1)}k_1(S_h^{n-1},t_{n-1})\\
&+ \frac{\beta_1\beta_2 h}{B(\beta_1)}\left[\frac{3t_n^{\beta_2-1}}{2}k_1(S_h^n,t_n) - \frac{t_{n-1}^{\beta_2-1}}{2}k_1(S_h^{n-1},t_{n-1})\right].
\end{aligned} \tag{47}$$

Further, we simplify the equation (47), then we get

$$\begin{aligned}
S_h(t_{n+1}) = S_h(0) &+ \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_1(S_h^n,t_n)\\
&- \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_1(S_h^{n-1},t_{n-1}),
\end{aligned} \tag{48}$$

the previous similar process applies to other equations, then we get

$$\begin{aligned}
S_2(t_{n+1}) &= S_2(0) + \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_2(S_2^n, t_n) \\
&\quad - \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_2(S_2^{n-1}, t_{n-1}), \\
I_m(t_{n+1}) &= I_m(0) + \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_3(I_m^n, t_n) \\
&\quad - \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_3(I_m^{n-1}, t_{n-1}), \\
I_f(t_{n+1}) &= I_f(0) + \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_4(I_f^n, t_n) \\
&\quad - \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_4(I_f^{n-1}, t_{n-1}), \\
I_{mf}(t_{n+1}) &= I_{mf}(0) + \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_5(I_{mf}^n, t_n) \\
&\quad - \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_5(I_{mf}^{n-1}, t_{n-1}), \\
T(t_{n+1}) &= T(0) + \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_6(T^n, t_n) \\
&\quad - \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_6(T^{n-1}, t_{n-1}), \\
S_\upsilon(t_{n+1}) &= S_\upsilon(0) + \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_7(S_\upsilon^n, t_n) \\
&\quad - \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_7(S_\upsilon^{n-1}, t_{n-1}), \\
E_\upsilon(t_{n+1}) &= E_\upsilon(0) + \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_8(E_\upsilon^n, t_n) \\
&\quad - \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_8(E_\upsilon^{n-1}, t_{n-1}), \\
I_\upsilon(t_{n+1}) &= I_\upsilon(0) + \beta_2 t_n^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{3\beta_1 h}{2B(\beta_1)}\right)k_9(I_\upsilon^n, t_n) \\
&\quad - \beta_2 t_{n-1}^{\beta_2-1}\left(\frac{1-\beta_1}{B(\beta_1)} + \frac{\beta_1 h}{2B(\beta_1)}\right)k_9(I_\upsilon^{n-1}, t_{n-1}).
\end{aligned} \tag{49}$$

5.3. Error analysis with fractal fractional in Caputo sense

We use the equation (38), and then we get

$$S_h(t_{n+1}) = S_h(0) + \frac{\beta_2}{\Gamma(\beta_1)}\sum_{q=0}^{n}\int_{t_q}^{t_{q+1}} u^{\beta_2-1}(t_{n+1}-u)^{\beta_1-1}k_1(S_h, u)du, \tag{50}$$

with the help of the Lagrangian polynomial for finding the approximate function $k_1(S_h, u)$ in the interval $[t_q, t_{q+1}]$ into equation (50), then we get

$$\begin{aligned}
k_1(S_h, u) &= \mathbb{P}_q(u) + \mathbb{E}_1(u), \\
&= \frac{k_1(S_{h,q}, t_q)}{\Delta t}(u - t_{q-1}) - \frac{k_1(S_{h,q-1}, t_{q-1})}{\Delta t}(u - t_q) \\
&\quad + \frac{(u-t_q)(u-t_{q-1})}{2!}\frac{\partial^2}{\partial u^2}[k_1(S_h, u)]_{u=y_q}.
\end{aligned} \tag{51}$$

Therefore, the error can be evaluated as

$$\mathbb{E}_{1,u}^{\beta_1}(\epsilon_u) = \frac{\beta_2}{\Gamma(\beta_1)}\sum_{q=0}^{n}\int_{t_q}^{t_{q+1}} u^{\beta_2-1}\frac{(u-t_q)(u-t_{q-1})}{2!}\frac{\partial^2}{\partial u^2}[k_1(S_h, u)]_{u=y_q}(t_{n+1}-u)^{\beta_1-1}du. \tag{52}$$

Taking the absolute value on both sides, we have

$$\begin{aligned}
&\left|\mathbb{E}_{1,u}^{\beta_1}(\epsilon_u)\right| \\
&= \left|\frac{\beta_2}{\Gamma(\beta_1)}\sum_{q=0}^{n}\int_{t_q}^{t_{q+1}} u^{\beta_2-1}\frac{\partial^2}{\partial u^2}[k_1(S_h, u)]_{u=y_q}(t_{n+1}-u)^{\beta_1-1}du\right|, \\
&\leq \frac{\beta_2}{\Gamma(\beta_1)}\sum_{q=0}^{n}\sup_{t\in\mathscr{I}}\frac{(u-t_q)(u-t_{q-1})}{2!} \\
&\times\left|\frac{\partial^2}{\partial u^2}[k_1(S_h, u)]_{u=y_q}\right|\left|\int_{t_q}^{t_{q+1}} u^{\beta_2-1}(t_{n+1}-u)^{\beta_1-1}du.\right.
\end{aligned} \tag{53}$$

We have

$$\begin{aligned}
&\left|\int_{t_q}^{t_{q+1}} u^{\beta_2-1}(t_{n+1}-u)^{\beta_1-1}du\right| \\
&< 2t_{n+1}^{\beta_1+\beta_2-1}B(\beta_1, \beta_2), \\
&< 2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1, \beta_2).
\end{aligned} \tag{54}$$

We use equation (54) in equation (53), then we present the following error:

$$\begin{aligned}
\left|\mathbb{E}_{1,u}^{\beta_1}(\epsilon_u)\right| &< \frac{\beta_2}{\Gamma(\beta_1)}\sum_{q=0}^{n}\frac{(u-t_q)(u-t_{q-1})}{2!} \\
&\quad \left|\frac{\partial^2}{\partial u^2}[k_1(S_h, u)]_{u=y_q}\right|2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1, \beta_2), \\
&< \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1, \beta_2)}{\Gamma(\beta_1)} \\
&\quad \times\sum_{q=0}^{n}\frac{(u-t_q)(u-t_{q-1})}{2!}\sup_{0\leq u\leq t_{n+1}}\left|\frac{\partial^2}{\partial u^2}[k_1(S_h, u)]_{u=y_q}\right|.
\end{aligned} \tag{55}$$

The previous similar process applies to other equations, then we get

$$\begin{aligned}
\left|\mathbb{E}_{2,u}^{\beta_1}(\epsilon_u)\right| &< \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1, \beta_2)}{\Gamma(\beta_1)} \\
&\quad \times\sum_{q=0}^{n}\frac{(u-t_q)(u-t_{q-1})}{2!}\sup_{0\leq u\leq t_{n+1}}\left|\frac{\partial^2}{\partial u^2}[k_2(S_2, u)]_{u=y_q}\right|,
\end{aligned} \tag{56}$$

$$\begin{aligned}
\left|\mathbb{E}_{3,u}^{\beta_1}(\epsilon_u)\right| &< \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1, \beta_2)}{\Gamma(\beta_1)} \\
&\quad \times\sum_{q=0}^{n}\frac{(u-t_q)(u-t_{q-1})}{2!}\sup_{0\leq u\leq t_{n+1}}\left|\frac{\partial^2}{\partial u^2}[k_3(I_m, u)]_{u=y_q}\right|,
\end{aligned} \tag{57}$$

$$\begin{aligned}
\left|\mathbb{E}_{4,u}^{\beta_1}(\epsilon_u)\right| &< \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1, \beta_2)}{\Gamma(\beta_1)} \\
&\quad \times\sum_{q=0}^{n}\frac{(u-t_q)(u-t_{q-1})}{2!}\sup_{0\leq u\leq t_{n+1}}\left|\frac{\partial^2}{\partial u^2}[k_4(I_f, u)]_{u=y_q}\right|,
\end{aligned} \tag{58}$$

$$\left|\mathbb{E}^{\beta_1}_{5,u}(\epsilon_u)\right| < \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1,\beta_2)}{\Gamma(\beta_1)} \times \sum_{q=0}^{n} \frac{(u-t_q)(u-t_{q-1})}{2!} \sup_{0\le u\le t_{n+1}} \left|\frac{\partial^2}{\partial u^2}[k_5(I_{mf},u)]_{u=y_q}\right|, \tag{59}$$

$$\left|\mathbb{E}^{\beta_1}_{6,u}(\epsilon_u)\right| < \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1,\beta_2)}{\Gamma(\beta_1)} \times \sum_{q=0}^{n} \frac{(u-t_q)(u-t_{q-1})}{2!} \sup_{0\le u\le t_{n+1}} \left|\frac{\partial^2}{\partial u^2}[k_6(T,u)]_{u=y_q}\right|, \tag{60}$$

$$\left|\mathbb{E}^{\beta_1}_{7,u}(\epsilon_u)\right| < \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1,\beta_2)}{\Gamma(\beta_1)} \times \sum_{q=0}^{n} \frac{(u-t_q)(u-t_{q-1})}{2!} \sup_{0\le u\le t_{n+1}} \left|\frac{\partial^2}{\partial u^2}[k_7(S_\upsilon,u)]_{u=y_q}\right|, \tag{61}$$

$$\left|\mathbb{E}^{\beta_1}_{8,u}(\epsilon_u)\right| < \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1,\beta_2)}{\Gamma(\beta_1)} \times \sum_{q=0}^{n} \frac{(u-t_q)(u-t_{q-1})}{2!} \sup_{0\le u\le t_{n+1}} \left|\frac{\partial^2}{\partial u^2}[k_8(E_\upsilon,u)]_{u=y_q}\right|, \tag{62}$$

$$\left|\mathbb{E}^{\beta_1}_{9,u}(\epsilon_u)\right| < \frac{2\beta_2((n+1)\Delta t)^{\beta_1+\beta_2-1}B(\beta_1,\beta_2)}{\Gamma(\beta_1)} \times \sum_{q=0}^{n} \frac{(u-t_q)(u-t_{q-1})}{2!} \sup_{0\le u\le t_{n+1}} \left|\frac{\partial^2}{\partial u^2}[k_9(I_\upsilon,u)]_{u=y_q}\right|. \tag{63}$$

6. Numerical simulation and result discussion

In this study, an investigation has been conducted to determine the dynamic of antenatal compliant susceptible pregnant women, antenatal non-compliant susceptible pregnant women, malaria-infected pregnant women, filariasis infected pregnant women, malaria-filariasis infected pregnant women, treatment of malaria-filariasis infected pregnant women, susceptible mosquitoes, exposed mosquito and infected mosquito using various fractional orders. To show the efficiency of the proposed strategy, simulations have been carried out using MATLAB (The MathWorks Inc, 2016). Through simulations, we have been able to gain a better understanding of the model's dynamics and perform a more comprehensive analysis by observing how changes in the parameters and initial conditions affected the model's predictions. The results from the fractional order analysis were found to be more informative and generalizable than those obtained from other related works. We have used the Toufik-Atanagana (TA) numerical techniques to solve the proposed co-infection model. $S_h(0) = 50$, $S_2(0) = 30$, $I_m(0) = 10$, $I_f(0) = 15$, $I_{mf}(0) = 20$, $T(0) = 10$, $S_v(0) = 25$, $E_v(0) = 20, I_v(0) = 10$ are denoted as the initial conditions of proposed co-infection model and the parameter values are presented in Table as follows.

Parameters	Numerical values	Description
μ_h	0.00004	Per capita natural mortality rate of pregnant women
μ_v	0.05	Per capita natural mortality rate of mosquito's
δ_1	0.12	Progression rate from compliant
τ	0.0006	Rate of loss of immunity
σ_h	0.8331	Rate of malaria symptoms in pregnant women
ϵ	0.7	Modification parameter
α	0.2	Biting rate of mosquito's
θ	1.2	Modification parameter
a	0.8333	Transmission probability of malaria in humans
ρ	1.4	Modification parameter
b	0.09	Transmission probability of malaria in mosquito's
δ	0.00211	Modification parameter
βl_m	0.0183	Rate of mosquito bite leading to malaria infection
k	0.011	Modification parameter
βl_f	0.000036	Rate of mosquito bite leading to filariasis infection
η	0.003	Modification parameter
βl_{mf}	0.000027	Rate of mosquito bite leading to co-disease infection
ν	0.0036	Modification param ter
γ	0.00231	Disease induced death rate of the co-disease
ψ	1.0	Progression rate of malaria leading to filariasis symptoms
ϕ_1	0.00341	Treatment rate of malaria
ϕ_2	0.00061	Treatment rate of filariasis
ϕ_3	0.00072	Treatment rate of malaria – filariasis
σ_v	0.0112	Progression rate of exposed to infected mosquito's
K_f	0.011	Transmission rate
μ_υ	0.1429	Natural death rate of mosquito's
K_v	0.502	Transmission rate between susceptible and infected mosquitos
λ_υ	1000	Per capita birth rate of mosquito's
K_{ma}	0.011	Transmission rate
πA_v	0.0000421	Recruitment rate of susceptible pregnant women
$(1-\pi)A_h$	0.00003	Proportion of non-compliant susceptible
σ	0.0112	Progression rate of exposed to infected mosquito's

Figures 1–9 show the dynamics of antenatal compliant susceptible pregnant women, antenatal non-compliant susceptible pregnant women, malaria-infected pregnant women, filariasis infected pregnant women, malaria-filariasis infected pregnant women, treatment of malaria-filariasis infected pregnant women, susceptible mosquitos, exposed mosquitos and infected mosquitos respectively when fractal dimension $\beta_2 = 1, 0.95, 0.90, 0.85$ and fractional order $\beta_1 = 1, 0.95, 0.90, 0.85$. Further, when fractional order $\beta_1 = 1$ is fixed and fractal dimension is varied then Figures 10–18 show the dynamics of antenatal compliant susceptible pregnant women, antenatal non-compliant susceptible pregnant women, malaria-infected pregnant women, filariasis

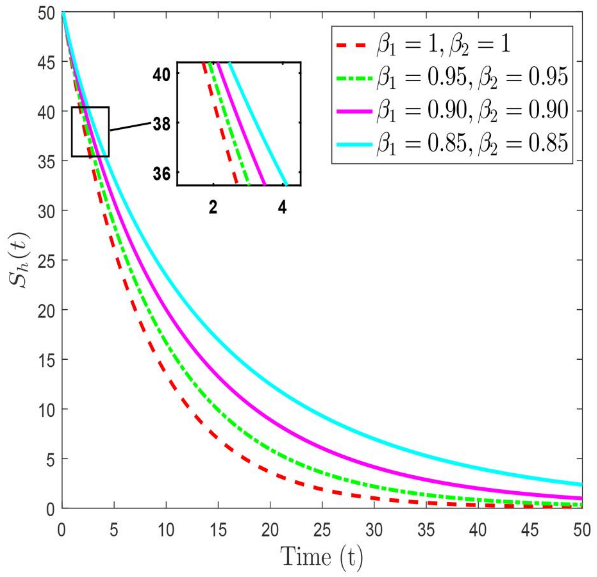


Figure 1. Plot for antenatal compliant susceptible pregnant women.

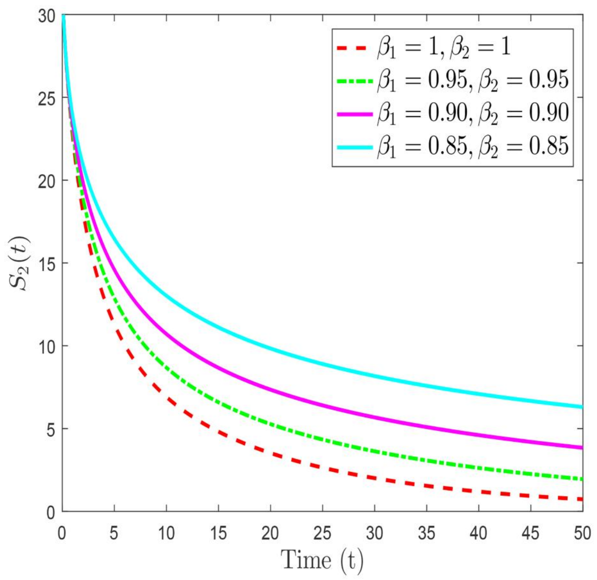


Figure 2. Plot for antenatal non-compliant susceptible pregnant women.

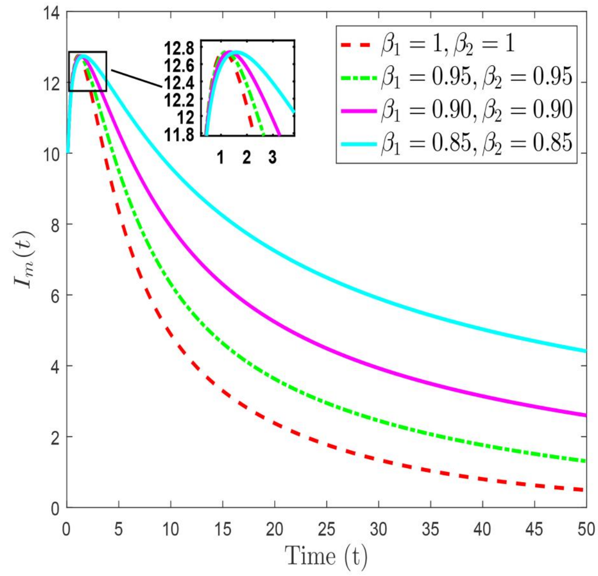


Figure 3. Plot for malaria infected pregnant women.

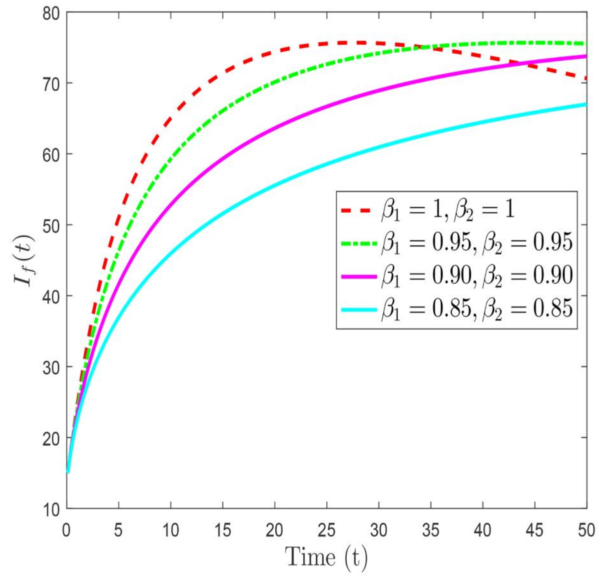


Figure 4. Plot for filariasis infected pregnant women.

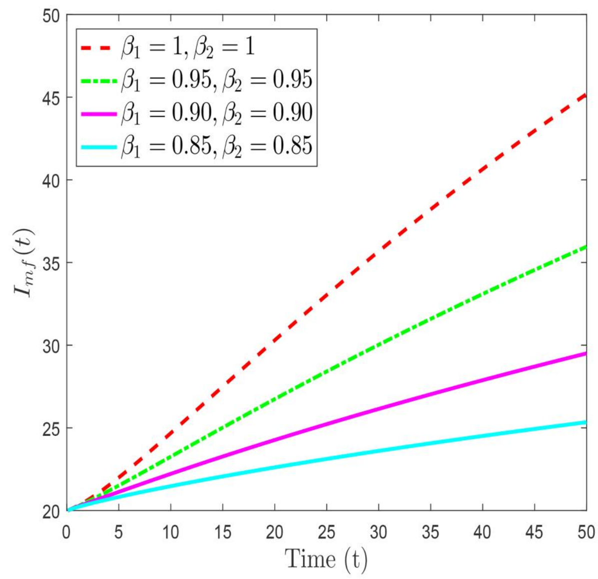


Figure 5. Plot for malaria-filariasis infected pregnant women.

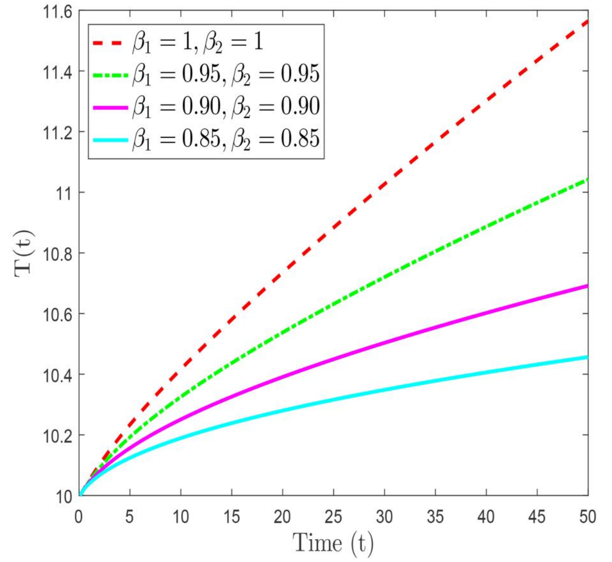


Figure 6. Plot for treatment of malaria-filariasis infected pregnant women.

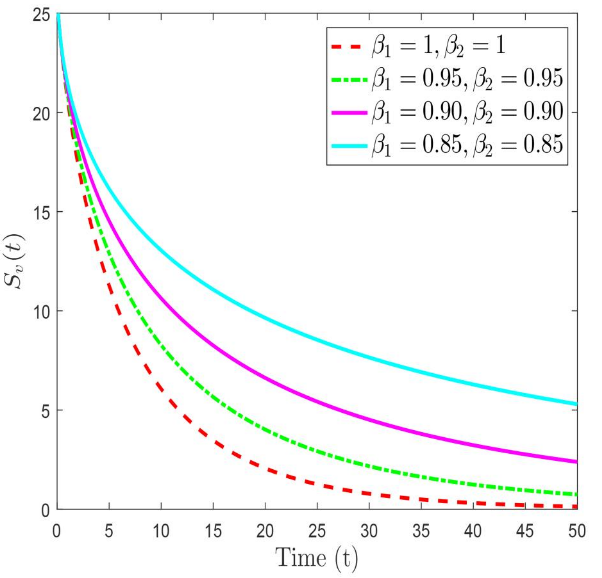


Figure 7. Plot for susceptible mosquito.

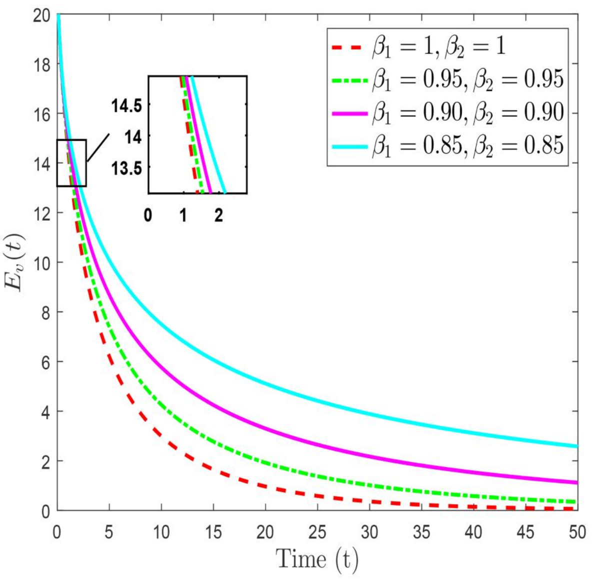


Figure 8. Plot for exposed mosquito.

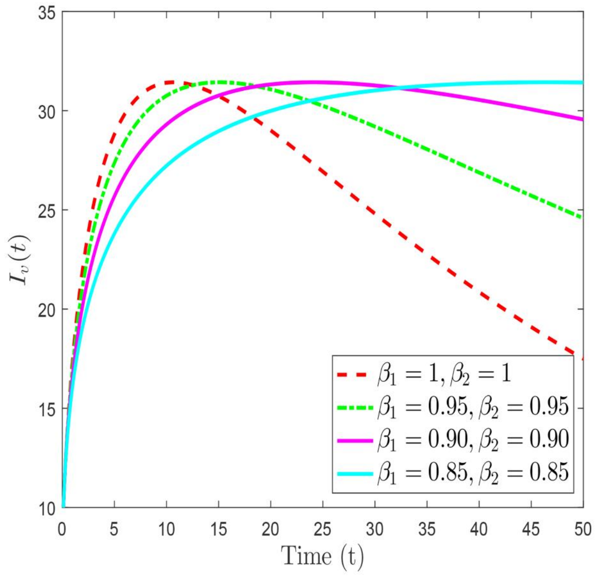


Figure 9. Plot for infected mosquito.

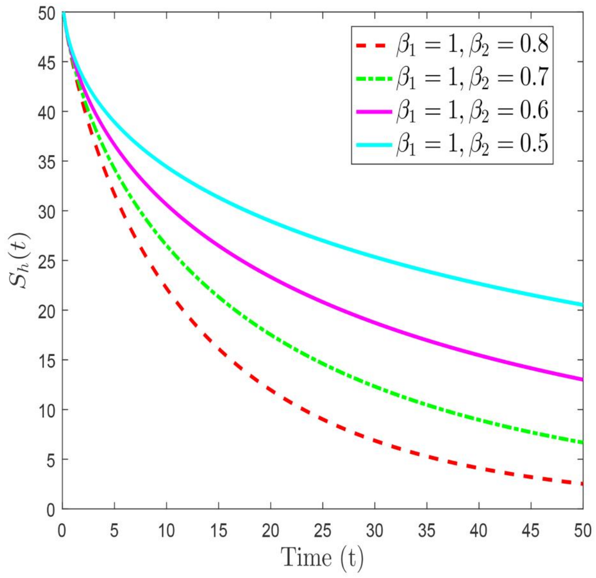


Figure 10. Plot for antenatal compliant susceptible pregnant women.

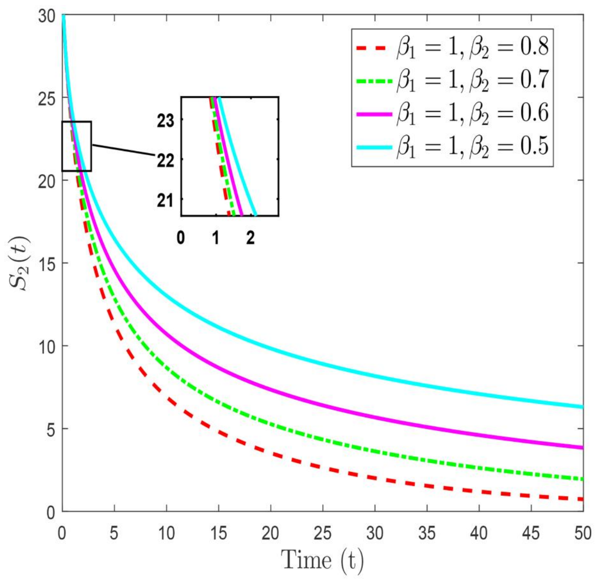


Figure 11. Plot for antenatal non-compliant susceptible pregnant women.

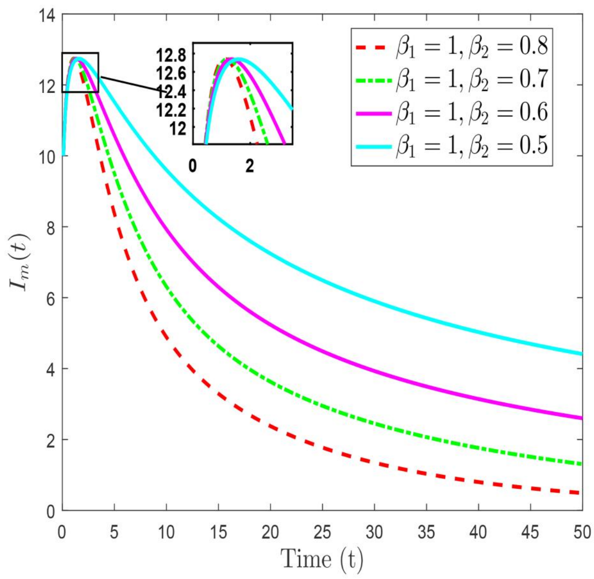


Figure 12. Plot for malaria infected pregnant women.

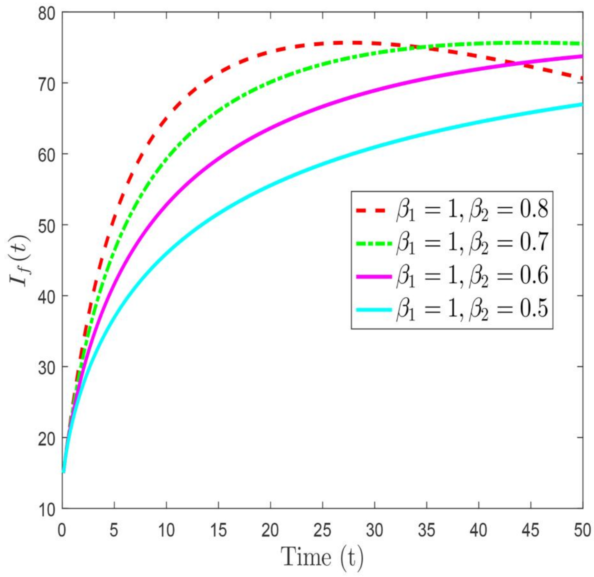


Figure 13. Plot for filariasis infected pregnant women.

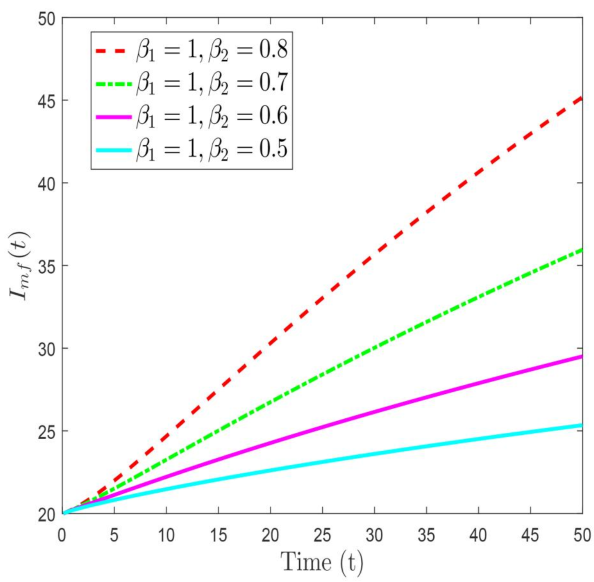


Figure 14. Plot for malaria-filariasis infected pregnant women.

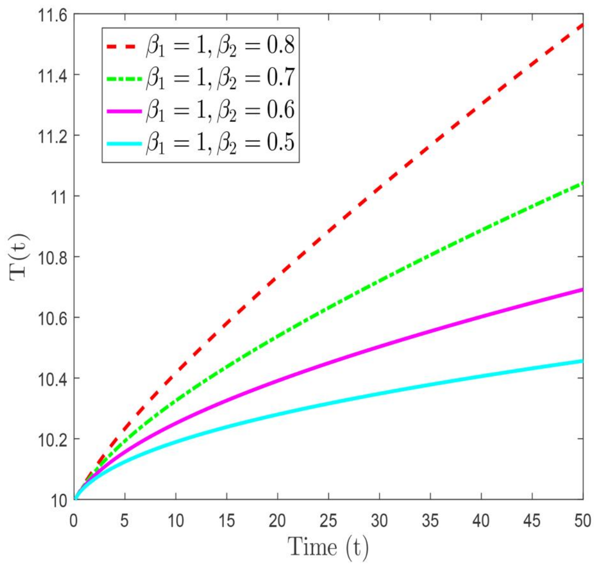


Figure 15. Plot for treatment of malaria-filariasis infected pregnant women.

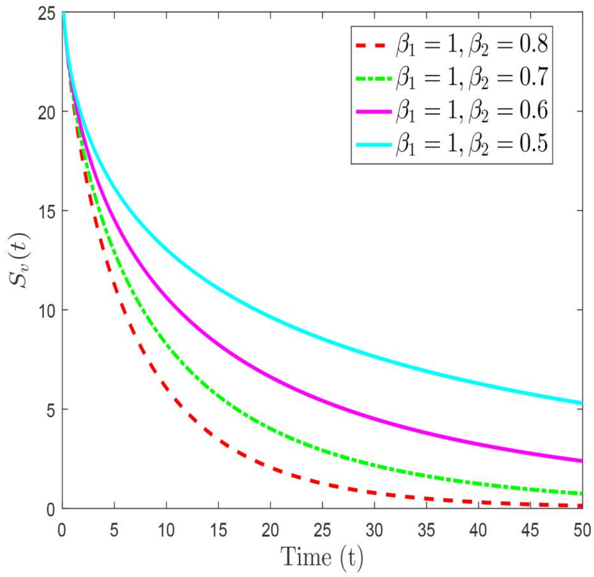


Figure 16. Plot for susceptible mosquito.

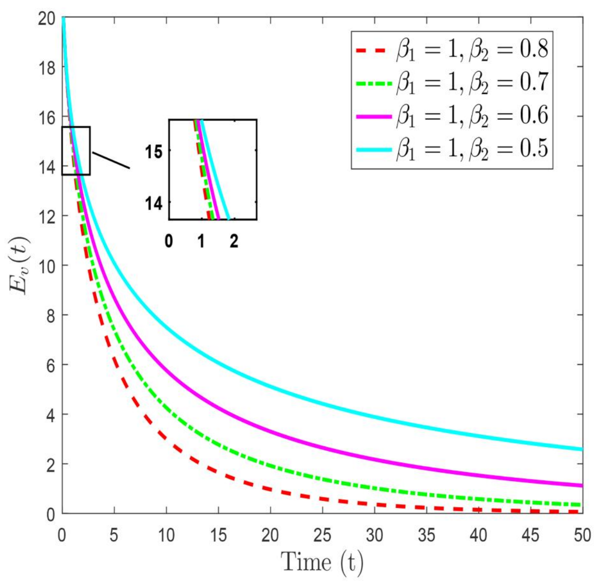


Figure 17. Plot for exposed mosquito.

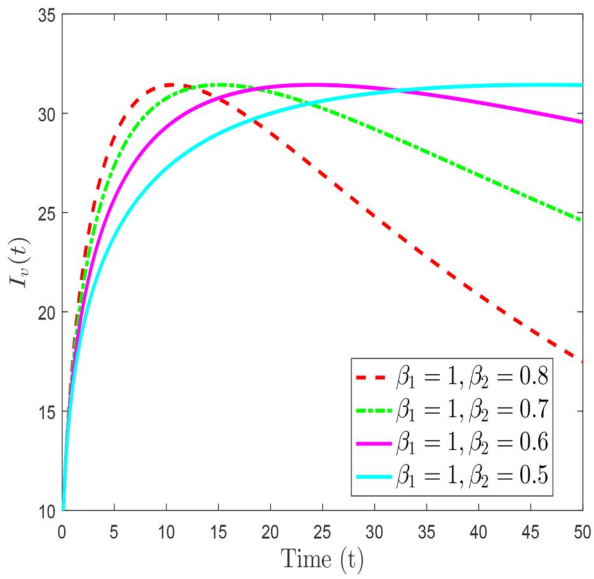


Figure 18. Plot for infected mosquito.

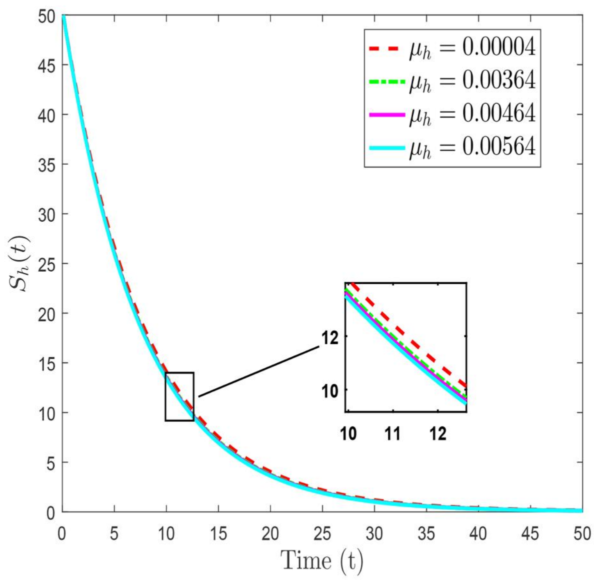


Figure 19. Plot of antenatal compliant susceptible pregnant women $S_h(t)$ for different values of μ_h.

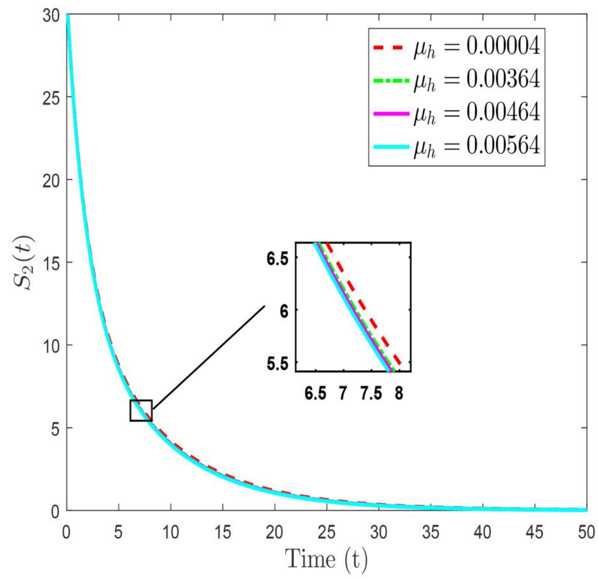


Figure 20. Plot of antenatal non-compliant susceptible pregnant women $S_2(t)$ for different values of μ_h.

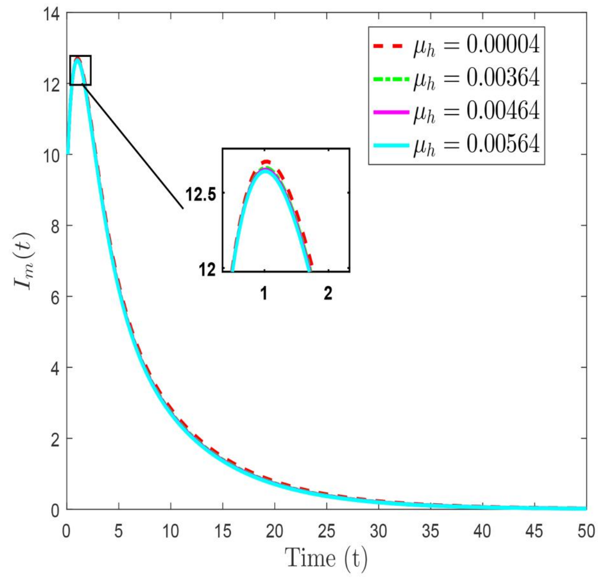


Figure 21. Plot of malaria infected pregnant women $I_m(t)$ for different values of μ_h.

infected pregnant women, malaria-filariasis infected pregnant women, treatment of malaria-filariasis infected pregnant women, susceptible mosquitos, exposed mosquitos and infected mosquitos respectively. Figures 19–24 show the dynamics of antenatal compliant susceptible pregnant women, antenatal non-compliant susceptible pregnant women, malaria-infected pregnant women, filariasis infected pregnant women, malaria-filariasis infected pregnant women andtreatment of malaria-filariasis infected pregnant women respectively when values of natural mortality rate (μ_h) of pregnant women is varied. Figures 25–27 show the dynamics of susceptible mosquitos, exposed mosquitos and infected mosquitos when values of natural mortality rate (μ_v) of mosquitos is varied. Figures 28 and 29 represent the comparison of numerical schemes (Toufik-Atangana schemes (33) and (43) with respect to Caputo and Caputo-Fabrizio operators) for the state variables antenatal compliant susceptible pregnant women, antenatal non-compliant susceptible pregnant women, malaria-infected pregnant women, filariasis infected pregnant women, malaria-filariasis infected pregnant women, treatment of malaria-filariasis infected pregnant women, susceptible mosquitos, exposed mosquitos and infected mosquitos respectively when fractal dimension $\beta_2 = 0.99$ and fractional order $\beta_1 = 0.99$. It has become clear that the fractal-

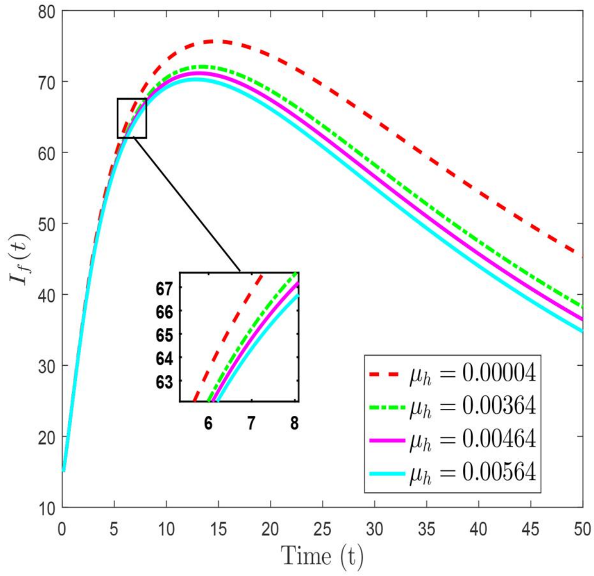


Figure 22. Plot of filariasis infected pregnant women $I_f(t)$ for different values of μ_h.

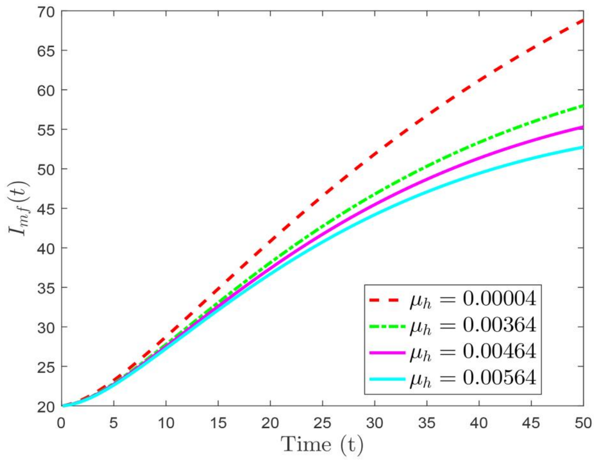


Figure 23. Plot of malaria-filariasis infected pregnant women $I_{mf}(t)$ for different values of μ_h.

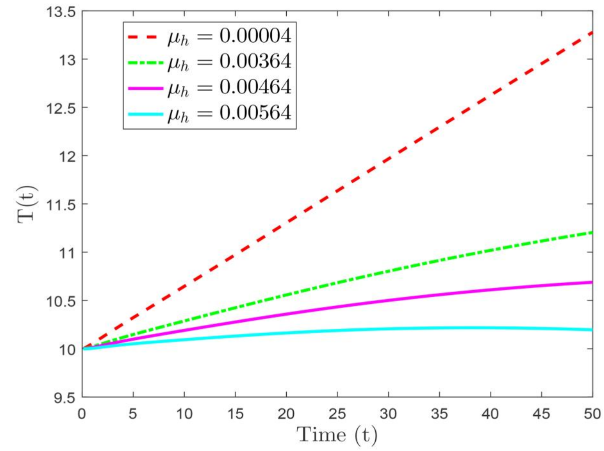


Figure 24. Plot of susceptible mosquito $T(t)$ for different values of μ_h.

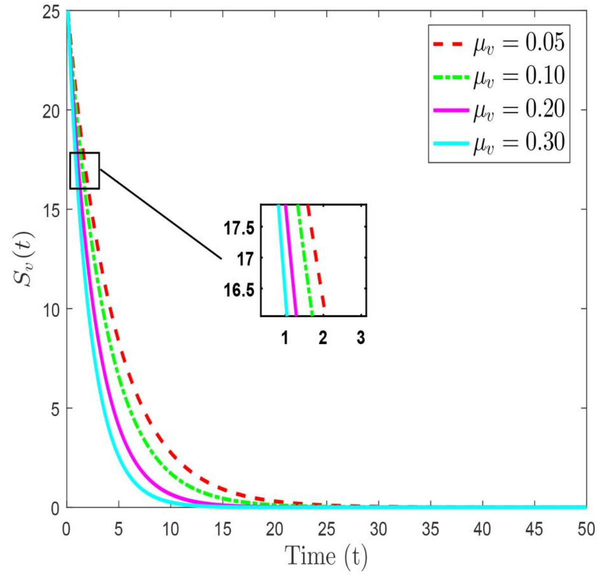


Figure 25. Plot of antenatal compliant susceptible pregnant women $S_v(t)$ for different values of μ_v.

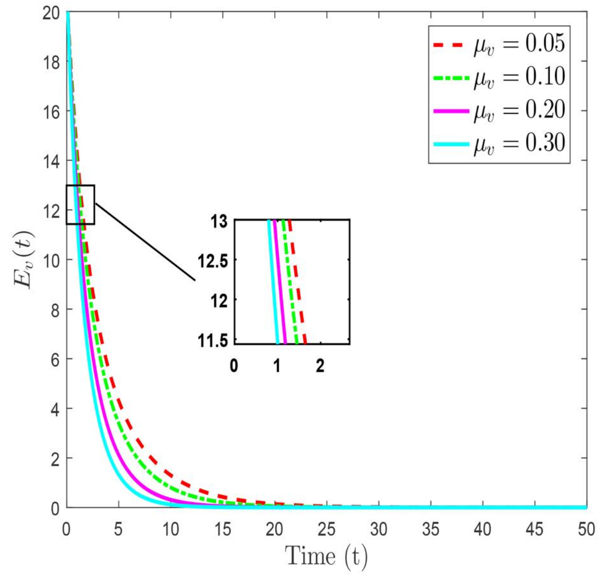


Figure 26. Plot of antenatal non-compliant susceptible pregnant women $E_v(t)$ for different values of μ_v.

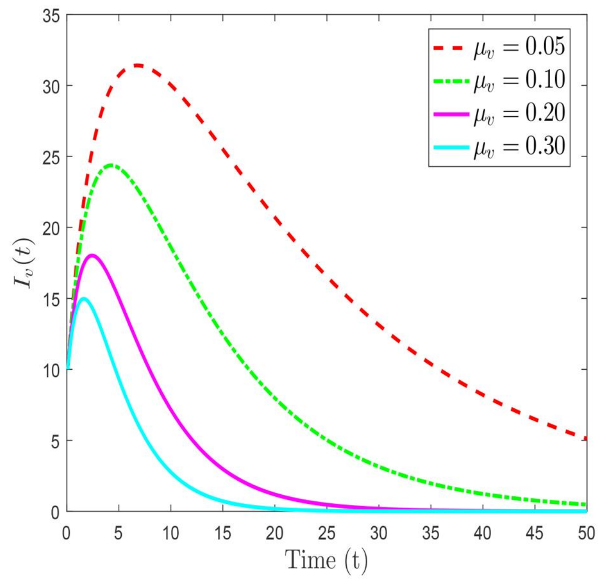
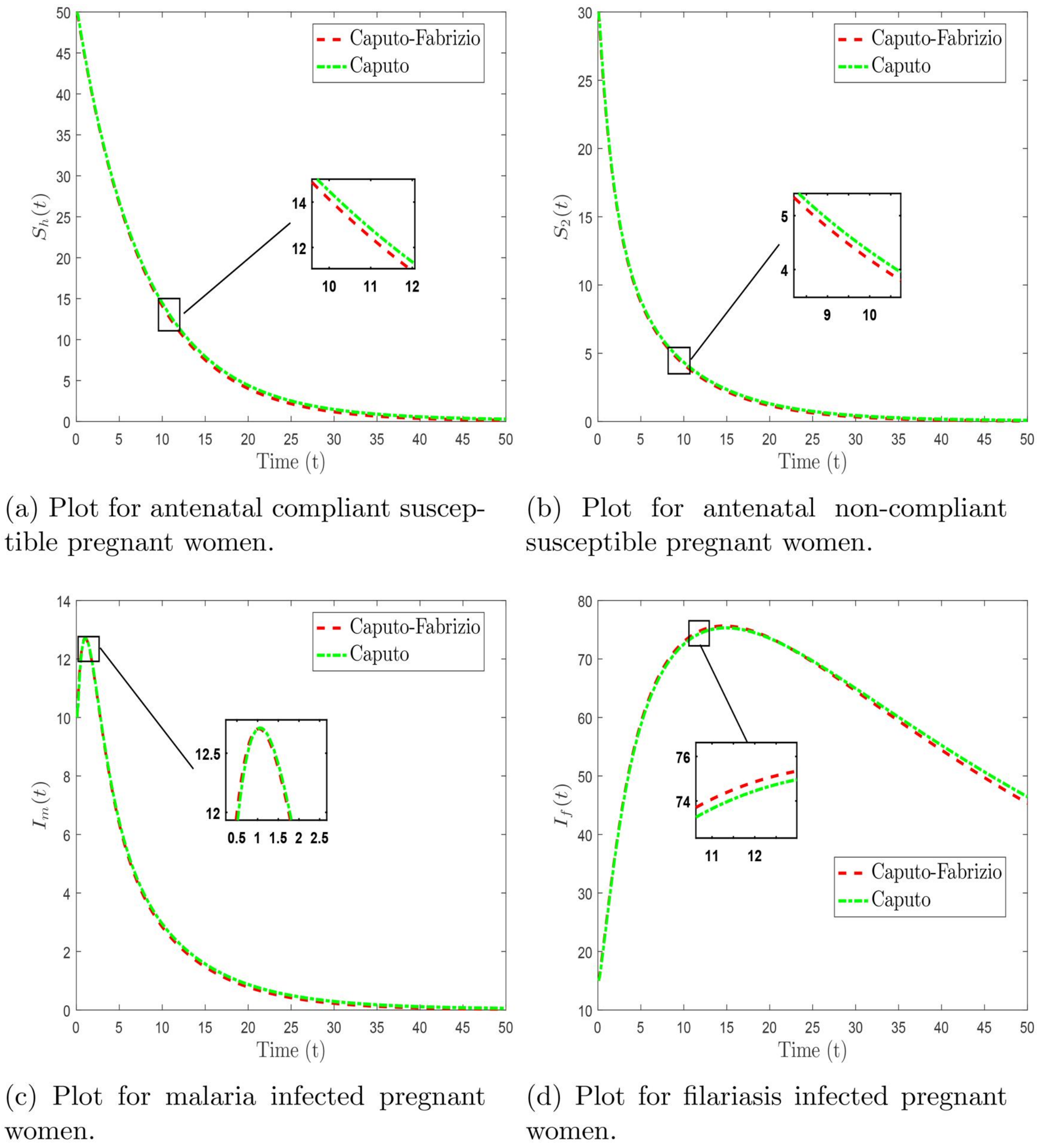


Figure 27. Plot of malaria infected pregnant women $I_v(t)$ for different values of μ_v.



(a) Plot for antenatal compliant susceptible pregnant women.

(b) Plot for antenatal non-compliant susceptible pregnant women.

(c) Plot for malaria infected pregnant women.

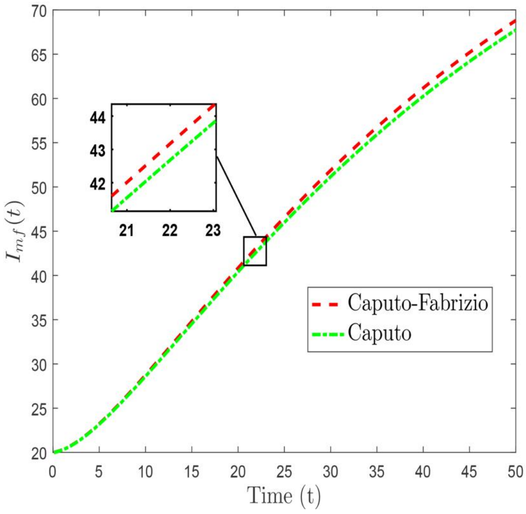
(d) Plot for filariasis infected pregnant women.

Figure 28. Comparison of numerical schemes (33) and (43) for co-infection malaria-filariasis model.

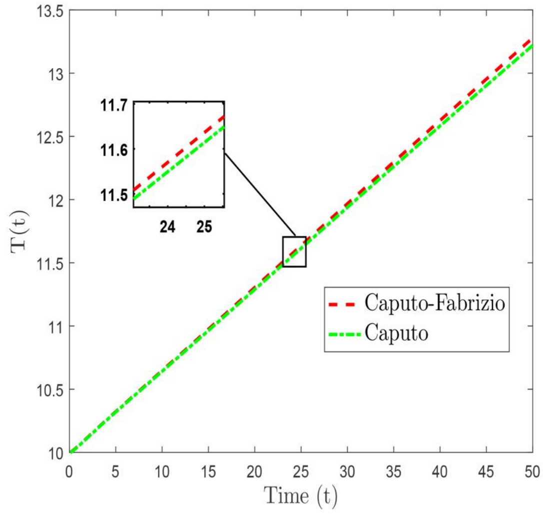
fractional approach is the most reliable for explaining the disease model, compared to the regular fractional and classical order cases.

7. Conclusion

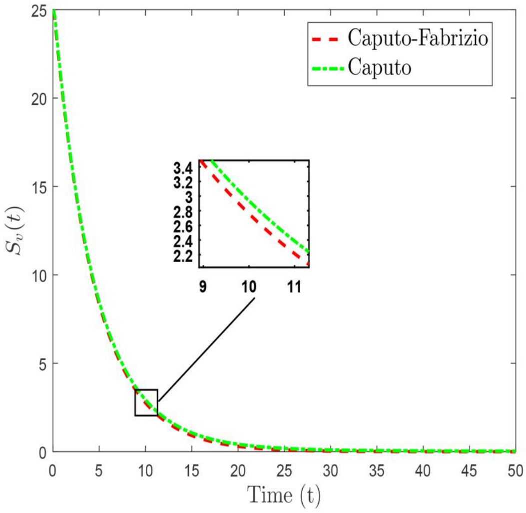
In this paper, The fractal-fractional derivative/integral has been used to examine the nonlinear dynamics of the malaria-filariasis model. We have used the Banach contraction theorem to analyze a malaria-filariasis model. The proposed model solution is examined using fixed-point theory to determine its existence and uniqueness. Applying the Ulam-Hyers stability technique, the stability analysis is conducted. We employ the numerical Toufik-Atanagana (TA) approach to offer an analytically estimated solution. We analyze the behaviour of the numerical solution and how it responds to various transmission parameters for a sort of arbitrary order with fractal dimensions. Variations in the state variable and model parameter values give information about the nature of the proposed model. With the help of a graphical representation, it has been demonstrated how the system's parameters and the order of derivatives will have a significant impact. Climatic variables, ideal controls, and time strategies have affected this disease. The study offers a distinctive perspective on the interactions between malaria and filariasis that will provide valuable insights for readers and public health authorities. Based on the numerical results presented, it can be concluded that the fractal-fractional principle yields higher efficiency than the fractional principle. Therefore, the fractal-fractional approach can be applied to a range of real-world problems to achieve better results. Moving forward, we plan to utilize this approach to analyze



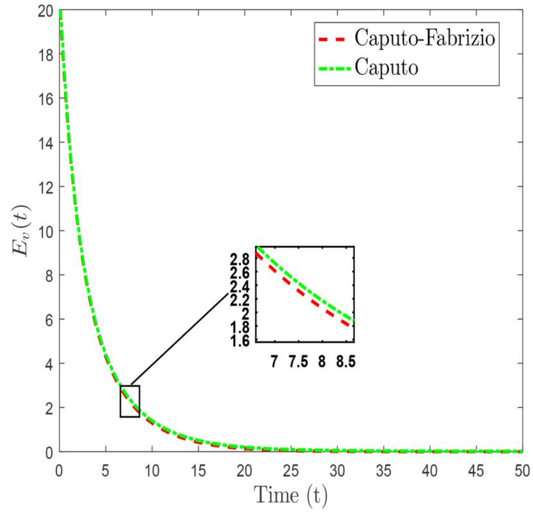
(a) Plot for malaria-filariasis infected pregnant women.



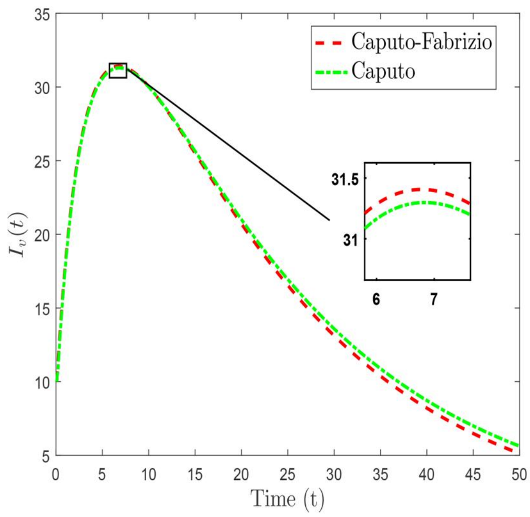
(b) Plot for treatment of malaria-filariasis infected pregnant women.



(c) Plot for susceptible mosquito.



(d) Plot for exposed mosquito.



(e) Plot for infected mosquito.

Figure 29. Comparison of numerical schemes (33) and (43) for co-infection malaria-filariasis model.

other real-world problems. In future, comparing the proposed models' numerical solution with additional numerical approaches could have potential benefits. To further improve and analyze the model, it can be extended to consider the impact of optimal controls and climatic factors.

Authors contributions

All authors contributed equally and significantly in writing this paper and typed, read, and approved the final manuscript.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Data availability statements

Data available on request from the authors.

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