Hindawi
Journal of Mathematics
Volume 2023, Article ID 8332760, 22 pages
https://doi.org/10.1155/2023/8332760

Research Article

A New Iteration Scheme for Approximating Common Fixed Points in Uniformly Convex Banach Spaces

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Received 20 July 2022; Revised 2 August 2023; Accepted 14 November 2023; Published 12 December 2023

Academic Editor: Ji Gao

In this paper, firstly, we introduce a method for finding common fixed point of L-Lipschitzian and total asymptotically strictly pseudo-non-spreading self-mappings and L-Lipschitzian and total asymptotically strictly pseudo-non-spreading non-self-mappings in the setting of a real uniformly convex Banach space. Secondly, the demiclosedness principle for total asymptotically strictly pseudo-non-spreading non-self-mappings is established. Thirdly, the weak convergence theorems of the proposed method to the common fixed point of the above mappings are proved. Our results improved, extended, and generalized some corresponding results in the literature.

1. Introduction and Preliminaries

Optimization theory (convex, nonconvex, and discrete) is an important field that has applications in almost every technical and nontechnical field, including wireless communication, networking, machine learning, security, transportation systems, finance (portfolio management), and operation research (supply chain and inventory). Numerous theoretical and practical areas, including variational and linear inequalities, approximation theory, nonlinear analysis, integral and differential equations and inclusions, dynamic systems theory, mathematics of fractals, mathematical economics (game theory, equilibrium problems, and optimization problems), mathematical modelling, and nonlinear analysis, rely on the fixed-point theory. Let Z be a Banach space (BS), $Z^\star$ the dual of Z, and $\varnothing \neq D \subset Z$ is a closed and convex subset of Z. The mapping $J\colon Z \longrightarrow 2^{Z^\star}$ defined by

$$J(\varpi) = \left\{\varpi^\star \in Z\colon \langle \varpi, \varpi^\star \rangle = \|\varpi\| \left\|\varpi^\star\right\|, \|\varpi\| = \left\|\varpi^\star\right\|\right\}, \quad (1)$$

is said to be normalized duality mapping.

Let $\Gamma\colon D \longrightarrow D$ be a nonlinear mapping. The symbols $\mathcal{N}, \mathbb{R}, \longrightarrow, \rightharpoonup, F(\Gamma)$ and $\mathcal{F} = \cap_{i=1}^N F(\Gamma_i)$ will be used to denote the set of natural numbers, the set of real numbers, strong convergence (SC), weak convergence (WC), the set of fixed points of Γ, and the set of common fixed points of Γ, respectively.

Definition 1. Recall that

(a) A mapping Γ is said to be nonspreading if there exists $j(s) \in J(s)$ such that, for all $s \in D$,

$$\phi(\Gamma s, \Gamma t) + \phi(\Gamma t, \Gamma s) \leq \phi(\Gamma s, t) + \phi(\Gamma t, s), \quad (2)$$

where $\phi(s,t) = \|s\|^2 - 2\langle s, j(t)\rangle + \|t\|^2$, for all s, t $\in$ Z and J is the duality mapping on D. Note that in real Hilbert spaces (H), the J is an identity mapping and

$\phi(\varpi, y) = \|\varpi - y\|^2$. Thus, in real Hilbert spaces, (2) is equivalent to

$$\|\Gamma s - \Gamma t\|^2 \leq \|s - t\|^2 + 2\langle s - \Gamma s, t - \Gamma t\rangle. \quad (3)$$

In 2008, Kohasaka and Takahashi [1] established this class of mapping in a smooth, strictly convex, and reflexive Banach space (RBS).

(b) A mapping Γ is called asymptotically nonspreading (ANS) if there exists $j(\varpi - y) \in J(\varpi - y)$ such that, for all $s, t \in D$,

$$\left\|\Gamma^n(s) - \Gamma^n(t)\right\|^2 \leq \|s - t\|^2 + 2\langle s - \Gamma^n s, j(t - \Gamma^n t)\rangle, \quad \text{for all } n \in \mathbb{N}. \quad (4)$$

Naraghirad [2] established the class of ANS mapping as a generalization of the class of nonspreading mapping. In addition, he proved that if K is a nonempty closed convex subset of a real BS and Γ is an ANS mapping of K, then Γ has a fixed point.

(c) A mapping Γ is said to be uniformly Lipschitzian with the Lipschitz constant $L > 0$ if

$$\left\|\Gamma^n(s) - \Gamma^n(t)\right\| \leq L\|s - t\|, \text{for all } s, t \in D \text{ and } n \in \mathbb{N}. \quad (5)$$

(d) A mapping Γ is called asymptotically strictly pseudo-non-spreading if there exist $k_n \subseteq [1, \infty]$ with $k_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $j(s - t) \in J(s - t)$ such that

$$\begin{aligned}\langle \Gamma^n s - \Gamma^n t, j(s - t)\rangle \leq \sigma_n\|s - t\|^2 - \gamma\left\|s - t - (\Gamma^n s - \Gamma^n t)\right\|^2 \\ + \langle s - \Gamma^n s, t - \Gamma^n t\rangle, \quad \text{for all } s, t \in \mathrm{D}, \text{for all } n \in \mathbb{N},\end{aligned} \quad (6)$$

where $\gamma = 1/2(1 - \beta) \in (0, 1)$, for all $\beta \in (0, 1)$ and $\sigma_n = 1/2(1 + k_n)$. Observe that $\sigma_n \longrightarrow 1$ as $k_n \longrightarrow 1$ and $n \longrightarrow \infty$. In a real Hilbert space (H) (see [3]), (6) is equivalent to

$$\begin{aligned}\left\|\Gamma^n s - \Gamma^n t\right\|^2 \leq k_n\|s - t\|^2 + \beta\left\|s - t - (\Gamma^n s - \Gamma^n t)\right\|^2 \\ + 2\langle s - \Gamma^n s, t - \Gamma^n t\rangle, \quad \text{for all } n \in \mathbb{N}.\end{aligned} \quad (7)$$

Remark 2. It is obvious from (4) and (7) that every ANS mapping is a subclass of the class of asymptotically strictly pseudo-non-spreading mapping with $\beta = 0$ and $k_n = 1$. Again, the class of k-asymptotically strictly pseudo-non-spreading mappings is more general than the classes of k-strictly pseudo-non-spreading mappings and k-asymptotically pseudocontractions (see [4], for more detail).

Example 1 (see [4]). Let $\Gamma\colon R \longrightarrow R$ be a mapping defined by

$$\Gamma s = \begin{cases} s, & \text{if } s \in (-\infty, 0), \\ -2s, & \text{if } s \in [0, \infty). \end{cases} \quad (8)$$

It was shown in [4] that Γ is β-strictly pseudo-non-spreading (i.e., a mapping $T\colon D \subseteq H \longrightarrow H$ such that for all $s, t \in D(T)$, there exists $\beta \in [0, 1]$ for which the inequality $\|\Gamma s - \Gamma t\|^2 \leq \|s - t\|^2 + \beta\|s - \Gamma s - (t - \Gamma t)\|^2 + 2\langle s - \Gamma s, t - \Gamma t\rangle$ holds but not nonspreading.

Observe that for all integer $n \geq 2$, we have

$$\Gamma^n s = \begin{cases} s, & \text{if } s \in (-\infty, 0), \\ -2s, & \text{if } s \in [0, \infty). \end{cases} \quad (9)$$

Clearly, Γ is asymptotically strictly pseudo-non-spreading mapping (see [3] for details).

Example 2. Let $Z = \ell^2$ with the usual norm $\|.\|$ defined by

$$\|s\| = \sqrt{\sum_{n=1}^{\infty} s_i^2}, \quad \text{for all } (s_1, s_2 \ldots) \in Z, \quad (10)$$

and $D = \{s = (s_1, s_2, \ldots, s_n, \ldots)\}$ be an orthogonal subspace of Z (i.e., for all $\varpi, y \in D \subset Z$, we have $(\langle s, t\rangle = 0)$). For each $s = (s_1, s_2 \ldots, s_n, \ldots) \in D$, define the mapping $\Gamma\colon D \longrightarrow D$ by

$$\Gamma^n s = \begin{cases} (s_1, s_2, \ldots, s_n, \ldots), & \text{if } \prod_{i=1}^{\infty} s_i < 0, \\ (-s_1, -s_2, \ldots, -s_n, \ldots), & \text{if } \prod_{i=1}^{\infty} s_i \geq 0. \end{cases} \quad (11)$$

Then, Γ is asymptotically strictly pseudo-non-spreading mapping (see [5] for details).

Remark 3. In the above discussion, each of the mappings considered is from a subset of a given space into itself. However, there are so many real-life problems in which the domain of the mapping under consideration is taken into the whole space (and not its subset). When that happens, the aforementioned mappings and their generalizations (assuming self-mappings) become irrelevant. Consequently, there is a need to consider another set of mappings (called non-self-mappings) that will bridge this gap.

The following definition will be required in the sequel.

Definition 4 (see [6]). Let Z be a BS and $\curlywedge\colon Z \longrightarrow D$ a continuous mapping. Then, $D \subset Z$ is called a retract of Z such that $\curlywedge(s) = s$, $\forall s \in D$. Further, if $\curlywedge$ is nonexpansive, then it is said to be a nonexpansive retraction (non-ER) of Z. Note that if $\curlywedge\colon Z \longrightarrow D$ is a retraction, then $\curlywedge^2 = \curlywedge$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space (UCBS) is a retract.

Example 3 (see [6]). Suppose $H = R^n$ with an inner product $\langle s,t\rangle = \sum_{i=1}^n s_i t_i$ and the usual norm $\|s\| = (\sum_{i=1}^n s_i^2)^{1/2}$, then H is a Hilbert space. Let $D = \{s \in H\colon \|s\| \le 1\}$. Define $\curlywedge\colon H \longrightarrow D$ by

$$\curlywedge s = \begin{cases} s, & \text{if } s \in D, \\ \dfrac{1}{\|s\|}, & \text{if } H - D. \end{cases} \tag{12}$$

Then, $\curlywedge$ is a non-ER of H onto D.

Definition 5. Let D be a nonempty, closed, and convex subset of a BS Z and $\Gamma\colon D \longrightarrow Z$ a non-self-mapping. Then,

(1) Γ is said to be ANS non-self-mapping if there exists $j(\varpi) \in J(\varpi)$ such that, for all $\varpi, y \in D$,

$$\left\|\Gamma(\curlywedge\Gamma)^{n-1}(\varpi) - \Gamma(\curlywedge\Gamma)^{n-1}(y)\right\|^2 \le \|\varpi - y\|^2 + 2\langle \varpi - \Gamma(\curlywedge\Gamma)^{n-1}\varpi, y - \Gamma(\curlywedge\Gamma)^{n-1}y\rangle, \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

(2) Γ is uniformly Lipschitzian with the Lipschitz constant $L > 0$ if

$$\left\|\Gamma(\curlywedge\Gamma)^{n-1}(\varpi) - \Gamma(\curlywedge\Gamma)^{n-1}(y)\right\| \le L\|\varpi - y\|, \quad \text{for all } \varpi, y \in C, n \in \mathbb{N}. \tag{14}$$

(3) Γ is said to be strictly asymptotically pseudo-non-spreading non-self-mapping if there exist $k_n \subseteq (1,\infty)$ with $k_n \longrightarrow 1$ as $n \longrightarrow \infty$ and $j(\varpi - y) \in J(\varpi - y)$ such that for all $\varpi \in C$,

$$\begin{aligned} \langle \Gamma(\curlywedge\Gamma)^{n-1}\varpi - \Gamma(\curlywedge\Gamma)^{n-1}y, j(\varpi - y)\rangle \le{} & -\gamma\left\|\varpi - \Gamma(\curlywedge\Gamma)^{n-1}\varpi - \left(y - \Gamma(\curlywedge\Gamma)^{n-1}y\right)\right\|^2 \\ & + \langle \varpi - \Gamma(\curlywedge\Gamma)^{n-1}\varpi, y - \Gamma(\curlywedge\Gamma)^{n-1}y\rangle \\ & + \sigma_n\|\varpi - y\|^2, \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{15}$$

where $\gamma = 1/2(1-\beta) \in (0,1)$, for all $\beta \in (0,1)$ and $\sigma_n = 1/2(1 + k_n)$. Observe that $\sigma_n \longrightarrow 1$ as $k_n \longrightarrow 1$ and $n \longrightarrow \infty$. Note that if Γ is a self-mapping, then $\curlywedge$ becomes the identity mapping so that (15) reduces to (7).

The above study of various nonlinear mappings is quite interesting. However, if there is no means to approximate their respective fixed points, then the time spent in the study would be a waste. Over the years, several researchers have constructed varying iterative schemes to achieve approximate fixed points of different nonlinear mappings. Chidume and Adamu [7] attained convergence via their modified iteration scheme for the common solution of split generalized mixed equality equilibrium and split equality fixed-point problems. Thianwan [8] established a new iteration scheme for mixed-type asymptotically nonexpansive mappings in hyperbolic spaces. Taiwo et al. [9] studied a simple strong convergent method for solving split common fixed-point problems. Shehu [10] investigated an iterative approximation for zeros of the sum of accretive operators, and Suantai et al. [11] worked on nonlinear iterative methods for solving the split common null point problem in Banach spaces. Still on the construction of the fixed-point iteration method, Saleem et al. [12, 13] proved several fixed-point results, by utilizing some novel iterative methods, in the context of intuitionistic extended fuzzy b-metric-like spaces and uniformly convex Banach space, respectively. Saleem et al. [14], while working on graphical fuzzy metric spaces, employed a new iterative method with the graphical structure to solve fractional differential equations. Again, in 2006, Wang [15] generalized the scheme studied in [16] (see below) for the case of two asymptotically nonexpansive non-self-mappings (ANENSMs), which was subsequently improved to a hybrid mixed-type iterative scheme involving two asymptotically nonexpansive self-mappings ANESMs and two ANENSMs in [17], in UCBS. Agwu et al. [18] generalized the scheme studied in [17] to hybrid mixed-type iteration method involving three total ANESMs and three ANENSMs (which simultaneously included the scheme studied in [17]) in UCBS, and Agwu and Igbokwe [19] generalized the scheme in [18] to hybrid mixed-type iteration method involving finite family of total ANESMs and finite family of total ANENSMs in real UCBS. Albert et al. [20] did work on the approximation of fixed point of nonexpansive mappings. Agwu et al. [18] proved the convergence of a three-step iteration scheme to the common fixed points of mixed-type total asymptotically nonexpansive mappings in UCBSs. Acedo and Xu [21] gave iteration methods for strict pseudocontractions in Hilbert space. Other works concerning the formulation and implementation of effective iteration techniques for fixed-point problems are readily available in [22] and [23].

Chidume et al. [16] established the following iterative scheme:

$$\begin{cases} \varpi_1 = \varpi \in D, \\ \varpi_{n+1} = \curlywedge\left(\alpha_n \Gamma (\curlywedge\Gamma)^{n-1} \varpi_n + (1-\alpha_n)\varpi_n\right), n \geq 1, \end{cases} \quad (16)$$

where α_n is a sequence in (0,1), D is a nonempty closed convex subset of a real UCBS Z, and $\curlywedge$ is a non-ER of Z onto D and proved several SC and WC theorems for ANENSMs in the context of UCBSs.

In [15], Wang generalized the iterative process (16) as follows:

$$\begin{cases} \varpi_1 = \varpi \in D, \\ \varpi_{n+1} = \curlywedge\left((1-\alpha_n)\varpi_n + \alpha_n \Gamma_1 (\curlywedge\Gamma_1)^{n-1} y_n\right), \\ y_n = \curlywedge\left((1-\beta_n)\varpi_n + \beta_n \Gamma_2 (\curlywedge\Gamma_2)^{n-1} \varpi_n\right), n \geq 1, \end{cases} \quad (17)$$

where $\Gamma_1, \Gamma_2\colon D \longrightarrow Z$ are two ANENSMs and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in [0, 1) and proved several WC and SC theorems for ANENSMs.

In 2012, Guo et al. [17] generalized the iterative process (16) as follows:

$$\begin{cases} \varpi_1 = \varpi \in D, \\ \varpi_{n+1} = \curlywedge\left((1-\alpha_n)G_1^n\varpi_n + \alpha_n \Gamma_1 (\curlywedge\Gamma_1)^{n-1} y_n\right), \\ y_n = \curlywedge\left((1-\beta_n)G_2^n\varpi_n + \beta_n \Gamma_2 (\curlywedge\Gamma_2)^{n-1} \varpi_n\right), n \geq 1, \end{cases} \quad (18)$$

where $G_1, G_2\colon D \longrightarrow D$ are two ANESMs, $\Gamma_1, \Gamma_2\colon K \longrightarrow Z$ are two ANENSMs, and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in [0, 1) and proved several WC and SC theorems for the mixed-type ANENSMs.

Recently, Saluja [24] generalized the iterative process (16) as follows:

$$\begin{cases} \varpi_1 = \varpi \in D, \\ \varpi_{n+1} = \curlywedge\left((1-\alpha_n)G_1^n\varpi_n + \alpha_n \Gamma_1 (\curlywedge\Gamma_1)^{n-1} y_n\right), \\ y_n = \curlywedge\left((1-\beta_n)G_2^n\varpi_n + \beta_n \Gamma_2 (\curlywedge\Gamma_2)^{n-1} \varpi_n\right), n \geq 1, \end{cases} \quad (19)$$

where $G_1, G_2\colon D \longrightarrow D$ are two total ANESMs, $\Gamma_1, \Gamma_2\colon D \longrightarrow Z$ are two total ANENSMs, and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in [0, 1) and proved some weak SC theorems for the mixed-type ANENSMs.

For the papers studied, it was discovered that a lot of attention has been given to fixed-point results for asymptotically nonexpansive mappings and some of its generalizations (Wang [15] studied convergence behavior of two ANENSMs in UCBS, Guo et al. [17] examined convergence character of four (two self and two nonself) asymptotically nonexpansive mappings, Saluja [24] investigated convergence behavior of four (two self and two nonself) total asymptotically nonexpansive mappings, Agwu and Igbokwe [19] understudied the nature of fixed point for a finite family of total ANESMs and ANENSMs, and Chima [25] examined fixed point for total asymptotically pseudocontractive mappings in the setup of a real Hilbert space), and almost all the results were communicated in the setup of a real Hilbert space. It is worth mentioning that there are other nonlinear mappings (ANS and asymptotically strict pseudo-non-spreading mappings; see, for instance, [3, 5]) that share the same parents (asymptotically quasi-non-expansive and asymptotically demicontractive mappings) with asymptotically nonexpansive mappings and asymptotically strict pseudocontractive mappings. Unlike nonexpansive-type mappings and their various generalization, the ANS-type mappings (especially, the class of total asymptotically strictly pseudo-non-spreading non-self-mappings) have not received much attention in the setup of a real BS as compared to those of the mappings studied above, perhaps due to unavailability of some working instruments in this area. Consequently, the following questions become necessary.

Question 6

(1) Is it possible to develop a demiclosedness principle for total asymptotically strict pseudo-non-spreading mappings in the setup of a real BS?

(2) Can one construct an independent mixed-type iterative scheme for the approximation of a common fixed point for a finite family of certain nonlinear mappings?

Motivated and inspired by the works of Ma and Wang [5] and Wojtaszczyk [26], inadequate iteration method for the class ANS-type mappings and the indispensable nature of weak convergence theorems in applications, in this paper, we study a new independent mixed-type iteration scheme (27) and then provide some WC theorems of this new iterative scheme (27) for mixed-type total asymptotically strictly pseudo-non-spreading self-mapping and total asymptotically strictly pseudo-non-spreading non-self-mapping in the setup of real UCBSs. Also, an affirmative answer is given to (1) and (2) in Question 6.

2. Relevant Preliminaries

In this section, we shall use the following definitions, lemmas, and known results in order to prove the main theorems of this paper: given a BS Z whose dimension is greater than or equal to 2. The mapping $\delta_Z(\varepsilon)\colon (0,2] \longrightarrow (0,2]$ represented by

$$\delta_Z(\varepsilon) = \inf\left\{1 - \left\|\frac{1}{2}(s+t)\right\| \colon \|s\| = 1, \|t\| = 1, \varepsilon = \|\varpi - y\|\right\}, \quad (20)$$

for all $s, t \in Z$, is called the modulus of convexity of Z. Note that if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0,2]$, then Z is called uniformly convex.

We recall the following definitions and lemmas which will be needed in what follows.

Definition 7 (see [27]). Let Z be a BS, Z^* its dual and $\mathscr{V} = \{s \in Z\colon \|s\| = 1\}$. If $\lim_{n\longrightarrow\infty} \|s + xt\| - \|\varpi\|/t$ exists for all $s, t \in \mathscr{V}$, then Z is given the Gateaux differentiable norm.

Definition 8 (see [27]). If the limit in Definition 7 exists and is attained uniformly for each $s \in \mathscr{V}$ (and for all $y \in \mathscr{V}$), then Z is given the Frechet differentiable norm (see [28] for more details). Consequent upon this, we have

$$\langle h, J(s)\rangle + \frac{1}{2}\|s\|^2 \leq \frac{1}{2}\|s+h\|^2 \leq \langle h, J(s)\rangle + \frac{1}{2}\|s\|^2 + d(\|s\|), \tag{21}$$

for all s, t ∈ Z, where functional $1/2\|\cdot\|^2$ at $s \in Z$, withJis the Frechet derivative $\langle\cdot\rangle$ is the pairing between Z and Z^* and d is an increasing function defined on $[0,\infty]$ such that $\lim_{\varpi \longrightarrow \infty} d(\varpi)/\varpi = 0$.

Definition 9. The BS Z is given Opial condition [29] if, for any sequence $\{s_n\} \in$ ZWCtos for each $s \in Z$, it follows that $\liminf_{n\longrightarrow\infty} \|s_n - s\| < \liminf_{n\longrightarrow\infty} \|s_n - t\|$ and equivalently $\limsup_{n\longrightarrow\infty} \|s_n - s\| < \limsup_{n\longrightarrow\infty} \|s_n - t\|$ for all $t \in Z$ with $s \neq t$. Whereas Hilbert spaces and all spaces l^p $(1 < p < \infty)$ satisfy Opial conditions, the space $L^p[0,\pi]$ with $1 < p \neq 2$ does not satisfy the Opial condition.

Definition 10 (see [5]). Let $\Gamma: D \longrightarrow D$ be a nonlinear mapping. Then, Γ is said to be demiclosed at 0, if, for any sequence $\{s_n\} \in D$, the condition that $s_n \rightharpoonup s \in D$ and $\Gamma s_n \longrightarrow 0$ implies $\Gamma s = 0$.

Definition 11. Let Z be a real BS. If, for every sequence $s_n \in Z$, $s_n \rightharpoonup s$ and $\|s_n\| \longrightarrow \|s\|$ imply $\|\varpi_n - \varpi\| \longrightarrow 0$. Then, Z is given the Kadec–Klec property [30].

Lemma 12 (see [31, 32]). *Let Z be a real BS. Then, for all* $s, t \in Z$, $j(s-t) \in J(s-t)$,

$$\|s+t\|^2 \leq \|s\|^2 + 2\langle t, j(s+t)\rangle. \tag{22}$$

Lemma 13 (see [33]). *Let the sequences* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \in [0,\infty]$ *and satisfying the inequality:*

$$\alpha_{n+1} \leq (1+\beta_n)\alpha_n + \gamma_n, \text{for all } n \geq 1. \tag{23}$$

If $\sum_{n=1}^{\infty}\beta_n < \infty$ *and* $\sum_{n=1}^{\infty}\gamma_n < \infty$, *then*

(1) $\lim_{n\longrightarrow\infty} \alpha_n$ *exists*
(2) ln particular, if $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to* 0, *then* $\lim_{n\longrightarrow\infty} = 0$.

Lemma 14 (see [30]). *Let Z be a UCBS and* $0 < p \leq \lambda_n \leq q < 1$ *for each* $n \geq 1$. *Suppose that* $\{s_n\}$ *and* $\{t_n\}$ *are sequences in Z such that*

$$\limsup_{n\longrightarrow\infty} \|s_n\| \leq r, \limsup_{n\longrightarrow\infty} \|t_n\| \leq r, \quad \lim_{n\longrightarrow\infty} \|\lambda_n s_n + (1-\lambda_n)t_n\| = r, \tag{24}$$

hold for some $r \geq 0$. *Then,* $\lim_{n\longrightarrow\infty} \|s_n - t_n\| = 0$.

Lemma 15 (see [30]). *Let Z be a real RBS such that its dual* Z^* *has the Kadec–Klec property. Let* $\{s_n\}$ *be a bounded sequence in Z and* $\gamma, \xi \in \omega_\omega(s_n)$ *(where* $\omega_\omega(s_n)$ *denotes the set of all weak subsequential limits of* $\{s_n\}$*). Suppose* $\lim_{n\longrightarrow\infty} \|\lambda s_n + (1-\lambda)\gamma - \xi\|$ *exists for all* $\lambda \in [0,1]$. *Then,* $\gamma = \xi$.

Lemma 16 (see [30]). *Let Z be a real UCBS and* $\varnothing \neq D \subset Z$ *be convex. Then, there exists a strictly increasing continuous convex function* $\phi: [0,\infty) \longrightarrow [0,\infty)$ *with* $\phi(0) = 0$ *such that for each Lipschitzian mapping* $\Gamma: D \longrightarrow D$ *with the Lipschitz constant* $L > 0$,

$$\|t\Gamma\varpi - (1-t)\Gamma y - \Gamma(\text{tx} - (1-t)y)\| \leq L\phi^{-1}\left(\|\varpi - y\| - \frac{1}{L}\|\Gamma\varpi - \Gamma y\|\right), \tag{25}$$

for all $\varpi, y \in D$ *and for all* $t \in [0.1]$.

Lemma 17 (see [34]). *Let Z be a real UCBS and* $\varnothing \neq D \subset Z$ *bounded close and convex. Then, there exists a strictly increasing continuous convex function* $\phi: [0,\infty) \longrightarrow [0,\infty)$ *with* $\phi(0) = 0$ *such that for any Lipschitzian mapping* $\Gamma: D \longrightarrow Z$ *with Lipschitz constant* $L \geq 1$ *and elements* $\{\varpi_n\}_{j=i}^{n}$ *in D and any nonnegative numbers* $\{t_j\}_{j=1}^{n}$ *with* $\sum_{j=1}^{n} t_j = 1$, *the following inequality holds:*

$$\left\|\Gamma\left(\sum_{j=1}^{n} t_j\varpi_j\right) - \sum_{j=1}^{n} t_j\Gamma\varpi_j\right\| \leq L\phi^{-1}\left\{\max_{1\leq j,k\leq n}\left(\left\|\varpi_j - \varpi_k\right\| - L^{-1}\left\|\Gamma\varpi_j - \Gamma\varpi_k\right\|\right)\right\}. \tag{26}$$

Lemma 18 (see [26]). *If the sequence* $\{\varpi_n\}_{n=1}^{\infty}$ *WC to* ϖ, *then there exists a sequence of convex combination* $y_j = \sum_{k=1}^{n(j)} \lambda_k^{(j)}\varpi_{k+j}$, $\lambda_k^{(j)} \geq 0$ *and* $\sum_{k=1}^{n(j)}\lambda^{(j)} = 1$, *such that* $\|y_j - \varpi\| \longrightarrow 0$. *as* $n \longrightarrow \infty$.

3. Main Results

Let Z a real normed space and $\varnothing \neq D \subset Z$ be closed and convex. Let $\Gamma_i: D \longrightarrow Z$ be a finite family of total asymptotically strictly pseudo-non-spreading non-self-mappings and $G_i: D \longrightarrow D$ be a finite family of total asymptotically strictly pseudo-non-spreading self-mappings. We define an iterative scheme generated by $\{\varpi_n\}_{n\geq 1}$ as follows:

$$\begin{cases} \varpi_1 \in K, \\ \varpi_{n+1} = \curlywedge\left[\eta_n y_{in} + (1-\eta_n)\left(\alpha_n\varpi_n + \frac{1}{2}(1-\alpha_n)\right)\left(G_i^n\varpi_n + \Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right)\right]; \\ y_{in} = (\curlywedge\beta_n)G_{i+1}^n\varpi_n + (1-\beta_n)\Gamma_{i+1}\left((\curlywedge\Gamma_{i+1})^{n-1}y_{(i+1)n}\right), \\ y_{(i+1)n} = \curlywedge\left(\nu_n G_{i+2}^n\varpi_n + (1-\nu_n)\Gamma_{i+2}(\curlywedge\Gamma_{i+2})^{n-1}\varpi_n\right), \end{cases} \tag{27}$$

where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}, \{\gamma_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \in (0,1)$ and $i = 1, 2, \ldots, m$.

Definition 19. Let Z be an arbitrary BS and $\varnothing \neq D \subset Z$ be closed and convex. Let $\Gamma: D \longrightarrow Z$ be nonlinear mapping. Following the terminology of Alber et al. [20], Γ is called total asymptotically strictly pseudo-non-spreading if for every $\varpi, y \in D$, $\gamma \in (0,1)$, and $j(\varpi - y) \in J(\varpi - y)$, there exist sequences $\{\sigma_n\}_{n\geq 1}, \{\xi_n\}_{n\geq 1} \subset (1,\infty)$: $\sigma_n \longrightarrow 1$ with $\xi_n \longrightarrow 0$ as $n \longrightarrow \infty$ and a strictly increasing continuous function $\phi: R^+ \longrightarrow R^+$, R^+ denoting the set of positive real numbers, with $\phi(0) = 0$ and $\lim_{n\longrightarrow\infty} \psi(t) = \infty$ such that

$$\begin{aligned}\langle \Gamma(\curlywedge\Gamma)^{n-1}\varpi - \Gamma(\curlywedge\Gamma)^{n-1}y, j(\varpi - y)\rangle \leq \|\varpi - y\|^2 - \gamma\left\|\varpi - \Gamma(\curlywedge\Gamma)^{n-1}\varpi - \left(y - \Gamma(\curlywedge\Gamma)^{n-1}y\right)\right\|^2 \\ + \langle \varpi - \Gamma(\curlywedge\Gamma)^{n-1}\varpi, y - \Gamma(\curlywedge\Gamma)^{n-1}y\rangle + \sigma_n\phi(\|\varpi - y\|) + \xi_n.\end{aligned} \tag{28}$$

If $F(\Gamma) \neq \varnothing$ and $q \in F(\Gamma)$, then (28) reduces to

$$\langle \Gamma(\curlywedge\Gamma)^{n-1}\varpi - q, j(\varpi - q)\rangle \leq \|\varpi - q\|^2 - \gamma\left\|\varpi - \Gamma(\curlywedge\Gamma)^{n-1}\varpi\right\|^2 + \sigma_n\phi(\|\varpi - q\|) + \xi_n. \tag{29}$$

Lemma 20 (demiclosed principle for total asymptotically strictly pseudo-non-spreading non-self-maps). *Let Z be a UCBS, $\varnothing \neq D \subset Z$ be closed, convex, and bounded and $\Gamma: D \longrightarrow Z$ be L-Lipschitz continuous and total asymptotically strictly pseudo-non-spreading mapping with $\phi: R^+ \longrightarrow R^+$ and the sequences $\{\mu_n\}_{n\geq 1}, \{\xi_n\}_{n\geq 1}$ such that $\mu_n, \xi_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, $I-\Gamma$ is demiclosed at zero.*

Proof. Suppose $\{\varpi_n\}_{n=1}^{\infty}$ WC to $\omega \in D$ and $\{\varpi_n - \Gamma\varpi_n\}$ SC to 0. We show that $(I-\Gamma)\omega = 0$. It is clear that $\{\varpi_n\}_{n\geq 1}$ is bounded. Hence, there exists $\rho > 0$ such that $\{\varpi_n\}_{n\geq 1} \subset C = D \cap \overline{B_\rho}$ is a closed ball in Z with center 0 and radius ρ. Thus, C is nonempty closed bounded and convex subset in D.

$\Gamma(\curlywedge\Gamma)^{n-1}\omega \longrightarrow \omega$ claimed as $n \in \longrightarrow \infty$. In fact, since $\{\varpi_n\}_{n\geq 1}$ CW to ω, by Lemma 18 (see, e.g., [14]), we get that, for all $n > 1$, there exists a convex combination

$$y_n = \sum_{n=1}^{m(n)} t_i^{(n)}\varpi_{i+n}, t_i^{(n)} \geq 0, \qquad \sum_{n=1}^{m(n)} t_i^{(n)} = 1 \text{ such that } \|y_n - \omega\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{30}$$

Since $\{\varpi_n - \Gamma\varpi_n\}$ converges to 0, it follows that for any positive integer $m \geq 1$, and given any $\epsilon > 0$, there exists $N_1 = N(\epsilon) > 0$ such that

$$\|(I-\Gamma)\varpi_n\| < \frac{\epsilon}{1+m}, \text{for all } n \geq N_1. \tag{31}$$

Hence, for all $n \geq N_1$, using Definition 19 and $\curlywedge$ is nonexpansive, we deduce, for any fixed $k \geq 1$, utilizing the well-known inequality

$$\|\varpi + y\|^2 \leq \|\varpi\|^2 + 2\langle y, j(y + \varpi)\rangle, \tag{32}$$

which holds for all $\varpi, y \in E$ and for all $j(\varpi + y) \in J(\varpi + y)$, we have

$$\begin{aligned}
\left\|\left(I - \Gamma(\curlywedge\Gamma)^{k-1}\right)\varpi_n\right\|^2 &= \left\|\varpi_n - \Gamma\varpi_n + \Gamma\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|^2 \\
&\leq \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\langle\Gamma\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&= \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\langle\Gamma\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n + \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&= \left\|\varpi_n - \Gamma_n\right\|^2 + 2\langle\Gamma\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&\quad - 2\langle\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&= \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\langle\Gamma\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&\quad - 2\langle\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \varpi_n + \varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&= \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\langle\Gamma\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&\quad - 2\langle\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&- 2\langle\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&= \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\langle\Gamma\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}Tx_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&\quad - 2\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n\right\|\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\| \\
&\quad - 2\left\|\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \varpi_n\right\|\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\| \\
&\leq \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\langle\Gamma x_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&\quad - 2\left\|\Gamma(P\Gamma)^{k-1}\varpi_n - \varpi_n\right\|\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\| \\
&= \left\|\varpi_n - \mathrm{Tx}_n\right\|^2 + 2\langle\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n \\
&\quad -\left(\Gamma(\curlywedge\Gamma)^{k-1}\Gamma\varpi_n - \mathrm{Tx}_n\right) \\
&\quad -\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&\quad - 2\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|^2 \\
&\leq \left\|\varpi_n - \Gamma\varpi_n\right\|^2 \\
&\quad + 2\langle\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n, j\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\rangle \\
&\quad - 2\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|^2 \\
&\leq \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|^2 \\
&\quad - \gamma\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \left(\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right)\right\|^2 \\
&\quad + \langle\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n, \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\rangle \\
&\quad + \sigma_n\psi\left(\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|\right) + \xi_n\Big\} - 2\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|^2 \\
&= \left\|\varpi_n - \Gamma\varpi_n\right\|^2 - 2\left\|\gamma\varpi - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|^2 \\
&\quad - 2\langle\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n, \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\rangle \\
&\quad + 2\sigma_n\psi\left(\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right) + 2\xi_n \\
&= \left\|\varpi_n - \Gamma\varpi_n\right\|^2 - 2\gamma\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&- 2\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|\left\|\Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\| \\
&+ 2\sigma_n\psi\left(\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|\right) + 2\xi_n \\
&\le \left\|\varpi_n - \Gamma\varpi_n\right\|^2 - 2\gamma\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|^2 + 2\sigma_n\psi\left(\left\|\varpi_n - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_n\right\|\right) + 2\xi_n \\
&\le \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + \sigma_n\phi\left(\left\|(I-\Gamma)\varpi_n\right\| + \left\|(\Gamma - \Gamma(\curlywedge\Gamma))\varpi_n\right\| + \left\|\left(\Gamma(\curlywedge\Gamma) - \Gamma(\curlywedge\Gamma)^2\varpi_n\right)\right\|\right. \\
&\left. + \left\|\left(\Gamma(\curlywedge\Gamma)^2 - \Gamma(\curlywedge\Gamma)^3\right)\varpi_n\right\| + \cdots + \left\|\left(\Gamma(\curlywedge\Gamma)^{k-2} - \Gamma(\curlywedge\Gamma)^{k-1}\right)\varpi_n\right\|\right) + 2\xi_n \\
&\le \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\sigma_n\phi\left(L\sum_{k=1}^{m-1}\left\|(I-\Gamma)\varpi_n\right\|\right) + 2\xi_n \\
&= \left\|\varpi_n - \Gamma\varpi_n\right\|^2 + 2\sigma_n\phi\left((m-1)L\left\|(I-\Gamma)\varpi_n\right\|\right) + 2\xi_n.
\end{aligned}
\tag{33}
$$

From (31) and (33) and the condition on the function ϕ, we obtain

$$
\left\|\left(I - (\curlywedge\Gamma)^{k-1}\right)\varpi_n\right\| < \epsilon. \tag{34}
$$

In addition,

$$
\begin{aligned}
\left\|\Gamma(\curlywedge\Gamma)^{k-1}y_n - y_n\right\| &\le \left\|\Gamma(\curlywedge\Gamma)^{k-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+n}\right\| \\
&+ \sum_{i=1}^{m(n)} t_i^{(n)}\left\|\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+n} - \varpi_{i+n}\right\|.
\end{aligned}
\tag{35}
$$

Moreover, with the help of Lemma 17, and for all $n \ge N$, there exists ϕ: $[0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ that is increasing function, and we obtain

$$
\begin{aligned}
\left\|\Gamma(\curlywedge\Gamma)^{k-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)}\Gamma(P\Gamma)^{k-1}\varpi_{i+n}\right\| &= \left\|\Gamma(\curlywedge\Gamma)^{k-1}\sum_{i=1}^{m(n)} t_i^{(n)}\varpi_{i+1}\right. \\
&\left. - \sum_{i=1}^{m(n)} t_i^{(n)}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+n}\right\| \\
&\le u_k\phi^{-1}\left\{\max_{1\le k,u\le n}\left(\varpi_{i+n} - \varpi_{i+u}\right.\right. \\
&\left.\left. - u_k^{-1}\left\|vx_{i+1} - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{u+n}\right\|\right)\right\} \\
&= u_k\phi^{-1}\left\{\max_{1\le k,u\le n}\left(\left\|\varpi_{i+n} - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+n}\right.\right.\right. \\
&+ \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+n} - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{u+n} \\
&\left. + \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{u+n} - \varpi_{i+u}\right\| \\
&\left.\left. - u_k^{-1}\left\|\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+1} - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{u+n}\right\|\right)\right\} \\
&\le u_k\phi^{-1}\left\{\max_{1\le k,u\le n}\left(\varpi_{i+n} - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+n}\right.\right. \\
&+ \left\|\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+n} - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{u+n}\right\| \\
&+ \left\|\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{u+n} - \varpi_{i+u}\right\|
\end{aligned}
$$

$$
\begin{aligned}
&- u_k^{-1}\left\|\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+1} - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{u+n}\right\|\Big)\Big\}\\
&\leq u_k\phi^{-1}\Big\{\max_{1\leq k,u\leq n}(\epsilon+\epsilon\\
&\quad +\left(1-u_k^{-1}\right)\left\|\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+1} - \Gamma(\curlywedge\Gamma)^{k-1}\varpi_{u+n}\right\|\Big)\Big\}\\
&\leq u_k\phi^{-1}\Big\{\max_{1\leq k,u\leq n}\Big(\epsilon+\epsilon+\left(1-u_k^{-1}\right)u_k\left\|\varpi_{i+1}-\varpi_{u+n}\right\|\Big)\Big\}\\
&\leq u_k\phi^{-1}\Big\{\max_{1\leq k,u\leq n}(\epsilon+\epsilon\\
&\quad +\left(1-u_k^{-1}\right)u_k\left(\left\|\varpi_{i+1}\right\|+\left\|\varpi_{u+n}\right\|\right)\Big\}\\
&< u_k\phi^{-1}\Big\{\max_{1\leq k,u\leq n}(2\epsilon+2r\left(1-u_k^{-1}\right)u_k\Big\},
\end{aligned}
\tag{36}
$$

since $\varpi_{i+1}, \varpi_{u+n} \in C$ and $u_k = 1+\sigma_n$.

Thus,

$$
\left\|\Gamma(\curlywedge\Gamma)^{k-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)}\Gamma(\curlywedge\Gamma)^{k-1}\varpi_{i+n}\right\| \leq u_k\phi^{-1}\Big\{\max_{1\leq k,u\leq n}\Big(2\epsilon+2r\left(1-u_k^{-1}\right)u_k\Big\}. \tag{37}
$$

Equations (34), (35), and (37) imply that

$$
\left\|\Gamma(\curlywedge\Gamma)^{k-1}y_n - y_n\right\| \leq u_k\phi^{-1}\Big\{\max_{1\leq k,u\leq n}(2\epsilon+2r\left(1-u_k^{-1}\right)u_k\Big\}+\epsilon. \tag{38}
$$

On the other hand, for any $k\geq 1$, it follows that (using (34))

$$
\begin{aligned}
\left\|\Gamma(\curlywedge\Gamma)^{k-1}\omega-\omega\right\| &\leq \left\|\Gamma(\curlywedge\Gamma)^{k-1}\omega - \Gamma(\curlywedge\Gamma)^{k-1}y_n\right\| + \left\|\Gamma(\curlywedge\Gamma)^{k-1}y_n - y_n\right\| + \left\|y_n-\omega\right\|\\
&\leq u_k\left\|\omega-y_n\right\| + u_k\phi^{-1}\Big\{\max_{1\leq k,u\leq n}(2\epsilon+2r\left(1-u_k^{-1}\right)u_k\Big\}+\epsilon\\
&\quad +\left\|y_n-\omega\right\|.
\end{aligned}
\tag{39}
$$

Taking $\limsup_{k\longrightarrow\infty}$ of both sides of (39), using (30) and for an arbitrary $\epsilon>0$, we deduce

$$
\left\|\Gamma(\curlywedge\Gamma)^{k-1}\omega-\omega\right\| \leq \phi^{-1}(0) = 0. \tag{40}
$$

That is, $\|\Gamma(\curlywedge\Gamma)^{k-1}\omega-\omega\| \longrightarrow 0$ as $k\longrightarrow\infty$. By the continuity of TP, we get

$$
\lim_{k\longrightarrow\infty} \Gamma(\curlywedge\Gamma)^{k-1}\omega = \Gamma\curlywedge\omega = \Gamma\omega = \omega. \tag{41}
$$

This completes the proof. □

Remark 21. The result of Lemma 12 still holds true if $\lambda = 1$. Thus, Lemma 12 can as well serve as a proof for the demiclosedness principle for total asymptotically pseudo-contractive non-self-mappings in UCBSs with $\lambda = 1$.

Lemma 22. *Let Z be a UCBS, $\varnothing \neq D\subset Z$ be closed and convex, $\Gamma_i\colon D\longrightarrow Z$ be a finite family of uniformly L''-Lipschitzian and total asymptotically strictly pseudo-non-spreading non-self-mappings with sequences $\{\mu_n\}_{n\geq1}$, $\{\xi_n\}_{n\geq1}\subset[0,\infty)$: $\mu_n, \xi_n \longrightarrow 0$ as $n\longrightarrow\infty$ and $G_i\colon D\longrightarrow D$ be a finite family of uniformly L'-Lipschitzian and total asymptotically strictly pseudo-non-spreading self-mappings with sequences $\{\gamma_n\}_{n\geq1}, \{\sigma_n\}_{n\geq1}\subset[0,\infty)$: $\gamma_n, \sigma_n\longrightarrow 0$ as $n\longrightarrow\infty$. Let $\{\alpha_n\}_{n\geq1}, \{\eta_n\}_{n\geq1}, \{\nu_n\}_{n\geq1}$ and $\{\beta_n\}_{n\geq1}$ be real sequences such that $\alpha_n, \eta_n, \nu_n, \beta_n \in (0,1)$. Suppose $\mathscr{F} = \cap_{i=1}^m F(G_i)\cap\cap_{i=1}^m F(\Gamma_i)\neq\varnothing$. If the following conditions are satisfied,*

- *(i)* $0<\eta\leq\eta_n\leq\nu_n\leq\beta_n\leq\alpha_n\leq\alpha<1$, $\sum_{n=1}^\infty\eta_n^2<\infty$, $\sum_{n=1}^\infty(1-\eta_n)=\infty$, $\sum_{n=1}^\infty(1-\eta_n)^2<\infty$
- *(ii)* $\sum_{n=1}^\infty\mu_n<\infty$, $\sum_{n=1}^\infty\xi_n<\infty$, $\sum_{n=1}^\infty\gamma_n<\infty$, $\sum_{n=1}^\infty\sigma_n<\infty$
- *(iii) There exist constants M' and M'' and a strictly increasing and continuous functions ψ, ϕ: $[0,\infty)\longrightarrow[0,\infty)$ with $\psi(0)=0=\psi(0)$ such that $\psi(t)=M''t^2$ and $\phi(s)=M's^2$, for all* t, s > 0

Then, $\lim_{n\longrightarrow\infty}\|\varpi_n - q\|$ and $\lim_{n\longrightarrow\infty} d(\varpi_n, F)$ both exist for all $q\in F$, where $\{\varpi_n\}_{n\geq1}$ is as defined by (27).

Proof. Set $\tau_n = \max_{1\le n<\infty}\{\mu_n, \gamma_n\}, M = \max\{M', M''\}$, $L = \max\{L', L''\}$ and $\theta_n = \max_{1\le n<\infty}\{\xi_n, \sigma_n\}$. Then, $\sum_{n=1}^{\infty}\tau_n < \infty$ and $\sum_{n=1}^{\infty}\theta_n < \infty$. Suppose $q\in \mathscr{F}$ is arbitrary, with the help of (27), we get

$$\begin{aligned}
\|y_{1n} - q\| &\le \left\|\beta_n G_2^n \varpi_n + (1-\beta_n)\Gamma_2(\curlywedge\Gamma_2)^{n-1} y_{2n} - q\right\| \\
&= \left\|\beta_n (G_2^n \varpi_n - q) + (1-\beta_n)(\Gamma_2(\curlywedge\Gamma_2)^{n-1} y_{2n} - q)\right\| \\
&\le \beta_n \|G_2^n\varpi_n - q\| + (1-\beta_n)\left\|\Gamma_2(\curlywedge\Gamma_2)^{n-1} y_{2n} - q\right\| \\
&\le \beta_n L' \|\varpi_n - q\| + (1-\beta_n)\text{Ł}'' \|y_{2n} - q\| \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)L\left\|P\left(\beta_n G_3^n\varpi_n + (1-\beta_n)\Gamma_3(\curlywedge\Gamma_3)^{n-1} y_{3n}\right) - q\right\| \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)L\left\|\beta_n G_3^n\varpi_n + (1-\beta_n)\Gamma_3(\curlywedge\Gamma_3)^{n-1} y_{3n} - q\right\| \\
&= \beta_n L\|\varpi_n - q\| + (1-\beta_n)L\left\|\beta_n (G_3^n\varpi_n - q) + (1-\beta_n)\left(\Gamma_3(\curlywedge\Gamma_3)^{n-1} y_{3n} - q\right)\right\| \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n \mathrm{LL}'\|\varpi_n - q\| + (1-\beta_n)^2 \mathrm{LL}''\|y_{3n} - q\| \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| + (1-\beta_n)^2 L^2\|y_{3n} - q\| \\
&= \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| \\
&\quad + (1-\beta_n)^2 L^2\left\|P\left(\beta_n G_4^n\varpi_n + (1-\beta_n)\Gamma_4(\curlywedge\Gamma_4)^{n-1} y_{4n} - q\right\|\right. \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| \\
&\quad + (1-\beta_n)^2\beta_n L^2\|G_4^n\varpi_n - q\| + (1-\beta_n)^3 L^2\left\|\Gamma_4(\curlywedge\Gamma_4)^{n-1} y_{4n} - q\right\| \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| + (1-\beta_n)^2\beta_n L^3\|\varpi_n - q\| \\
&\quad + (1-\beta_n)^3 L^3\|y_{4n} - q\| \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| + (1-\beta_n)^2\beta_n L^3\|\varpi_n - q\| \\
&\quad + (1-\beta_n)^3 L^3\left\|\beta_n G_5^n\varpi_n + (1-\beta_n)\Gamma_5(\curlywedge\Gamma_5)^{n-1} y_{5n} - q\right\| \\
&= \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| + (1-\beta_n)^2\beta_n L^3\|\varpi_n - q\| \\
&\quad + (1-\beta_n)^3 L^3\left\|\beta_n (G_5^n\varpi_n - q) + (1-\beta_n)\left(\Gamma_5(\curlywedge\Gamma_5)^{n-1} y_{4n} - q\right)\right\| \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| + (1-\beta_n)^2\beta_n L^3\|\varpi_n - q\| \\
&\quad + (1-\beta_n)^3\beta_n L^3 L'\|\varpi_n - q\| + (1-\beta_n)^4 L^3 L''\|y_{5n} - q\| \\
&\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| + (1-\beta_n)^2\beta_n L^3\|\varpi_n - q\| \\
&\quad + (1-\beta_n)^3\beta_n L^4\|\varpi_n - q\| + (1-\beta_n)^4 L^4\|y_{5n} - q\|.
\end{aligned} \tag{42}$$

By continuing in this manner, we obtain that

$$\begin{aligned}
\|y_{1n} - q\| &\le \beta_n L\|\varpi_n - q\| + (1-\beta_n)\beta_n L^2\|\varpi_n - q\| + (1-\beta_n)^2\beta_n L^3\|\varpi_n - q\| \\
&\quad + (1-\beta_n)^3\beta_n L^4\|\varpi_n - q\| + (1-\beta_n)^4\beta_n L^5\|\varpi_n - q\| \\
&\quad + \cdots + (1-\beta_n)^{m-1}\beta_n L^m\|\varpi_n - q\| \\
&\le \left(\beta_n L + (1-\beta_n)\beta_n L^2 + (1-\beta_n)^2\beta_n L^3 + (1-\beta_n)^3\beta_n L^4 + (1-\beta_n)^4\beta_n L^5\right. \\
&\quad \left. + \cdots + (1-\beta_n)^{m-1}\beta_n L^m\right)\|\varpi_n - q\| \\
&= \frac{\beta_n L(1 - (1-\beta_n)^m L^m)}{1 - (1-\beta_n)L}\|\varpi_n - q\|.
\end{aligned} \tag{43}$$

Following the same method as above, we get

$$\|y_{2n}-q\| \leq \frac{\beta_n L(1| - ((1-\beta_n)^m L^m)}{1-(1-\beta_n)L}\|\varpi_n - q\|. \quad (44)$$

In general,

$$\|y_{\text{in}}-q\| \leq \Phi\|\varpi_n - q\|, (i=1,2,\ldots,m), m \in N, \quad (45)$$

where $\Phi = \beta_n L(1- (1-\beta_n)^m L^m)/1-(1-\beta_n)L$.

Also, for $i=1$, we obtain the following estimation using (27):

$$\begin{aligned}
\|\varpi_{n+1}-y_{1n}\| &= \left\|P\left[\eta_n y_{1n} + (1-\eta_n)\left(\alpha_n\varpi_n + \frac{1}{2}(1-\alpha_n)\left(G_1^n\varpi_n + \Gamma_1(\measuredangle\Gamma_1)^{n-1}y_{1n}\right)\right] - P(y_{1n})\right\| \\
&\leq \left\|\eta_n y_{1n} + (1-\eta_n)\left(\alpha_n\varpi_n + \frac{1}{2}(1-\alpha_n)(G_1^n\varpi_n + \Gamma_1(\measuredangle\Gamma_1)^{n-1}y_{1n}\right) - y_{1n}\right\| \\
&= \left\|(1-\eta_n)\left[\alpha_n(\varpi_n - y_{1n}) + \frac{1}{2}(1-\alpha_n)(G_1^n\varpi_n + \Gamma_1(\measuredangle\Gamma_1)^{n-1}y_{1n} - y_{1n}\right\| \\
&\leq (1-\eta_n)\alpha_n\|\varpi_n - y_{1n}\| + \frac{1}{2}(1-\alpha_n)(1-\eta_n)\left\|\left(G_1^n\varpi_n + \Gamma_1(\measuredangle\Gamma_1)^{n-1}y_{1n} - y_{1n}\right)\right\| \\
&\leq (1-\eta_n)\alpha_n\|\varpi_n - y_{1n}\| + \frac{1}{2}(1-\eta_n)^2\left\|\left(G_1^n\varpi_n + \Gamma_1(\measuredangle\Gamma_1)^{n-1}y_n - y_{1n}\right)\right\|.
\end{aligned} \quad (46)$$

By following the same method as above for i= 2, we get

$$\|\varpi_{n+1}-y_{2n}\| \leq (1-\eta_n)\alpha_n\|\varpi_n - y_{2n}\| + \frac{1}{2}(1-\eta_n)^2\left\|\left(G_2^n\varpi_n + \Gamma_1(\measuredangle\Gamma_1)^{n-1}y_{2n} - y_{2n}\right)\right\|, \quad (47)$$

and in general,

$$\|\varpi_{n+1}-y_{\text{in}}\| \leq (1-\eta_n)\alpha_n\|\varpi_n - y_{\text{in}}\| + \frac{1}{2}(1-\eta_n)^2\left\|\left(G_i^n\varpi_n + \Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{\text{in}} - y_{\text{in}}\right)\right\|. \quad (48)$$

In addition, using (27) and Lemma 12, we obtain, for all $q \in \mathcal{F}$ and $i=1,2,\ldots,m$, that

$$\begin{aligned}
\|\varpi_{n+1}-q\|^2 &\leq \left\|\left[\eta_n y_{\text{in}} + (1-\eta_n)\left[\alpha_n\varpi_n + \frac{1}{2}(1-\alpha_n)\left(G_i^n\varpi_n + \Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{\text{in}}\right] - q\right\|^2 \\
&= \left\|\eta_n(y_{\text{in}}-q) + (1-\eta_n)\left[\alpha_n(\varpi_n - q) + \frac{1}{2}(1-\alpha_n)((G_i^n\varpi_n - q)\right.\right. \\
&\quad \left. + \left(\Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{\text{in}} - q\right)\right\|^2 \\
&\leq \eta_n^2\|y_{\text{in}}-q\|^2 + 2(1-\eta_n)\left\langle\left[\alpha_n(\varpi_n - q) + \frac{1}{2}(1-\alpha_n)((G_i^n\varpi_n - q)\right.\right. \\
&\quad \left. + \left(\Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{in} - q\right)\right], j(\varpi_{n+1}-q)\rangle \\
&= \eta_n^2\Phi^2\|\varpi_n - q\|^2 + 2\alpha_n(1-\eta_n)\langle\varpi_n - q, j(\varpi_{n+1}-q)\rangle \\
&\quad + (1-\eta_n)(1-\alpha_n)\langle G_i^n\varpi_{n+1} - q, j(\varpi_{n+1}-q)\rangle \\
&\quad + (1-\eta_n)(1| - \alpha_n\langle G_i^n\varpi_n - q - (q - \varpi_{n+1}) - (G_i^n\varpi_{n+1} - \varpi_{n+1})
\end{aligned}$$

$$
\begin{aligned}
&-2(\varpi_{n+1}-q)-G_i^n\varpi_{n+1}, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha_n)\langle\Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha_n)\langle\Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{\mathrm{in}}-q-(q-\varpi_{n+1}) \\
&-\left(\Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}-\varpi_{n+1}\right)-2(\varpi_{n+1}-q), j(\varpi_{n+1}-q)\rangle \\
&=\eta_n^2\Phi^2\|\varpi_n-q\|^2+2\alpha_n(1-\eta_n)\|\varpi_n-q\|\|\varpi_{n+1}-q\| \\
&+(1-\eta_n)(1-\alpha_n)\langle G_i^n\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha)\|G_i^n\varpi_n-q\|\|\varpi_{n+1}-q\| \\
&-(1-\eta_n)(1-\alpha)\|q-\varpi_{n+1}\|\|\varpi_{n+1}-q\| \\
&-(1-\eta_n)(1-\alpha)\|G_i^n\varpi_{n+1}-\varpi_{n+1}\|\|\varpi_{n+1}-q\| \\
&-2(1-\eta_n)(1-\alpha)\|\varpi_{n+1}-q\|\|\varpi_{n+1}-q\| \\
&+(1-\eta_n)(1-\alpha_n)\langle\Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha_n)\left\|\Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{in}-q\right\|\|\varpi_{n+1}-q\| \\
&-(1-\eta_n)(1-\alpha)\|q-\varpi_{n+1}\|\|\varpi_{n+1}-q\| \\
&-(1-\eta_n)(1-\alpha)\left\|\Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}-\varpi_{n+1}\right\|\|\varpi_{n+1}-q\| \\
&-2(1-\eta_n)(1-\alpha)\|\varpi_{n+1}-q\|\|\varpi_{n+1}-q\| \\
&\leq\eta_n^2\Phi^2\|\varpi_n-q\|^2+2\alpha_n(1-\eta_n)\|\varpi_n-q\|\|\varpi_{n+1}-q\| \\
&+(1-\eta_n)(1-\alpha_n)\langle G_i^n\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha)L'\|\varpi_n-q\|\|\varpi_{n+1}-q\|-4(1-\eta_n)(1-\alpha)\|\varpi_{n+1}-q\|^2 \\
&+(1-\eta_n)(1-\alpha_n)\langle\Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha_n)L''\|y_{in}-q\|\|\varpi_{n+1}-q\|.
\end{aligned}
\tag{49}
$$

From (45) and (49) and the fact that $2\mathrm{ab}\leq a^2+b^2$, we obtain

$$
\begin{aligned}
\|\varpi_{n+1}-q\|^2 &\leq \eta_n^2\Phi^2\|\varpi_n-q\|^2+2\alpha_n(1-\eta_n)\|\varpi_n-q\|\|\varpi_{n+1}-q\| \\
&+(1-\eta_n)(1-\alpha_n)\langle G_i^n\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\alpha)(1-\eta_n)L\|\varpi_n-q\|\|\varpi_{n+1}-q\|-4(1-\eta_n)(1-\alpha)\|\varpi_{n+1}-q\|^2 \\
&+(1-\eta_n)(1-\alpha_n)\langle\Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha_n)\Phi L\|\varpi_n-q\|\|\varpi_{n+1}-q\| \\
&=\eta_n^2\Phi^2\|\varpi_n-q\|^2+2\alpha_n(1-\eta_n)\|\varpi_n-q\|\|\varpi_{n+1}-q\| \\
&+(1-\eta_n)(1-\alpha_n)\langle G_i^n\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha)L\|\varpi_n-q\|\|-(\varpi_{n+1}-\varpi_n)+\varpi_n-q\| \\
&-4(1-\eta_n)(1-\alpha)\|\varpi_{n+1}-q\|^2 \\
&+(1-\eta_n)(1-\alpha_n)\langle\Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}-q, j(\varpi_{n+1}-q)\rangle \\
&+(1-\eta_n)(1-\alpha)\Phi L\|\varpi_n-q\|\|-(\varpi_{n+1}-\varpi_n)+\varpi_n-q\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \|\varpi_n - q\|^2 + \alpha_n^2 (1-\eta_n)^2 \|\varpi_n - q\|^2 + \|\varpi_{n+1} - q\|^2 \\
&+ (1-\eta_n)(1-\alpha_n)\langle G_i^n \varpi_{n+1} - q, j(\varpi_{n+1} - q)\rangle \\
&+ (1-\eta_n)(1-\alpha)L\|\varpi_n - q\|^2 + (1-\eta_n)(1-\alpha)L\|\varpi_{n+1} - \varpi_n\|\|\varpi_n - q\| \\
&- 4(1-\eta_n)(1-\alpha)\|\varpi_{n+1} - q\|^2 \\
&+ (1-\eta_n)(1-\alpha_n)\langle \Gamma_i (\measuredangle\Gamma_i)^{n-1}\varpi_{n+1} - q, j(\varpi_{n+1} - q)\rangle \\
&+ (1-\eta_n)(1-\alpha)\Phi L\|\varpi_n - q\|^2 + (1-\eta_n)(1-\alpha)\Phi L\|\varpi_{n+1} - \varpi_n\|\|\varpi_n - q\|.
\end{aligned} \tag{50}$$

Since each $\Gamma_i\colon D \longrightarrow Z$ and each $G_i\colon D \longrightarrow D$, for $i = 1, 2, \ldots, m$, is total asymptotically strictly pseudo-non-spreading mappings, the last inequality becomes

$$\begin{aligned}
\|\varpi_{n+1} - q\|^2 &\leq \left[1 + \left(\eta_n^2\Phi + (1-\eta)^2\left(1+\Phi^2\right)L\right)\right]\|\varpi_n - q\|^2 + \|\varpi_{n+1} - q\|^2 \\
&+ (1-\eta)(1-\alpha_n)\left[\|\varpi_{n+1} - q\|^2 - \lambda'\|\varpi_{n+1} - G_i^n\varpi_{n+1}\|^2 + \gamma_n\phi\left(\|\varpi_{n+1} - q\|\right) + \sigma_n\right] \\
&- 4(1-\eta)(1-\alpha)\|\varpi_{n+1} - q\|^2 + (1-\eta)(1-\alpha_n)\left[\|\varpi_{n+1} - q\|^2\right. \\
&\left.-\lambda''\|\varpi_{n+1} - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}\|^2 + \mu_n\psi\left(\|\varpi_{n+1} - q\|\right) + \xi_n\right] \\
&+ (1-\eta_n)^2(1+\Phi)L\|\varpi_{n+1} - \varpi_n\|\|\varpi_n - q\| \\
&= \left[1 + \left(\eta_n^2\Phi + (1-\eta)^2\left(1+\Phi^2\right)L\right)\right]\|\varpi_n - q\|^2 + \|\varpi_{n+1} - q\|^2 \\
&- (1-\eta)(1|-\alpha_n\lambda'\|\varpi_{n+1} - G_i^n\varpi_{n+1}\|^2 + (1-\eta)(1-\alpha_n)\gamma_n\phi\left(\|\varpi_{n+1} - q\|\right) \\
&+ (1-\eta)(1-\alpha_n)\sigma_n - 2(1-\eta)(1-\alpha)\|\varpi_{n+1} - q\|^2 \\
&- (1-\eta)(1-\alpha_n)\lambda''\|\varpi_{n+1} - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}\|^2 \\
&+ (1-\eta)(1-\alpha_n)\mu_n\psi\left(\|\varpi_{n+1} - q\|\right) + (1-\eta)(1-\alpha_n)\xi_n \\
&+ (1-\eta_n)^2(1+\Phi)L\|\varpi_{n+1} - \varpi_n\|\|\varpi_n - q\| \\
&\leq \left[1 + \left(\eta_n^2\Phi + (1-\eta)^2\left(1+\Phi^2\right)L\right)\right]\|\varpi_n - q\|^2 + \|\varpi_{n+1} - q\|^2 \\
&- (1-\eta)(1|-\alpha_n\lambda'\|\varpi_{n+1} - G_i^n\varpi_{n+1}{}^2\| + (1-\eta)(1-\alpha_n)\gamma_n M'\|\varpi_{n+1} - q\|^2 \\
&+ (1-\eta)(1-\alpha_n)(\sigma_n + \xi_n) - 2(1-\eta)(1-\alpha)\|\varpi_{n+1} - q\|^2 \\
&- (1-\eta)(1-\alpha_n)\lambda''\|\varpi_{n+1} - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}\|^2 \\
&+ (1-\eta)(1-\alpha_n)\mu_n M''\|\varpi_{n+1} - q\|^2 + \\
&+ (1-\eta_n)^2(1+\Phi)L\|\varpi_{n+1} - \varpi_n\|\|\varpi_n - q\| \\
&\leq \left[1 + \left(\eta_n^2\Phi + (1-\eta)^2\left(1+\Phi^2\right)L\right)\right]\|\varpi_n - q^2\| + \|\varpi_{n+1} - q\|^2 \\
&- (1-\eta)(1|-\alpha_n\lambda'\|\varpi_{n+1} - G_i^n\varpi_{n+1}\|^2 + (1-\eta)(1-\alpha_n)\gamma_n M\|\varpi_{n+1} - q\|^2 \\
&+ (1-\eta)(1-\alpha_n)(\sigma_n + \xi_n) - 2(1-\eta)(1-\alpha)\|\varpi_{n+1} - q\|^2 \\
&- (1-\eta)(1-\alpha_n)\lambda''\|\varpi_{n+1} - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_{n+1}\|^2 \\
&+ (1-\eta)(1-\alpha_n)\mu_n M\|\varpi_{n+1} - q\|^2 + (1-\eta_n)^2(1+\Phi)L\|\varpi_{n+1} - \varpi_n\|\|\varpi_n - q\| \\
&= \left[1 + \left(\eta_n^2\Phi + (1-\eta)^2\left(1+\Phi^2\right)L\right)\right]\|\varpi_n - q\|^2 + \|\varpi_{n+1} - q\|^2
\end{aligned}$$

$$
\begin{aligned}
&-(1-\eta)\Big(1-\alpha_n\lambda'\|\varpi_{n+1}-G_i^n\varpi_{n+1}\|^2+(1-\eta)^2\Big(\gamma_n+\mu_n M\|\varpi_{n+1}-q\|^2\\
&+(1-\eta)^2(\sigma_n+\xi_n)-(1-\eta)(1-\alpha)\|\varpi_{n+1}-q\|^2\\
&-(1-\eta)(1-\alpha_n)\lambda''\big\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\big\|^2\\
&+(1-\eta_n)^2(1+D\Phi)L\|\varpi_{n+1}-\varpi_n\|\|\varpi_n-q\|\\
\le{}&\Big[1+\Big(\eta_n^2\Phi+(1-\eta)^2\big(1+\Phi^2\big)L\Big)\Big]\|\varpi_n-q\|^2+\|\varpi_{n+1}-q\|^2\\
&-(1-\eta)\Big(1-\alpha_n\lambda'\|\varpi_{n+1}-G_i^n\varpi_{n+1}{}^2\|+2(1-\eta)^2\tau_n M\|\varpi_{n+1}-q\|^2\\
&-2(1-\eta)(1-\alpha)\|\varpi_{n+1}-q^2\|-(1-\eta)(1-\alpha_n)\lambda''\big\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\big\|^2\\
&+(1-\eta_n)^2\big[2\theta_n+(1+\Phi)L\|\varpi_{n+1}-\varpi_n\|\|\varpi_n-q\|\big].
\end{aligned}
\tag{51}
$$

Set $\Upsilon=[(1-\eta_n)(1-\alpha_n)-2(1-\eta_n)^2\tau_n M]\|\varpi_{n+1}-q\|^2$. Then, we obtain from (51) that

$$
\begin{aligned}
\Upsilon\le{}&\Big[1+\Big(\eta_n^2\Phi+(1-\eta)^2\big(1+\Phi^2\big)L\Big)\Big]\|\varpi_n-q\|^2\\
&-(1-\eta)(1-\alpha_n)\lambda'\|\varpi_{n+1}-G_i^n\varpi_{n+1}\|^2\\
&+(1-\eta)^2\big[2\theta_n+(1+\Phi)L\|\varpi_n-q\|\|\varpi_{n+1}-q\|\big]\\
&-(1-\eta)(1-\alpha_n)\lambda''\\
&\times\big\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\big\|^2.
\end{aligned}
\tag{52}
$$

Observe that

$$
(1-\eta_n)(1-\alpha_n)-2(1-\eta_n)^2\tau_n M\le 1-2(1-\alpha_n)\tau_n M.
\tag{53}
$$

Thus,

$$
\begin{aligned}
\Big(1-2(1-\eta_n)^2\tau_n M\Big)\|\varpi_{n+1}-q\|^2\le{}&\Big[1+\Big(\eta_n^2\Phi+(1-\eta)^2\big(1+\Phi^2\big)L\Big)\Big]\|\varpi_n-q\|^2\\
&-(1-\eta)(1-\alpha_n)\lambda'\|\varpi_{n+1}-G_i^n\varpi_{n+1}\|^2\\
&+(1-\eta)^2\big(2\theta_n+(1+\Phi)L\|\varpi_n-q\|\|\varpi_{n+1}-q\|\big)\\
&-(1-\eta)(1-\alpha_n)\lambda''\\
&\times\big\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\big\|^2.
\end{aligned}
\tag{54}
$$

Since $\sum_{n=1}^{\infty}\tau_n<\infty$, it follows that $\lim_{n\longrightarrow\infty}2(1-\eta_n)^2\tau_n M=0$. Consequently, for any $\epsilon>0$, there exists a natural number n_0 such that $2(1-\eta_n)^2\tau_n M<\epsilon$. Without loss of generality, let $\epsilon=1/2$ so that $2(1-\eta_n)^2\tau_n M<1/2$. Let

$$
\begin{aligned}
b_n&=\frac{1+\eta_n^2\Phi+(1-\eta)^2\big(1+\Phi^2\big)L}{1-2(1-\eta_n)^2M\tau_n}-1=\frac{\eta_n^2\Phi+(1-\eta)^2\big[\big(1+\Phi^2\big)L+2M\tau_n\big]}{1-2(1-\eta_n)^2M\tau_n}.\\
c_n&=\frac{(1-\eta)^2\big(2\theta_n+(1+\Phi)L\|\varpi_n-q\|\|\varpi_{n+1}-q\|\big)}{1-2(1-\eta_n)^2M\tau_n}.
\end{aligned}
\tag{55}
$$

Thus, when $n\ge n_0$, we have

$$
\begin{aligned}
&0\le b_n\le 2\Big\{\eta_n^2\Phi+(1-\eta)^2\big[\big(1+\Phi^2\big)L+2M\tau_n\big]\Big\},\\
&0\le c_n\le 2(1-\eta)^2\big(2\theta_n+(1+\Phi)L\|\varpi_n-q\|\|\varpi_{n+1}-q\|\big).
\end{aligned}
\tag{56}
$$

By condition (ii), $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$.

From (51)–(53), we have

$$a_{n+1} \leq (1 + b_n) a_n + c_n. \tag{57}$$

Again, from (57) and Lemma 13, it follows that $\lim_{n\longrightarrow\infty} \|\varpi_n - q\|$ exists so that there exists a constant Q such that $\|\varpi_n - q\| \leq Q$.

By utilizing the infimum for all $q \in F$ in (57), we obtain

$$d(\varpi_{n+1}, F) \leq (1 + b_n) d(\varpi_n, F) + c_n, \text{for all } n \in N. \tag{58}$$

Moreover, $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, by utilizing Lemma 13 that $\lim_{n\longrightarrow\infty} d(\varpi_n, F)$ exists. This completes the proof. □

Lemma 23. *Let $Z, D, \{\Gamma_i\}_{i=1}^{\mathbb{N}}, \{G_i\}_{i=1}^{\mathbb{N}}$ and $\mathcal{F}$ be as stated in Lemma 13. If, in addition to the assumptions of Lemma 13, the following conditions are satisfied:*

(a) $\|\varpi_n - \Gamma_i (\curlywedge\Gamma_i)^{n-1}\varpi_n\| \leq \|G_i^n \quad \varpi_n - \Gamma_i (\curlywedge\Gamma_i)^{n-1}\varpi_n\|$, $i = 1, 2, \ldots, m$,

(b) $\|\varpi_n - \Gamma_i (\curlywedge\Gamma_i)^{n-1} \quad y_n\| \leq \|G_i^n \varpi_n - \Gamma_i (\curlywedge\Gamma_i)^{n-1} y_n\|$, $i = 1, 2, \ldots, m$,

then $\lim_{n\longrightarrow\infty} \|\varpi_n - \Gamma_i\varpi_n\| = 0$ *and* $\lim_{n\longrightarrow\infty} \|\varpi_n - G_i\varpi_n\| = 0$, $i = 1, 2, \ldots, m$.

Proof. Set $\tau_n = \max_{1\leq n<\infty}\{\mu_n, \gamma_n\}, M = \max\{M_i', M_i''\}$, $L = \max\{L', L''\}$ and $\theta_n = \max_{1\leq n<\infty}\{\xi_n, \sigma_n\}$. Then, $\sum_{n=1}^{\infty}\tau_n < \infty$ and $\sum_{n=1}^{\infty}\theta_n < \infty$. With the help of arbitrary $q \in F$, $\lim_{n\longrightarrow\infty} \|\varpi_n - q\|$ exists by Lemma 22. Now, assume that $\lim_{n\longrightarrow\infty} \|\varpi_n - q\| = c$, then using the fact that $\sum_{n=1}^{\infty}\tau_n < \infty$ and $\sum_{n=1}^{\infty}\theta_n < \infty$, we get

$$\lim_{n\longrightarrow\infty} \left\| y_{(i+1)n} - q \right\| = \lim_{n\longrightarrow\infty} \left\| \nu_n (G_{i+2}^n \varpi_n - q) + (1 - \nu_n)\left(\Gamma_{i+2} (\curlywedge\Gamma_{i+2})^{n-1} \varpi_n - q\right) \right\| = c. \tag{59}$$

Also, we have

$$\begin{aligned}
\left\| G_{i+2}^n \varpi_n - q \right\|^2 &\leq \left\| \varpi_n - q^2 - \lambda' \right\| \varpi_n - G_{i+2}^n \varpi_n{}^2 + \gamma_n \Psi\left(\left\|\varpi_n - q\right\|\right) + \sigma_n \\
&\leq \left\|\varpi_n - q\right\|^2 + \gamma_n \psi\left(\left\|\varpi_n - q\right\|\right) + \sigma_n. \\
&\Rightarrow \limsup_{n\longrightarrow\infty} \left\| G_{i+2}^n \varpi_n - q \right\| \leq c, i = 1, 2, \ldots, m.
\end{aligned} \tag{60}$$

Furthermore,

$$\begin{aligned}
\left\| \Gamma_{i+2} (\curlywedge\Gamma_{i+2})^{n-1} \varpi_n - q \right\|^2 &\leq \left\|\varpi_n - q\right\|^2 - \lambda'' \left\| \varpi_n - \Gamma_{i+2} (\curlywedge\Gamma_{i+2})^{n-1} \varpi_n \right\|^2 + \mu_n \phi\left(\left\|\varpi_n - q\right\|\right) + \xi_n \\
&\leq \left\|\varpi_n - q\right\|^2 + \mu_n \phi\left(\left\|\varpi_n - q\right\|\right) + \xi_n \\
\Rightarrow \limsup_{n\longrightarrow\infty} \left\| \Gamma_{i+2} (\curlywedge\Gamma_{i+2})^{n-1} \varpi_n - q \right\| &\leq c, i = 1, 2, \ldots, m.
\end{aligned} \tag{61}$$

Therefore, from (59)–(61) and Lemma 22, we obtain

$$\lim_{n\longrightarrow\infty} \left\| G_{i+2}^n \varpi_n - \Gamma_{i+2} (\curlywedge\Gamma_{i+2})^{n-1} \varpi_n \right\| = 0, \quad i = 1, 2, \ldots, m. \tag{62}$$

From (62) and condition (ii) (i.e., $\|\varpi_n - \Gamma_{i+2} (\curlywedge\Gamma_{i+2})^{n-1}\varpi_n\| \leq \|G_{i+2}^n\varpi_n - \Gamma_{i+2} (\curlywedge\Gamma_{i+2})^{n-1}\varpi_n\|$) of Lemma 23, we get

$$\lim_{n\longrightarrow\infty} \left\| \varpi_n - \Gamma_{i+2} (\curlywedge\Gamma_{i+2})^{n-1} \varpi_n \right\| = 0, \quad i = 1, 2, \ldots, m. \tag{63}$$

Again, from (27), we get

$$\lim_{n\longrightarrow\infty} \left\| y_{in} - q \right\| = \lim_{n\longrightarrow\infty} \left\| \beta_n (G_{i+1}^n \varpi_n - q) + (1 - \beta_n)\left(\Gamma_{i+1} (\curlywedge\Gamma_{i+1})^{n-1} y_{(i+1)n} - q\right) \right\| = c. \tag{64}$$

Using (59) and (64), condition (b), Lemma 14, and following the same technique as above, we obtain

$$\lim_{n\longrightarrow\infty}\left\|G_{i+1}^{n}\varpi_{n}-\Gamma_{i+1}\left(\curlywedge\Gamma_{i+1}\right)^{n-1}y_{(i+1)n}\right\|=0,\quad i=1,2,\ldots,m, \tag{65}$$

$$\lim_{n\longrightarrow\infty}\left\|\varpi_{n}-\Gamma_{i+1}\left(\curlywedge\Gamma_{i+1}\right)^{n-1}y_{(i+1)n}\right\|=0,\quad i=1,2,\ldots,m. \tag{66}$$

Moreover, from (54), we get

$$\begin{aligned}(1-\alpha_n)(1-\eta_n)\lambda''\left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\right\|^2 &\leq\left[1+\left(\eta_n^2\Phi+(1-\eta)^2\left(1+\Phi^2\right)L\right)\right]\left\|\varpi_n-q\right\|^2\\ &-\left(1-2(1-\eta_n)^2\tau_n M\right)\left\|\varpi_{n+1}-q\right\|^2\\ &-(1-\eta_n)(1-\alpha_n)\lambda'\left\|\varpi_{n+1}-G_i^n\varpi_{n+1}\right\|^2\\ &+2(1-\eta_n)^2\left(2\theta_n+(1+\Phi)L\left\|\varpi_n-q\right\|\times\left\|\varpi_{n+1}-q\right\|\right).\\ &\Rightarrow\sum_{n=1}^{\infty}(1-\eta_n)(1-\alpha_n)\lambda''\left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\right\|^2\leq\left\|\varpi_1-q\right\|^2+\left(1+\Phi^2\right)Q^2\sum_{n=1}^{\infty}(1-\eta_n)^2\\ &+\Phi Q^2\sum_{n=1}^{\infty}\eta^2+2MQ^2\sum_{n=1}^{\infty}(1-\eta_n)^2\tau_n\\ &+2\sum_{n=1}^{\infty}(1-\eta_n)^2\theta_n+LQ^2\sum_{n=1}^{\infty}(1-\eta_n)^2<\infty,\end{aligned} \tag{67}$$

$$\Rightarrow\liminf_{n\longrightarrow\infty}\left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\right\|=0. \tag{68}$$

Similarly, we obtain from (54) that

$$\lim_{n\longrightarrow\infty}\inf\left\|\varpi_{n+1}-S_i^n\varpi_{n+1}\right\|=0. \tag{69}$$

Now, from (65) and (66) and the inequality

$$\begin{aligned}\left\|y_{in}-\varpi_n\right\|&=\left\|\curlywedge\left(\beta_nG_{i+1}^n\varpi_n+(1-\beta_n)\Gamma_{i+1}\left(\curlywedge\Gamma_{i+1}\right)^{n-1}y_{(i+1)n}\right)-\varpi_n\right\|\\ &\leq\left\|\beta_nG_{i+1}^n\varpi_n+(1-\beta_n)\Gamma_{i+1}\left(\curlywedge\Gamma_{i+1}\right)^{n-1}y_{(i+1)n}-\varpi_n\right\|\\ &\leq\left\|\varpi_n-\Gamma_{i+1}\left(\curlywedge\Gamma_{i+1}\right)^{n-1}y_{(i+1)n}\right\|+\beta_n\left\|G_{i+1}^n\varpi_n-\Gamma_{i+1}\left(\curlywedge\Gamma_{i+1}\right)^{n-1}y_{(i+1)n}\right\|,\end{aligned} \tag{70}$$

we obtain that

$$\lim_{n\longrightarrow\infty}\left\|y_{\text{in}}-\varpi_n\right\|=0. \tag{71}$$

Again, (48) and (74) and utilizing $\sum_{n=1}^{\infty}(1-\eta_n)^2<\infty$, we obtain

$$\lim_{n\longrightarrow\infty}\left\|\varpi_{n+1}-y_{\text{in}}\right\|=0. \tag{72}$$

Also, from (71) and (72) and the inequality

$$\left\|\varpi_{n+1}-\varpi_n\right\|\leq\left\|\varpi_{n+1}-y_{\text{in}}\right\|+\left\|y_{\text{in}}-\varpi_n\right\|, \tag{73}$$

we obtain that

$$\lim_{n\longrightarrow\infty}\left\|\varpi_{n+1}-\varpi_n\right\|=0. \tag{74}$$

Observe that

$$\begin{aligned}\left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\| &\leq \left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\right\|+\left\|\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}-T_i(\mathrm{P}T_i)^{n-1}y_{in}\right\|\\ &\leq \left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\right\|+L\left\|\varpi_{n+1}-y_{in}\right\|\\ &\leq \left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_{n+1}\right\|+L\left\|\varpi_{n+1}-\varpi_n\right\|+L\left\|\varpi_n-y_{in}\right\|.\end{aligned}\tag{75}$$

Equations (68), (71), (74), and (75) imply that

$$\lim_{n\longrightarrow\infty}\left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\|=0,\quad i=1,2,\ldots,m.\tag{76}$$

Furthermore,

$$\begin{aligned}\left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_n\right\| &\leq \left\|G_1^n\varpi_n-G_i^n\varpi_{n+1}\right\|+\left\|\varpi_{n+1}-G_i^n\varpi_{n+1}\right\|+\left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_n\right\|\\ &\leq L\left\|\varpi_n-\varpi_{n+1}\right\|+\left\|\varpi_{n+1}-G_i^n\varpi_{n+1}\right\|\\ &\quad+\left\|\varpi_{n+1}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_n\right\|,\end{aligned}\tag{77}$$

so that from (69), (74), and (77), we get

$$\lim_{n\longrightarrow\infty}\left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\|=0,\quad i=1,2,\ldots,m.\tag{78}$$

Observe that

$$\begin{aligned}\left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n\right\| &\leq \left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\|\\ &\quad+\left\|\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n\right\|\\ &\leq \left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_n\right\|+L\left\|y_{in}-\varpi_n\right\|,\end{aligned}\tag{79}$$

so that, using (71) and (78), we have

$$\lim_{n\longrightarrow\infty}\left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n\right\|=0,\quad i=1,2,\ldots,m.\tag{80}$$

Thus, from condition (a) of Lemma 14 and (80), we obtain

$$\lim_{n\longrightarrow\infty}\left\|\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n\right\|=0,\quad i=1,2,\ldots,m.\tag{81}$$

Again, observe that

$$\begin{aligned}\left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\| &\leq \left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n\right\|+\left\|\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\|\\ &\leq \left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n\right\|+L\left\|\varpi_n-y_{in}\right\|,\end{aligned}\tag{82}$$

so that from (71) and (80), we get

$$\lim_{n\longrightarrow\infty}\left\|G_i^n\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\|=0,\quad i=1,2,\ldots,m.\tag{83}$$

Also,

$$\begin{aligned}\left\|\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\| &\leq \left\|\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n\right\|+\left\|\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\|\\ &\leq \left\|\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}\varpi_n\right\|+L\left\|\varpi_n-y_{in}\right\|,\end{aligned}\tag{84}$$

so that from (71) and (81), we get

$$\lim_{n\longrightarrow\infty}\left\|\varpi_n-\Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{in}\right\|=0,\quad i=1,2,\ldots,m.\tag{85}$$

Now, we estimate $\|\varpi_n - \Gamma_i\varpi_n\|$ as follows:

$$
\begin{aligned}
\left\|\varpi_n - \Gamma_i\varpi_n\right\| &= \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n - \left(\Gamma_i(\measuredangle\Gamma_i)^{n-1}y_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right)\right. \\
&\quad \left. + \Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{in} - T_i\varpi_n\right\| \\
&\leq \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + \left\|\Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{\text{in}} - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| \\
&\quad + \left\|\Gamma_i(\measuredangle\Gamma_i)^{n-1}y_{\text{in}} - \Gamma_i\varpi_{\text{in}}\right\| \\
&\leq \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L\left\|y_{\text{in}} - \varpi_n\right\| + \left\|\Gamma_i(\measuredangle\Gamma_i)(\measuredangle\Gamma_i)^{n-2}y_{\text{in}} - \Gamma_i\varpi_n\right\| \\
&\leq \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L\left\|y_{\text{in}} - \varpi_n\right\| + L\left\|(\measuredangle\Gamma_i)(\measuredangle\Gamma_i)^{n-2}y_{\text{in}} - \varpi_n\right\| \\
&= \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L\left\|y_{\text{in}} - \varpi_n\right\| + L\left\|\Gamma_i(\measuredangle\Gamma_i)^{n-2}y_{\text{in}} - \varpi_n\right\| \\
&= \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L\left\|y_{\text{in}} - \varpi_n\right\| + L\left\|\Gamma_i(\measuredangle\Gamma_i)^{n-2}y_{\text{in}} - \Gamma_i(\measuredangle\Gamma_i)^{n-2}\varpi_{n-1}\right. \\
&\quad \left. + \varpi_{n-1} - \varpi_n - \left(\varpi_{n-1} - \Gamma_i(\measuredangle\Gamma_i)^{n-2}\varpi_{n-1}\right)\right\| \\
&\leq \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L\left\|y_{\text{in}} - \varpi_n\right\| + L\left\|\Gamma_i(\measuredangle\Gamma_i)^{n-2}y_{\text{in}} - \Gamma_i(\measuredangle\Gamma_i)^{n-2}\varpi_{n-1}\right\| \\
&\quad + L\left\|\varpi_{n-1} - \varpi_n\right\| + L\left\|\varpi_{n-1} - \Gamma_i(\measuredangle\Gamma_i)^{n-2}\varpi_{n-1}\right\| \\
&\leq \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L\left\|y_{\text{in}} - \varpi_n\right\| + L^2\left\|y_{\text{in}} - \varpi_{n-1}\right\| + L\left\|\varpi_{n-1} - \varpi_n\right\| \\
&\quad + L\left\|\varpi_{n-1} - \Gamma_i(\measuredangle\Gamma_i)^{n-2}\varpi_{n-1}\right\| \\
&= \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L\left\|y_{\text{in}} - \varpi_n\right\| + L^2\left\|y_{\text{in}} - \varpi_n + \varpi_n - \varpi_{n-1}\right\| \\
&\quad + L\left\|\varpi_{n-1} - \varpi_n\right\| + L\left\|\varpi_{n-1} - \Gamma_i(\measuredangle\Gamma_i)^{n-2}\varpi_{n-1}\right\| \\
&\leq \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L\left\|y_{\text{in}} - \varpi_n\right\| + L^2\left\|y_n - \varpi_n\right\| + L^2\left\|\varpi_n - \varpi_{n-1}\right\| \\
&\quad + L\left\|\varpi_{n-1} - \varpi_n\right\| + L\left\|\varpi_{n-1} - \Gamma_i(\measuredangle\Gamma_i)^{n-2}\varpi_{n-1}\right\| \\
&= \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + L(1+L)\left\|y_{\text{in}} - \varpi_n\right\| + L(1+L)\left\|\varpi_{n-1} - \varpi_n\right\| \\
&\quad + L\left\|\varpi_{n-1} - \Gamma_i(\measuredangle\Gamma_i)^{n-2}\varpi_{n-1}\right\|.
\end{aligned}
\tag{86}
$$

From (71), (74), (81), and (82), we obtain

$$\lim_{n\longrightarrow\infty}\left\|\varpi_n - \Gamma_i\varpi_n\right\| = 0, \quad i = 1, 2, \ldots, m. \tag{87}$$

Next, observe that

$$
\begin{aligned}
\left\|\varpi_n - G_i^n\varpi_n\right\| &= \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n - \left(G_i^n\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right)\right\| \\
&\leq \left\|\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| + \left\|G_i^n\varpi_n - \Gamma_i(\measuredangle\Gamma_i)^{n-1}\varpi_n\right\| \\
&\Rightarrow \lim_{n\longrightarrow\infty}\left\|\varpi_n - G_i^n\varpi_n\right\| = 0.
\end{aligned}
\tag{88}
$$

By (80) and (81), again, observe that

$$\begin{aligned}\left\|\varpi_n - G_i\varpi_n\right\| &= \left\|\varpi_n - G_i^n\varpi_n - \left(G_i^n\varpi_{n+1} - G_i^n\varpi_n\right) + G_i\left(G_i^{n-1}\varpi_{n+1}\right) - G_i\varpi_n\right\| \\ &\le \left\|\varpi_n - G_i^n\varpi_n\right\| + \left\|G_i^n\varpi_{n+1} - G_i^n\varpi_n\right\| + \left\|G_i\left(G_i^{n-1}\varpi_{n+1}\right) - G_i\varpi_n\right\| \\ &\le \left\|\varpi_n - G_i^n\varpi_n\right\| + L\left\|\varpi_{n+1} - \varpi_n\right\| + L\left\|G_i^{n-1}\varpi_{n+1} - \varpi_n\right\| \\ &= \left\|\varpi_n - G_i^n\varpi_n\right\| + L\left\|\varpi_{n+1} - \varpi_n\right\| + L\|G_i^{n-1}\varpi_{n+1} - G_i^{n-1}\varpi_n \\ &\quad + G_i^{n-1}\varpi_n - G_i^{n-1}\varpi_{n-1} + S_i^{n-1}\varpi_{n-1} - \varpi_{n-1} + \varpi_{n-1} - \varpi_n\| \\ &\le \left\|\varpi_n - G_i^n\varpi_n\right\| + L\left\|\varpi_{n+1} - \varpi_n\right\| + L\left\|G_i^{n-1}\varpi_{n+1} - G_i^{n-1}\varpi_n\right\| \\ &\quad + L\left\|G_i^{n-1}\varpi_n - S_i^{n-1}\varpi_{n-1}\right\| + L\left\|G_i^{n-1}\varpi_{n-1} - \varpi_{n-1}\right\| + L\left\|\varpi_{n-1} - \varpi_n\right\| \\ &\le \left\|\varpi_n - G_i^n\varpi_n\right\| + L\left\|\varpi_{n+1} - \varpi_n\right\| + L^2\left\|\varpi_{n+1} - \varpi_n\right\| \\ &\quad + L^2\left\|\varpi_n - \varpi_{n-1}\right\| + L\left\|G_i^{n-1}\varpi_{n-1} - \varpi_{n-1}\right\| + L\left\|\varpi_{n-1} - \varpi_n\right\| \\ &= \left\|\varpi_n - G_i^n\varpi_n\right\| + L(1+L)\left\|\varpi_{n+1} - \varpi_n\right\| + L(1+L)\left\|\varpi_n - \varpi_{n-1}\right\| \\ &\quad + L\left\|G_i^{n-1}\varpi_{n-1} - \varpi_{n-1}\right\|,\end{aligned} \tag{89}$$

so that from (69), (74), and (88), we get

$$\lim_{n\longrightarrow\infty}\left\|\varpi_n - G_i\varpi_n\right\| = 0, \quad i = 1, 2, \ldots, m. \tag{90}$$

□

Lemma 24. *Let $Z, D, \{\Gamma_i\}_{i=1}^{\mathbb{N}}, \{G_i\}_{i=1}^{\mathbb{N}}$ and $\mathcal{F}$ be as stated in Lemma 13. Under the conditions of Lemma 13, forall $\xi_i, \xi_j \in \mathcal{F}, i, j = 1.2. \cdots, \mathbb{N}$, with $i \ne j$, the $\lim_{n\longrightarrow\infty}\|ux_n + (1-u)\xi_i - \xi_j\|$ exists for all $u \in [0,1]$, where $\{\varpi_n\}$ is the sequence defined by (27).*

Proof. Clearly, $\lim_{n\longrightarrow\infty}\|\varpi_n - q\|$ exists for all $q \in \mathcal{F}$ (by Lemma 13), and hence, $\{\varpi_n\}$ is bounded. Let $a_n(u) = \|\mathrm{ux}_n + (1-u)\xi_i - \xi_j\|, i, j = 1.2. \cdots, \mathbb{N}$, with $i \ne j$ exists for all $u \in [0,1]$. Then, $\lim_{n\longrightarrow\infty} a_{(0)} = \|\xi_i - \xi_j\|$ and $\lim_{n\longrightarrow\infty} a_{(1)} = \|\varpi_n - \xi_j\|$ exist by Lemma 22. It remains, therefore, to prove Lemma 24 for $u \in (0,1)$. Now, for all $\varpi \in D$, define

$$\begin{cases}\mathcal{V}_n(\varpi) = \curlywedge\left[\eta_n y_{\mathrm{in}} + (1-\eta_n)\left(\alpha_n\varpi_n + \frac{1}{2}(1-\alpha_n)\left(G_i^n\varpi_n + \Gamma_i(\curlywedge\Gamma_i)^{n-1}y_{\mathrm{in}}\right)\right]; \\ \mathcal{Q}_{\mathrm{in}} = \curlywedge\left(\beta_n G_{i+1}^n\varpi_n + (1-\beta_n)\Gamma_{i+1}(\curlywedge\Gamma_{i+1})^{n-1}y_{(i+1)n}\right), \\ \mathcal{S}_{(i+1)n} = \curlywedge\left(\nu_n G_{i+2}^n\varpi_n + (1-\nu_n)\Gamma_{i+2}(\curlywedge\Gamma_{i+2})^{n-1}\varpi_n\right).\end{cases} \tag{91}$$

Then, it follows that $\varpi_{n+1} = \mathcal{V}_n\varpi_n, \mathcal{V}_n\xi = \xi$, for all $\xi \in \mathcal{F}$. Now, from (57) of Lemma 22, we see that

$$\left\|\mathcal{V}_n(\varpi) - \mathcal{V}_n(q)\right\|^2 \le (1+\delta_n)\|\varpi - y\|^2 + c_n = \ell_n\|\varpi - y\|^2 + f_n, \tag{92}$$

where $\delta_n = 2(1-\alpha_n)^2\tau_n$ and $f_n = 2(1-\alpha)\theta_n$ with $\sum_{n=1}^{\infty}\delta_n < \infty, \sum_{n=1}^{\infty} f_n < \infty$ and $\ell_n = 1 + \delta_n$. Since $\sum_{n=1}^{\infty}\delta_n < \infty$, it follows that $\ell_n \longrightarrow 1$ as $n \longrightarrow \infty$. Set

$$\begin{cases} S_{n,m} = \mathcal{V}_{n+m-1}\mathcal{V}_{n+m-2}\cdots\mathcal{V}_n, m \in N \\ z_{n,m} = \left\|S_{n,m}\left(\mathrm{ux}_n + (1-u)\xi_1\right) - S_{n.m}\left(\mathrm{ux}_m + (1-u)\xi_2\right)\right\|. \end{cases} \tag{93}$$

Then, it follows from the standard argument that $\lim_{n\longrightarrow\infty} a_n(u)$ exists; i.e., $\lim_{n\longrightarrow\infty}\|\mathrm{ux}_n + (1-u)\xi_i - \xi_j\|$, $i, j = 1.2. \cdots, \mathbb{N}$, with $i \ne j$ exists for all $u \in [0,1]$. This completes the proof. □

Lemma 25. *Let $Z, D, \{\Gamma_i\}_{i=1}^{\mathbb{N}}, \{G_i\}_{i=1}^{\mathbb{N}}$ and $\mathcal{F}$ be as stated in Lemma 13. If, in addition to the assumptions of Lemma 13, E has Frechet differentiable norm, then, for all $\xi_i, J\ \xi_j \in \mathcal{F}, i, j = 1, 2, \ldots, m; i < j$, the limit $\lim_{n\longrightarrow\infty}\langle(\varpi_n, J(\xi_i - \xi_j)\rangle$ exists, where $\{\varpi_n\}$ is the sequence defined by (27). If $\omega_\omega\{s_n\}$ denotes the set of all weak subsequential limits of $\{\varpi_n\}$,* *then* $\langle \eta_i - \eta_j, \xi_i - \xi_j \rangle = 0$ *for all* $\xi_i, \xi_j \in \mathscr{F}, i, j = i, j = 1, 2, \ldots, m; i < j$ *and for all* $\eta_i, \eta_j \in \omega_\omega(\varpi_n); i, j = i, j = 1, 2, \ldots, m; i < j.$

Proof. Set $\varpi = \xi_i - \xi_j$ with $\xi_i \neq \xi_j$ and $h = u(\varpi_n - \xi_i)$ in (21). Then, we have

$$\begin{aligned} t\langle \varpi_n - \xi_i, J(\xi_i - \xi_j)\rangle + \frac{1}{2}\left\|\xi_i xi-_j\right\|^2 &\leq \frac{1}{2}\left\|\mathrm{ux}_n + (1-u)\xi_i - \xi_j\right\|^2 \\ &\leq t\langle \varpi_n - \xi_i, J(\xi_i - \xi_j)\rangle + \frac{1}{2}\left\|\xi_i - \xi_j\right\|^2 + b\left(t\left\|\varpi_n - p_i\right\|\right). \end{aligned} \tag{94}$$

From $\sup_{n\geq 1}\|\varpi_n - \xi\| \leq \mathscr{K}^\star$ for some $\mathscr{K}^\star > 0$, we get

$$\begin{aligned} u \limsup_{n\longrightarrow\infty}\langle \varpi_n - \xi_i, J(\xi_i - \xi_j)\rangle + \frac{1}{2}\left\|\xi_i - \xi_j\right\|^2 &\leq \frac{1}{2}\lim_{n\longrightarrow\infty}\left\|\mathrm{ux}_n + (1-u)\xi_i - \xi_j\right\|^2 \\ & u \liminf_{n\longrightarrow\infty}\langle \varpi_n - \xi_i, J(\xi_i - \xi_j)\rangle + \frac{1}{2}\left\|\xi_i - \xi_j\right\|^2 \\ & + b(u\mathscr{K}^\star). \end{aligned} \tag{95}$$

That is,

$$u \limsup_{n\longrightarrow\infty}\langle \varpi_n - \xi_i, J(\xi_i - \xi_j)\rangle \leq u \liminf_{n\longrightarrow\infty}\langle \varpi_n - \xi_i, J(\xi_i - \xi_j)\rangle + b\left(\frac{u\mathscr{K}^\star}{\mathscr{K}^\star}\right)\mathscr{K}^\star. \tag{96}$$

If $u \longrightarrow 0$, then $\lim_{n\longrightarrow\infty}\langle \varpi_n - \xi_i, J(\xi_i - \xi_j)\rangle$ exists for all $\xi_i, \xi_j \in \mathscr{F}$ and for $\eta_i, \eta_j \in \omega_\omega \varpi_n, i, j = i, j = 1, 2, \ldots, m; i < j$. This completes the proof. ☐

Theorem 26. *Let* $Z, D, \{\Gamma_i\}_{i=1}^{\mathbb{N}}, \{G_i\}_{i=1}^{\mathbb{N}}$ *and* $\mathscr{F}$ *be as stated in Lemma 13. If, in addition to the assumptions of Lemma 13, Z has Frechet differentiable norm, then the sequence* $\{\varpi_n\}$ *is defined by (27) WC to a common fixed point in* $\mathscr{F}$.

Proof. With the help of Lemma 25, $\langle \eta_i - \eta_j, J(\xi_1 - \xi_2)\rangle = 0$, $\forall\, \eta_i, \eta_2 \in \omega_\omega(\varpi_n), i, j = 1, 2, \ldots, m;\ i < j.$ Hence, $\|\eta^\star - \xi\star\|^2 = \langle \eta^\star - \xi^\star, J(\eta^\star - \xi^\star)\rangle = 0$. Thus, $p^\star = q^\star$. Therefore, $\{\varpi_n\}$ WC to a common fixed point of $\mathscr{F}$. This completes the proof. ☐

Theorem 27. *Let* $Z, D, \{\Gamma_i\}_{i=1}^{\mathbb{N}}, \{G_i\}_{i=1}^{\mathbb{N}}$ *and* $\mathscr{F}$ *be as stated in Lemma 13. If, in addition to the assumptions of Lemma 13, the space* $Z^\star$ *of Z has the Kadec–Klec (KK) property and the mappings* $I - G_i$ *and* $I - \Gamma_i$ *for* $i = 1, 2, \ldots, m \in \mathbb{N}$*, where I is an identity mapping, are demiclosed at zero, then the sequence* $\{\varpi_n\}$ *described by (27) WC to a common fixed point in* $\mathscr{F}$.

Proof. By Lemma 14, $\{\varpi_n\}$ is bounded and Z is reflexive, there exists a subsequence $\{\varpi_{n_k}\}$ of $\{\varpi_n\}$ which WC to some $\eta^\star \in K$. With the help of Lemma 14, we deduce $\lim_{n\longrightarrow\infty}\|\varpi_{n_k} - G_i\varpi_{n_k}\| = 0$ and $\lim_{n\longrightarrow\infty}\|\varpi_{n_k} - \Gamma_i\varpi_{n_k}\| = 0,\ i = 1, 2, \ldots, m \in \mathbb{N}$. By the assumptions, the mappings $I - G_i$ and $I - \Gamma_i$ for $i = 1, 2, \ldots, m \in \mathbb{N}$, where I is an identity mapping, are demiclosed at zero, boundedness of $\{\varpi_n\}$, and the uniqueness of the limit of the weakly convergence sequence follows that the sequence $\{\varpi_n\}$ WC to $q^\star \in F$. This completes the proof. ☐

Theorem 28. *Let* $Z, D, \{\Gamma_i\}_{i=1}^{\mathbb{N}}, \{G_i\}_{i=1}^{\mathbb{N}}$ *and* $\mathscr{F}$ *be as stated in Lemma 13. If, in addition to the assumptions of Lemma 13, Z satisfies Opial's condition and the mappings* $I - G_i$ *and* $I - \Gamma_i$ *for* $i = 1, 2, \ldots, m \in \mathbb{N}$*, where I denotes the identity mapping, are demiclosed at zero, then the sequence* $\{\varpi_n\}$ *defined by (27) WC to a common fixed point in* $\mathscr{F}$.

Proof. Suppose $\eta^\star \in \mathscr{F}$. With the help of Lemma 22, the sequence $\{\|\varpi_n - \eta^\star\|\}$ exists and is convergent and $\{\varpi_n\}$ is bounded. By utilizing Lemma 23, we deduce that $\lim_{n\longrightarrow\infty}\|\varpi_{n_k} - G_i\varpi_{n_k}\| = 0$ and $\lim_{n\longrightarrow\infty}\|\varpi_{n_k} - \Gamma_i\varpi_{n_k}\| = 0,\ i = 1, 2, \ldots, m \in \mathbb{N}$. Finally, the demiclosed property of each $(I - S_i)$ and $(I - \Gamma_i)$, boundedness of $\{\varpi_n\}$, the uniqueness of the limit of the weakly convergence sequence, and the Opial property of the underlying space follows that the sequence $\{\varpi_n\}$ weakly converges to $\eta^\star \in \mathscr{F}$. ☐

Remark 29. The following iteration techniques are immediate consequences of our newly constructed iteration scheme:

(1) If $\alpha_n = 1$ in (27), we have

$$\begin{cases} \varpi_1 \in D, \\ \varpi_{n+1} = \eta_n y_{\text{in}} + (1-\eta_n)\varpi_n; \\ y_{\text{in}} = \curlywedge\left(\beta_n G_{i+1}^n \varpi_n + (1-\beta_n)\Gamma_{i+1}(\curlywedge\Gamma_{i+1})^{n-1} y_{(i+1)n}\right), \\ y_{(i+1)n} = \curlywedge\left(\nu_n G_{i+2}^n \varpi_n + (1-\nu_n)\Gamma_{i+2}(\curlywedge\Gamma_{i+2})^{n-1}\varpi_n\right), \end{cases} \tag{97}$$

where $\{\beta_n\}_{n\geq 1}, \{\gamma_n\}_{n\geq 1}$, $\{\eta_n\}_{n\geq 1} \in (0,1)$ and $i = 1, 2, \ldots, m \in \mathbb{N}$.

(2) If $\eta_n = 1, y_{(i+1)n} = y_n, G_{i+1}^n = G_1^n, G_{i+2}^n = G_2, \Gamma_{i+1} = \Gamma_2$ and $\Gamma_{i+2} = \Gamma_2$ in (97), we have

$$\begin{cases} \varpi_1 \in D, \\ \varpi_{n+1} = \curlywedge\left(\beta_n G_1^n \varpi_n + (1-\beta_n)\Gamma_1(\curlywedge\Gamma_1)^{n-1} y_n\right); \\ y_n = \curlywedge\left(\nu_n G_2^n \varpi_n + (1-\nu_n)\Gamma_2(\curlywedge\Gamma_2)^{n-1}\varpi_n\right), \end{cases} \tag{98}$$

where $\{\beta_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \in (0,1)$.

(3) If $G_1^n = G_2^n = I$ in (98), where I is an identity map on D, *we have*

$$\begin{cases} \varpi_1 \in D, \\ \varpi_{n+1} = \curlywedge\left(\beta_n \varpi_n + (1-\beta_n)\Gamma_1(\curlywedge\Gamma_1)^{n-1} y_n\right); \\ y_n = \curlywedge\left(\nu_n \varpi_n + (1-\nu_n)\Gamma_2(\curlywedge\Gamma_2)^{n-1}\varpi_n\right), \end{cases} \tag{99}$$

where $\{\beta_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \in (0,1)$.

(4) If $G_i^n = I = \curlywedge = I, \alpha_n = \eta_n = 0$ and $\Gamma_i = \Gamma, y_{\text{in}} = y_n$ and $y_{(i+1)n} = z_n$ in (27), we get

$$\begin{cases} \varpi_1 \in D, \\ \varpi_{n+1} = \frac{1}{2}(\varpi_n + \Gamma y_n)]; \\ y_n = \beta_n \varpi_n + (1-\beta_n)\Gamma z_n, \\ z_n = \nu_n \varpi_n + (1-\nu_n)\Gamma\varpi_n), \end{cases} \tag{100}$$

where $\{\beta_n\}_{n\geq 1}, \{\gamma_n\}_{n\geq 1} \in (0,1)$.

(5) If $\Gamma_1^n = \Gamma_2^n = \Gamma$ and $\curlywedge = I$ in (99), where I is an identity map on D, we have

$$\begin{cases} \varpi_1 \in D, \\ \varpi_{n+1} = \beta_n \varpi_n + (1-\beta_n)\Gamma y_n; \\ y_n = \nu_n \varpi_n + (1-\nu_n)\Gamma\varpi_n, \end{cases} \tag{101}$$

where $\{\beta_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \in (0,1)$.

(6) If $\beta_n = 1 = \nu_n$ in (100), we get

$$\begin{cases} \varpi_1 \in D, \\ \varpi_{n+1} = \frac{1}{2}(\varpi_n + \Gamma y_n). \end{cases} \tag{102}$$

4. Conclusion

In this manuscript,

(1) We established a new fixed-point algorithm for approximating the common fixed point of finite families of L-Lipschitzian and total asymptotically strictly pseudo-non-spreading self-mappings and L-Lipschitzian and total asymptotically strictly pseudo-non-spreading non-self-mappings in the setup of a real UCBS

(2) We introduce a new type of nonlinear mapping called total asymptotically strictly pseudo-non-spreading self-mappings in the setup of UCBS

(3) Demiclosedness principle for total asymptotically strictly pseudo-non-spreading self-mappings and several WC results were obtained using our newly constructed iteration scheme in the setup of a real UCBS

(4) A slight modification of our iteration scheme resulted in several well-known iteration schemes currently existing in the literature, see, for instance, (97)–(102)

(5) Our WC results improve, generalize, and extend several well-known WC results from the setup of real Hilbert spaces to those of real UCBSs

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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