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EXISTENCE AND ULAM-HYERS-RASSIAS STABILITY OF MILD SOLUTIONS FOR IMPULSIVE INTEGRO-DIFFERENTIAL SYSTEMS VIA RESOLVENT OPERATORS

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Abstract. The aim of this paper is to present existence, Ulam-Hyers-Rassias stability and continuous dependence on initial conditions for the mild solution of impulsive integro-differential systems via resolvent operators. Our analysis is based on fixed point theorem with generalized measures of noncompactness, this approach is combined with the technique that uses convergence to zero matrices in generalized Banach spaces. An example is presented to illustrate the efficiency of the result obtained.

1. **Introduction.** Milman and Myshkis [27] considered differential equations with impulses for the first time, which was followed by a period of active research culminating in the monograph by Halanay and Wexler [20]. Many phenomena and evolution processes in physics, chemical technology, population dynamics, and natural sciences can change state abruptly or be perturbed in the short term. These disturbances can be viewed as impulses. In addition to communications, mechanics (jump discontinuities in velocity), electrical engineering, medicine, and biology, impulsive problems arise in a variety of other applications. [1, 2, 6, 19, 40, 7, 21], and its references include current results for impulsive evolution equations.

In 1930, Kuratowski [22] proposed the concept of a measure of noncompactness. This concept is extremely useful in functional analysis, such as metric fixed point theory and operator equation theory in Banach spaces. This concept is also used to investigate the existence of solutions for ordinary and partial differential equations, as well as integral and integro-differential equations. In 1955, Darbo [11], an Italian mathematician, used the Kuratowski measure to investigate a class of operators

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(condensing operators) whose properties are intermediate between contraction and compact mappings. Darbo's fixed point theorem can be used to prove the existence result of various classes of operator equations. For more recent works on the subject, see [12].

In 2009, Precup [32] showed the importance of vector-valued metric convergence in the study of semilinear operator systems. Many authors have studied the existence of solutions for systems of differential equations using the vector version fixed point theorem in recent years (see, [10, 15, 29, 33]).

Recently, many authors combined the concept of a measure of noncompactness and matrices that converge to zero, R. Graef et al. in [15] gave the vector versions of Sadovskii's fixed point theorem. In [23], N. Laksaci et al. generalized Darbo's fixed point theorem for iterated Operators.

Another important aspect of the research that drew the attention of the researchers was Ulam stability and its many kinds, See ([25, 42, 24, 34, 35, 36]) for further information on recent developments in the Ulam type stability of differential equations.

Motivated by works [9, 15, 23], we will investigate the existence and stability of mild solutions for impulsive integro-differential system via resolvent operators of the form:

$$\begin{cases} \xi'(\theta) = A_1\xi(\theta) + f_1\left(\theta, \xi(\theta), \varphi(\theta), H_1(\xi(\theta), \varphi(\theta))\right) \\ \qquad\qquad + \displaystyle\int_0^\theta B_1(\theta - s)\xi(s)ds, \text{ for } \theta \in \widetilde{\Theta}, \\ \varphi'(\theta) = A_2\varphi(\theta) + f_2\left(\theta, \xi(\theta), \varphi(\theta), H_1(\xi(\theta), \varphi(\theta))\right) \\ \qquad\qquad + \displaystyle\int_0^\theta B_2(\theta - s)\varphi(s)ds, \text{ for } \theta \in \widetilde{\Theta}, \\ \xi(\theta_k^+) - \xi(\theta_k^-) = \aleph_k(\xi(\theta_k^-), \varphi(\theta_k^-)), \ k = 1, \ldots, m, \\ \varphi(\theta_k^+) - \varphi(\theta_k^-) = \widetilde{\aleph}_k(\xi(\theta_k^-), \varphi(\theta_k^-)), \ k = 1, \ldots, m, \\ (\xi(0), \varphi(0)) = (\xi_0, \varphi_0), \end{cases} \tag{1}$$

where $\Theta = [0, T]$, $\widehat{\Theta}_m = \{\theta_1, \ldots, \theta_m\}$, $\widetilde{\Theta} = \Theta \backslash \widehat{\Theta}_m$ with $\theta_0 = 0 < \theta_1 < \theta_2 < \ldots < \theta_m < \ldots < \theta_{m+1} = T$, and for $i = 1, 2$, $(E, \|\cdot\|)$ is a Banach space, $A_i : D(A_i) \subset E \to E$ are the infinitesimal generators of a strongly continuous semigroup $\{S_i(\theta)\}_{\theta \geq 0}$, $B_i(\theta)$ are a closed linear operator with domain $D(A_i) \subset D(B_i(\theta))$, $\aleph_k$, $\widetilde{\aleph}_k : E \times E \to E$ the operators H_i are defined by

$$H_i(\xi, \varphi)(\theta) = \int_0^a h_i(\theta, s, \xi(s), \varphi(s))ds, \ a > 0,$$

and the nonlinear term $f_i : \Theta \times E \times E \times E \to E$, are a given functions.

The paper is organized as follows. In Section 2, we recall some definitions and facts which will be needed in our analysis. In Section 3, we prove some existence and stability results. In the last section, we give an example that provides a relevant illustration.

2. **Preliminaries.** We introduce in this section some of the notations, definitions, fixed point theorems and preliminary facts that will be used in the remainder of this paper.

Let us denote by $C(\Theta, E)$ the space consisting of all functions defined and continuous on the interval Θ with values in the space $(E, \|\cdot\|)$, with the standard norm

$$\|\xi\|_\infty = \sup_{\theta\in\Theta} \|\xi(\theta)\|.$$

Next, we consider a division of Θ, i.e., a finite set $\{\theta_0, \dots, \theta_{m+1}\}$ and put $\Theta_0 = [0, \theta_1], \Theta_k = (\theta_k, \theta_{k+1}], \bar{\Theta}_k = [\theta_k, \theta_{k+1}]$ for $k = 1, \dots, m$, $\xi(\theta^+) = \lim_{\theta\to\theta^+} \xi(\theta)$. We define the space of piecewise continuous functions:

$$\mathrm{PC}(\Theta, E) = \Big\{ \xi : \Theta \to E : \xi|_{\Theta_k} \in C(\Theta_k, E), \text{ such that } \xi(\theta_k^-) \text{ and } \xi(\theta_k^+) \text{ exist and satisfy } \xi(\theta_k^-) = \xi(\theta_k), \text{ for } k = 1, \dots, m \Big\}.$$

Note that $(\mathrm{PC}(\Theta, E), \|\cdot\|_{PC})$ is a Banach space, with the norm

$$\|\xi\|_{\mathrm{PC}} = \sup_{\theta\in\Theta} \|\xi(\theta)\|.$$

Let the Banach space $PC^1(\Theta, E) = \{\xi \in PC(\Theta, E) \; : \; \xi' \in PC(\Theta, E)\}$, with norm

$$\|\xi\|_{PC^1} = \max\{\|\xi\|_{\mathrm{PC}}, \|\xi'\|_{\mathrm{PC}}\}.$$

2.1. **Generalized Banach space.**

Definition 2.1. Let X be a vector metric space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A map $\|\cdot\| : X \to \mathbb{R}_+^n$ is called a norm on X if it satisfies the following properties:

- If $\|\xi\| = 0$ then $\xi = (0, \dots, 0)$;
- $\|\lambda\xi\| = |\lambda|\|\xi\|$ for $\xi \in X, \lambda \in \mathbb{K}$;
- $\|\xi + v\| \le \|\xi\| + \|v\|$ for every $\xi, v \in X$.

Remark 2.2. The pair $(X, \|\cdot\|_X)$ is called a generalized normed space. If the generalized metric generated by $\|\cdot\|_X$ (*i.e* $d(\xi, v) = \|\xi - v\|_X$) is complete then the space $(X, \|\cdot\|_X)$ is called a generalized Banach space, where

$$\|\xi - v\|_X = \begin{pmatrix} \|\xi - v\|_1 \\ \vdots \\ \|\xi - v\|_n \end{pmatrix}.$$

Let $X\times X = PC(\Theta, E)\times PC(\Theta, E)$ be endowed with the vector norm $\|\cdot\|_{X\times X}$ defined by $\|v\|_{X\times X} = (\|u_1\|_{PC}, \|u_2\|_{PC})$ for $v = (u_1, u_2)$. It is clear that $(PC(\Theta, E)\times PC(\Theta, E), \|\cdot\|_{X\times X})$ is a generalized Banach space.

In the case of generalized Banach spaces in the sense of Perov, the notations of convergent sequence, Cauchy sequence, completeness, open and closed subset are similar to those for usual metric spaces.

Definition 2.3. A square matrix M of real numbers is said to be convergent to zero if and only if, $M^n \to 0$ as $n \to \infty$.

Theorem 2.4. *A square matrix M of real numbers is convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc i.e. $|\lambda| < 1$, for every $\lambda \in \mathrm{C}$ with det $(M - \lambda I) = 0$, where I denote the unit matrix of $\mathcal{M}_{n\times n}(\mathbb{R})$.*

Lemma 2.5 ([38])**.** *Let $M \in \mathcal{M}_{m\times m}(\mathbb{R}_+)$. Then, the following assertions are equivalent:*

- *M is convergent towards zero,*
- *$M^k \to 0$ as $k \to \infty$,*
- *The matrix $(I - M)$ is nonsingular and*
$$(I - M)^{-1} = I + M + M^2 + \ldots + M^k + \ldots$$
- *The matrix $(I - M)$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.*

Definition 2.6. Let $Q \in \mathcal{M}_{2\times 2}(\mathbb{R})$ is said to be order preserving (or positive) if
$$p_1 \leq p_0, \; q_1 \leq q_0,$$
imply
$$Q\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \geq Q\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}.$$
in the sense of components.

Lemma 2.7 ([32])**.** *Let*
$$Q = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix},$$
where $\alpha, \beta, \gamma, \delta \geq 0$ and $\det Q > 0$. Then Q^{-1} is order preserving.

2.1.1. *Resolvent operator.* We consider the following Cauchy problem
$$\begin{cases} w'(\theta) = Aw(\theta) + \int_0^\theta B(\theta - s)w(s)ds; & \text{for } \theta \geq 0, \\ w(0) = w_0 \in E. \end{cases} \tag{2}$$

The existence and properties of a resolvent operator have been discussed in [13, 16, 17].

In what follows, we suppose the following assumptions:

$(R1)$ A is the infinitesimal generator of a uniformly continuous semigroup $\{S(\theta)\}_{\theta>0}$,

$(R2)$ For all $\theta \geq 0, B(\theta)$ is closed linear operator from $D(A)$ to E and $B(\theta) \in B(D(A), E)$. For any $\xi \in D(A)$, the map $\theta \to B(\theta)\xi$ is bounded, differentiable and the derivative $\theta \to B'(\theta)\xi$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 2.8 ([16])**.** *Assume that $(R1)$-$(R2)$ hold, then there exists a unique resolvent operator for the Cauchy problem (2).*

2.1.2. *Measure of noncompactness.* Now, we give definitions and properties for a measure of noncompactness.

Definition 2.9. Let X be a generalized Banach space and $(\mathcal{A}, \leq)$ be a partially ordered set. A map $\beta : \mathcal{P}(X) \to \mathcal{A} \times \mathcal{A} \times \ldots \times \mathcal{A}$ is called a generalized measure of noncompactness $(M.N.C.)$ on X if
$$\beta(\overline{co}C) = \beta(C) \;\; \text{for every} \;\; C \in \mathcal{P}(X),$$
where
$$\beta(C) := \begin{pmatrix} \beta_1(C) \\ \vdots \\ \beta_n(C) \end{pmatrix}.$$

A typical example of a $M.N.C.$ is the Hausdorff measure of noncompactness χ defined, for all $\Omega \subset X$, by
$$\chi(\Omega) := \inf\left\{\epsilon \in \mathbb{R}_+^n : \text{there exists } n \in \mathbb{N} \text{ such that } \Omega \text{ has finite } \epsilon\text{-net } \right\}.$$

Lemma 2.10 ([26])**.** *Let $\Omega \subset C(a, b)$ be bounded and equicontinuous. Then, $\overline{\mathrm{co}}(\Omega) \subset C(a, b)$ is also bounded and equicontinuous.*

Lemma 2.11 ([18]). *Let $\Omega \subset C(a,b)$ be bounded and equicontinuous, and let β the Kuratowski's measure of noncompactness. Then, $\xi(\theta)=\beta(\Omega(\theta))$ is continuous and*

$$\beta\left(\int_a^b \Omega(s)ds\right) \leq \int_a^b \beta(\Omega(\varsigma))ds.$$

Definition 2.12. Let X, Y be two generalized normed spaces and a map $N: X \to Y$. N is called an M-contraction (with respect to β) if there exists $M \in M_{n\times n}(\mathbb{R}_+)$ converging to zero such that, for every $\Omega \in \mathcal{P}(X)$, we have

$$\beta(N(\Omega)) \leq M\beta(\Omega).$$

The next result is concerned with β-condensing or M-contractivity.

Theorem 2.13 ([15]). *Let $F \subset X$ be a bounded closed convex subset and $N: F \to F$ be a generalized β-condensing continuous mapping, where β is a nonsingular measure of noncompactress defined on the subsets of X. Then the set*

$$\operatorname{Fix}(N)=\{x \in F : x=N(x)\}$$

is nonempty.

Theorem 2.14 ([23]). *Let F be a closed, bounded, and convex subset of X, and let $N: F \to F$ be a continuous operator. For any subset Ω of F, set*

$$N^1\Omega = N\Omega, \quad N^p\Omega = N\left(\overline{\mathrm{co}}\left(N^{p-1}\Omega\right)\right), \quad p=2,3,\ldots \tag{3}$$

Suppose there exists a matrix M that approaches zero and a positive integer p_0 such that for any subset Ω of F, we have

$$\beta\left(N^{p_0}\Omega\right) \leq M\beta(\Omega),$$

where β is an arbitrary generalized measure of noncompactness. Then, N has at least one fixed point in F.

Lemma 2.15 ([4]). *Let $\xi(\theta)$ and $b(\theta)$ be nonnegative continuous function for $\theta \geq \alpha$, and let*

$$\xi(\theta) \leq \sigma + \int_\alpha^\theta b(s)\xi(s)ds, \quad \theta \geq \alpha,$$

where $\sigma \geq 0$ is a constant. Then

$$\xi(\theta) \leq \sigma \exp\left(\int_\alpha^\theta b(s)ds\right), \quad \theta \geq \alpha.$$

Lemma 2.16 ([37]). *Let $\xi(\theta)$ be a nonnegative piecewise continuous function that satisfies, for $\theta \geq \theta_0$, the inequality*

$$\xi(\theta) \leq C + \int_{\theta_0}^\theta V(s)\xi(s)\mathrm{d}s + \sum_{\theta_0<\theta_i<\theta} \beta_i \xi\left(\theta_i\right), \text{ for all } \theta \geq \theta_0,$$

where $C \geq 0, \beta_i \geq 0, V(\tau)>0$, and τ_i are the first kind discontinuity points of the function $\xi(\theta)$. Then the following estimate holds for the function $\xi(\theta)$,

$$\xi(\theta) \leq C \prod_{\theta_0<\tau_i<\theta}\left(1+\beta_i\right) \exp\left[\int_{\theta_0}^\theta V(\tau)\mathrm{d}\tau\right].$$

3. Main results.

In this section we discuss the existence of mild solution for the problem (1).

3.1. **Existence of solutions.** Let us recollect the following particular measure of noncompactness that derives from [3], and will be utilized in our main results in order to establish a measure of noncompactness in the space $PC(\Theta, E)\times PC(\Theta, E)$.

For $\Pi_1 \subset PC(\Theta, E)$, let us put

$$\begin{aligned}\Pi_1|_{\bar{\Theta}_k} =&\big\{\xi \in \mathrm{C}\left(\bar{\Theta}_k, E\right) : \xi\left(\theta_k\right) = x\left(\theta_k^+\right), \xi(\theta) = x(\theta), \theta \in \Theta_k,\\ &x \in \Pi_1;\ k = 1, \ldots, m\big\}.\end{aligned}$$

The set $\Pi_1 \subset \mathrm{PC}(\Theta, E)$ is relatively compact if and only if the set $\Pi_1|_{\bar{\Theta}_k}$ is relatively compact in $\mathrm{C}\left(\bar{\Theta}_k, E\right)$ for $k = 0, 1, \ldots, m+1$.

Now, for a nonempty bounded subset Π_1 in the space $PC(\Theta, E)$ and $I \subset \Theta$, let $\omega_0(\Pi_1)$ be the modulus of quasi-equicontinuity of the set of functions Π_1 denote

$$\omega_0(I, \Pi_1) = \lim_{\epsilon\to 0} \sup_{\xi\in\Pi_1} \sup\{\|\xi(\kappa) - \xi(\tau)\|\ ;\ \kappa, \tau \in I,\ |\kappa - \tau| \le \epsilon\}.$$

Given the Hausdorff measure of noncompactness $\wp$ and let $\widehat{\wp}$ be the real M.N.C. defined on bounded subsets on $PC(\Theta, E)$ by $\widehat{\wp}(H) = \sup_{\theta\in\Theta} \wp(H(\theta))$.

Finally, consider the function χ_{PC} defined on the family of subset of $H = H_1 \times H_2 \subset PC(\Theta, E) \times PC(\Theta, E)$ by the formula

$$\chi_{PC}(H_1 \times H_2) = \begin{pmatrix}\chi_1(H_1)\\ \chi_2(H_2)\end{pmatrix} = \begin{pmatrix}\max_{k=\overline{0:m}}(\omega_0(\Theta_k, H_1), \widehat{\wp}_1(H_1))\\ \max_{k=\overline{0:m}}(\omega_0(\Theta_k, H_2), \widehat{\wp}_2(H_2))\end{pmatrix}.$$

It can be shown similar to [3, 5, 15, 30] that the function χ_{PC} is monotone, regular, and nonsingular measure of noncompactness on the space $PC(\Theta, E) \times PC(\Theta, E)$.

Definition 3.1. A function $(\xi, \varphi) \in PC(\Theta, E)\times PC(\Theta, E)$ is called a mild solution of problem (1) if it satisfies

$$\begin{aligned}\xi(\theta) &= R_1(\theta)\xi_0 + \int_0^\theta R_1(\theta - s) f_1\left(s, \xi(s), \varphi(s), H_1(\xi(s), \varphi(s))\right) ds\\ &\quad + \sum_{0<\theta_k<\theta} R_1(\theta - \theta_k)\aleph_k(\xi(\theta_k^-), \varphi(\theta_k^-));\quad \theta \in \Theta,\\ \varphi(\theta) &= R_2(\theta)\varphi_0 + \int_0^\theta R_2(\theta - s) f_2\left(s, \xi(s), \varphi(s), H_2(\xi(s), \varphi(s))\right) ds\\ &\quad + \sum_{0<\theta_k<\theta} R_2(\theta - \theta_k)\widetilde{\aleph}_k(\xi(\theta_k^-), \varphi(\theta_k^-));\quad \theta \in \Theta.\end{aligned}$$

The following assumption will be needed throughout the paper:

$(H1)$ For $i = 1, 2$; $f_i : \Theta \times E \times E \times E \to E$ are Carathéodory functions and there exist $p_i,\ q_i \in L^1(\Theta, \mathbb{R}^+)$ and a continuous nondecreasing functions $\psi_i, \phi_i : \Theta \to (0, +\infty)$ such that :

$$\|f_i(\theta, \xi, v, w(\xi, \xi)) - f_i(\theta, \bar{\xi}, \bar{v}, w(\bar{\xi}, \bar{v}))\| \le p_i(\theta)\psi_i(\|\xi - \bar{\xi}\|) + q_i(\theta)\phi_i(\|v - \bar{v}\|),$$
$$\text{for } \xi,\ \bar{\xi}, v,\ \bar{v}, w \in E,$$

with

$$\psi_i(\theta) \le \theta,\ \phi_i \le \theta,\ \text{and } f_i^0 = \|f_i(\cdot, 0, 0, 0)\| \in L^1(\Theta, \mathbb{R}^+).$$

$(H2)$ For $i = 1, 2$; $h_i : D_{h_i}\times E\times E \to E$ are continuous and there exists a continuous functions $h_{c_i},\ \bar{h}_{c_i} :\ \Theta \to (0, +\infty)$ such that,

$$\|h_i(\theta, s, \xi, v) - h_i(\theta, s, \bar{\xi}, \bar{v})\| \le h_{c_i}(\theta)\|\xi - \bar{\xi}\| + \bar{h}_{c_i}(\theta)\|v - \bar{v}\|,$$

$$\text{for each } (\theta,s)\in D_{h_i} \text{ and } \xi,\ \bar{\xi}, v,\ \bar{v}\in E,$$

with

$$\max\Big\{\sup_{\theta\in\Theta}\{h_{c_i}(\theta)\},\sup_{\theta\in\Theta}\{\bar{h}_{c_i}(\theta)\},\sup_{(\theta,s)\in D_{h_i}}\{\|h(\theta,s,0,0)\|\}\Big\}=\max\{h^*_{c_i},\bar{h}^*_{c_i},h^*_i\}<\infty.$$

$(H3)$ $\aleph_k,\ \widetilde{\aleph}_k : E\times E\to E$ are continuous and there exist positive constants $m^i_k,\widetilde{m}^i_k,\ i=1,2$, such that,

$$\begin{aligned}\|\aleph_k(\xi,v)-\aleph_k(\bar{\xi},\bar{v})\| &\le m^1_k(\|\xi-\bar{\xi}\|)+\widetilde{m}^1_k(\|v-\bar{v}\|),\\ \|\widetilde{\aleph}_k(\xi,v)-\widetilde{\aleph}_k(\bar{\xi},\bar{v})\| &\le m^2_k(\|\xi-\bar{\xi}\|)+\widetilde{m}^2_k(\|v-\bar{v}\|),\end{aligned}$$

with

$$\sum_{k=0}^{m}\|\aleph_k(0,0)\|<\infty \text{ and } \sum_{k=0}^{m}\|\widetilde{\aleph}_k(0,0)\|<\infty.$$

$(H4)$ Assume that $(R1)-(R2)$ hold, and there exist $M_{R_i}\ge 1$ and $\beta_i\ge 0$ $i=1,2$, such that

$$\|R_i(\theta)\|_{B(E)}\le M_{R_i}e^{-\beta_i\theta}.$$

Theorem 3.2. *Assume that the conditions $(H1)-(H4)$ are satisfied, and the matrix*

$$\widehat{M}=\begin{pmatrix} M_{R_1}\left(\|p_1\|_{L^1}+\sum_{k=0}^{m}m^1_k\right) & M_{R_1}\left(\|q_1\|_{L^1}+\sum_{k=0}^{m}\widetilde{m}^1_k\right)\\ M_{R_2}\left(\|p_2\|_{L^1}+\sum_{k=0}^{m}m^2_k\right) & M_{R_2}\left(\|q_2\|_{L^1}+\sum_{k=0}^{m}\widetilde{m}^2_k\right)\end{pmatrix} \tag{4}$$

converges to zero. Then the system (1) has at least one mild solution.

Proof. Transform the problem (1) into a fixed point problem, consider the operator $\Upsilon : PC(\Theta,E)\times PC(\Theta,E)\to PC(\Theta,E)\times PC(\Theta,E)$ define by:

$$\Upsilon(\xi(\theta),\varphi(\theta))=(\Upsilon_1(\xi(\theta),\varphi(\theta)),\Upsilon_2(\xi(\theta),\varphi(\theta))),$$

where

$$\begin{aligned}\Upsilon_1(\xi(\theta),\varphi(\theta)) &= R_1(\theta)\xi_0+\textstyle\int_0^\theta R_1(\theta-s)f_1\left(s,\xi(s),\varphi(s),H_1(\xi(s),\varphi(s))\right)ds\\ &\quad+\textstyle\sum_{0<\theta_k<\theta}R_1(\theta-\theta_k)\aleph_k(\xi(\theta_k^-),\varphi(\theta_k^-)),\end{aligned}$$

$$\begin{aligned}\Upsilon_2(\xi(\theta),\varphi(\theta)) &= R_2(\theta)\varphi_0+\textstyle\int_0^\theta R_2(\theta-s)f_2\left(s,\xi(s),\varphi(s),H_2(\xi(s),\varphi(s))\right)ds\\ &\quad+\textstyle\sum_{0<\theta_k<\theta}R_2(\theta-\theta_k)\widetilde{\aleph}_k(\xi(\theta_k^-),\varphi(\theta_k^-)).\end{aligned}$$

We show that Υ was well defined. Let $(\xi,\varphi)\in X\times X,\ \theta\in\Theta$, then we have

$$\begin{aligned}\|\Upsilon_1(\xi(\theta),\varphi(\theta))\| &\le \|R_1(\theta)\|(\|\xi_0\|+\|Q_1(\xi)\|)\\ &\quad+\int_0^\theta\|R_1(\theta-s)\|\|f_1\left(s,\xi(s),\varphi(s),H_1(\xi(s),\varphi(s))\right)\|ds\\ &\quad+\sum_{0<\theta_k<\theta}\|R_1(\theta-\theta_k)\|\|\aleph_k(\xi(\theta_k^-),\varphi(\theta_k^-))\|.\end{aligned}$$

From $(H1)$, we have

$$\begin{aligned}||f_1(s,\xi(s),\varphi(s),H_1(\xi(s),\varphi(s)))|| &\le p_1(s)\psi_1(\|\xi\|_X)+q_1(s)\phi_1(\|v\|_X)\\ &\quad+\|f_1(s,0,0,0)\|.\end{aligned}$$

Also, we have

$$\|\aleph_k(\xi,v)\|\le m^1_k(\|\xi\|_X)+\widetilde{m}^1_k(\|v\|_X)+\|\aleph_k(0,0)\|.$$

Then, we get

$$\begin{aligned}\|\Upsilon_1(\xi(\theta),\varphi(\theta))\| \leq{}& M_{R_1}\|\xi_0\| + M_{R_1}\left(\|p_1\|_{L^1}\psi_1(\|\xi\|_X) + \|q_1\|_{L^1}\phi_1(\|\xi\|_X)\right)\\ &+ M_{R_1}\int_0^\theta f_1^0(s)ds + M_{R_1}\sum_{k=0}^{m}(m_k^1(\|\xi\|_X) + \widetilde{m}_k^1(\|v\|_X))\\ &+ M_{R_1}\sum_{k=0}^{m}\|\aleph_k(0,0)\|.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|\Upsilon_2(\xi(\theta),\varphi(\theta))\| \leq{}& M_{R_2}\|\varphi_0\| + M_{R_2}\left(\|p_2\|_{L^1}\psi_1(\|\xi\|_X) + \|q_2\|_{L^1}\phi_2(\|\varphi\|_X)\right)\\ &+ M_{R_2}\int_0^\theta f_2^0(s)ds + M_{R_2}\sum_{k=0}^{m}(m_k^2(\|\xi\|_X) + \widetilde{m}_k^2(\|v\|_X))\\ &+ M_{R_2}\sum_{k=0}^{m}\|\widetilde{\aleph}_k(0,0)\|.\end{aligned}$$

Then

$$\|\Upsilon(\xi,\varphi)\|_{X\times X} < \infty.$$

Obviously, the fixed points of operator Υ are mild solution of the problem (1). We use Theorem 2.13 to prove that Υ has a fixed point.

Step 1. Υ is continuous.

Let $(\xi_n,\varphi_n)_{n\in\mathbb{N}}$ be a couple of sequences such that $(\xi_n,\varphi_n)\to(\xi^*,\varphi^*)$, then for $\theta\in\Theta$, we have

$$\begin{aligned}&\|(\Upsilon_1(\xi_n,\varphi_n))(\theta) - (\Upsilon_1(\xi^*,\varphi^*))(\theta)\|\\ &\leq M_{R_1}\int_0^\theta \|f_1(s,\xi_n(s),\varphi_n(s),H(\xi_n(s),\varphi_n(s)))\\ &\quad - f_1(s,\xi^*(s),\varphi^*(s),H(\xi^*(s),\varphi^*(s)))\|ds\\ &\quad + M_{R_1}\sum_{k=0}^{m}\|\aleph_k(\xi_n(\theta),\varphi_n(\theta)) - \aleph_k(\xi^*(\theta),\varphi^*(\theta))\|.\end{aligned}$$

By the continuity of h_1, we get

$$h_1(\theta,s,\xi_n(s),\varphi_n(s)) \to h_1(\theta,s,\xi^*(s),\varphi^*(s)) \ as\ n\to+\infty.$$

And we have

$$\begin{aligned}\|h_1(\theta,s,\xi_n(s),\varphi_n(s)) - h_1(\theta,s,\xi^*(s),\varphi^*(s))\| \leq{}& h_{c_1}^*\|\xi_n(s)-\xi^*(s)\|\\ &+ \bar{h}_{c_1}^*\|\varphi_n(s)-\varphi^*(s)\|.\end{aligned}$$

By Lebesgue dominated convergence theorem, we obtain

$$\int_0^\theta h_1(\theta,s,\xi_n(s),\varphi_n(s))ds \to \int_0^\theta h_1(\theta,s,\xi^*(s),\varphi^*(s))ds,\ as\ n\to+\infty.$$

Also, the continuity of $\aleph_k$, give

$$\aleph_k(\xi_n(\theta),\varphi_n(\theta)) \to \aleph_k(\xi^*(\theta),\varphi^*(\theta)) \ as\ n\to+\infty.$$

Hence, from the continuity of the function f_1 and the Lebesgue dominated convergence theorem, we get

$$\|\Upsilon_1(\xi_n,\varphi_n) - \Upsilon_1(\xi^*,\varphi^*)\|_X \to 0,\ as\ n\to+\infty.$$

Similarly, we get

$$\|\Upsilon_2(\xi_n,\varphi_n)-\Upsilon_2(\xi^*,\varphi^*)\|_X \to 0,\ as\ n\to+\infty.$$

Thus, Υ is continuous.

Let B_δ be defined by $B_\delta=\{(\xi,\varphi)\in X\times X:(\|\xi\|_X,\|\varphi\|_X)\le(\delta_1,\delta_2)\}$, with $\delta_i>0$. The set B_δ is bounded, closed and convex of $X\times X$.

Step 2.

Claim 1. $\Upsilon(B_\delta)\subset(B_\delta)$.

Let $(\xi,\varphi)\in B_\delta$ and $\theta\in\Theta$, from $(H1)-(H3)$, it follows that

$$\begin{aligned}\|\Upsilon_1(\xi(\theta),\varphi(\theta))\| \le{}& M_{R_1}\left(\|\xi_0\|+\|p_1\|_{L^1}\psi_1(\delta_1)+\|q_1\|_{L^1}\phi_1(\delta_2)\right)\\ &+M_{R_1}\int_0^\theta f_1^0(s)ds\\ &+\sum_{k=0}^m\left(m_k^1(\delta_1)+\widetilde{m}_k^1(\delta_2)\right)+\sum_{k=0}^m\|\aleph_k(0,0)\|.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|\Upsilon_2(\xi(\theta),\varphi(\theta))\| \le{}& M_{R_2}\left(\|\varphi_0\|+\|p_2\|_{L^1}\psi_2(\delta_1)+\|q_2\|_{L^1}\phi_2(\delta_2)\right)\\ &+M_{R_2}\int_0^\theta f_2^0(s)ds\\ &+\sum_{k=0}^m\left(m_k^2(\delta_1)+\widetilde{m}_k^2(\delta_2)\right)+\sum_{k=0}^m\|\widetilde{\aleph}_k(0,0)\|.\end{aligned}$$

Then

$$\begin{aligned}\|\Upsilon(\xi,\varphi)(\theta)\|\le{}&\begin{pmatrix}M_{R_1}\left(\|p_1\|_{L^1}+\sum_{k=0}^m m_k^1\right) & M_{R_1}\left(\|q_1\|_{L^1}+\sum_{k=0}^m \widetilde{m}_k^1\right)\\ M_{R_2}\left(\|p_2\|_{L^1}+\sum_{k=0}^m m_k^2\right) & M_{R_2}\left(\|q_2\|_{L^1}+\sum_{k=0}^m \widetilde{m}_k^2\right)\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}\\ &+\begin{pmatrix}M_{R_1}(\|\xi_0\|+\int_0^\theta f_1^0(s)ds+\sum_{k=0}^m\|\aleph_k(0,0)\|)\\ M_{R_2}(\|\varphi_0\|+\int_0^\theta f_2^0(s)ds+\sum_{k=0}^m\|\widetilde{\aleph}_k(0,0)\|)\end{pmatrix}.\end{aligned}$$

Since $\widehat{M}$ converges to zero, then $\left(I-\widehat{M}\right)$ is invertible and its inverse $\left(I-\widehat{M}\right)^{-1}$ has nonnegative elements. Hence

$$\widehat{M}^{-1}\begin{pmatrix}M_{R_1}(\|\xi_0\|+\int_0^\theta f_1^0(s)ds+\sum_{k=0}^m\|\aleph_k(0,0)\|)\\ M_{R_2}(\|\varphi_0\|+\int_0^\theta f_2^0(s)ds+\sum_{k=0}^m\|\widetilde{\aleph}_k(0,0)\|)\end{pmatrix}\le\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}.$$

Thus

$$\|\Upsilon(\xi,\varphi)\|_{X\times X}\le\begin{pmatrix}\delta_1\\ \delta_2\end{pmatrix}.$$

Claim 2. The set $\Upsilon(B_\delta)$ is equicontinuous.

For $(\xi,\varphi)\in B_\delta$ and $\kappa_1,\kappa_2\in\Theta$, we have

$$\begin{aligned}&\|\Upsilon_1(\xi,\varphi)(\kappa_1)-\Upsilon_1(\xi,\varphi)(\kappa_2)\|\\ &\quad\le\|R(\kappa_1)-R(\kappa_2)\|(\|\xi_0\|)\end{aligned}$$

$$+\int_0^{\kappa_1}\|R(\kappa_1-s)-R(\kappa_2-s)\|\big(p_1(s)\psi_1(\delta_1)+q_1(s)\phi_1(\delta_2)\big)ds$$
$$+M_{R_1}\int_{\kappa_1}^{\kappa_2}\big(p_1(s)\psi_1(\delta_1)+q_1(s)\phi_1(\delta_2)\big)ds+M_{R_1}\sum_{\kappa_1<\theta_k<\kappa_2}\|\aleph_k(\xi(\theta_k^-),\varphi(\theta_k^-))\|$$
$$+\sum_{0<\theta_k<\kappa_1}\|R_1(\kappa_1-\theta_k)-R_1(\kappa_2-\theta_k)\|\|\aleph_k(\xi(\theta_k^-),\varphi(\theta_k^-))\|.$$

By the strong continuity of $R_1(\cdot)$ and $(H1)$-$(H3)$, we obtain

$$\|\Upsilon_1(\xi,\varphi)(\kappa_1)-\Upsilon_1(\xi,\varphi)(\kappa_2)\|\to 0 \ as\ \kappa_1\to\kappa_2.$$

Similarly, we get

$$\|\Upsilon_2(\xi,\varphi)(\kappa_1)-\Upsilon_2(\xi,\varphi)(\kappa_2)\|\to 0 \ as\ \kappa_1\to\kappa_2.$$

Hence, the set $\Upsilon(B_\delta)$ is equicontinuous.

Step 3. Υ is generalized χ_{PC}-condensing operator.

Let $\Omega\subset\Omega_1\times\Omega_2$, then $\Upsilon_i(\Omega), i=1,2$ are bounded and equicontinuous on Θ. From (H1) and (H3), we have

$$\wp_i\left\{f_i(\theta,\Omega(\theta),H_i(\Omega(\theta)))\right\}\le p_i(\theta)\wp_1\left\{\Omega_1(\theta)\right\}+q_i(\theta)\wp_2\left\{\Omega_2(\theta)\right\},$$

and

$$\wp_i\left\{\widehat{\aleph}_i^k(\Omega(\theta))\right\}\le m_k^1\wp_1\left\{\Omega_1(\theta)\right\}+\widetilde{m}_k^1\wp_2\left\{\Omega_2(\theta)\right\},$$

where

$$(\widehat{\aleph}_1^k,\widehat{\aleph}_2^k)=(\aleph_k,\widetilde{\aleph}_k).$$

Then, we have

$$\begin{aligned}\wp_i(\Upsilon_i\Omega(\theta))&\le M_{R_i}\int_0^\theta\wp_i\left\{f_i(s,\Omega(s),H_i(\Omega(s)))\right\}ds+M_{R_i}\sum_{0<\theta_k<\theta}\wp_i\left\{\widehat{\aleph}_i^k(\Omega(\theta_k^-))\right\}\\&\le M_{R_i}\int_0^\theta(p_i(s)\wp_1\left\{\Omega_1(s)\right\}+q_i(s)\wp_2\left\{\Omega_2(s)\right\})ds\\&\quad+M_{R_i}\sum_{k=0}^m\left(m_k^1\wp_1\left\{\Omega_1(\theta)\right\}+\widetilde{m}_k^1\wp_2\left\{\Omega_2(\theta)\right\}\right)\\&\le M_{R_i}\left(\|p_i\|+\sum_{k=0}^m m_k^1\right)\wp_1\left\{\Omega_1(\theta)\right\}\\&\quad+M_{R_i}\left(\|q_i\|+\sum_{k=0}^m\widetilde{m}_k^1\right)\wp_2\left\{\Omega_2(\theta)\right\}.\end{aligned}$$

Since $\Upsilon(B_\delta)$ is equicontinuous, we get

$$\chi_{PC}(\Upsilon\Omega)\le\begin{pmatrix}M_{R_1}\left(\|p_1\|_{L^1}+\sum_{k=0}^m m_k^1\right) & M_{R_1}\left(\|q_1\|_{L^1}+\sum_{k=0}^m\widetilde{m}_k^1\right)\\ M_{R_2}\left(\|p_2\|_{L^1}+\sum_{k=0}^m m_k^2\right) & M_{R_2}\left(\|q_2\|_{L^1}+\sum_{k=0}^m\widetilde{m}_k^2\right)\end{pmatrix}\begin{pmatrix}\chi_1(\Omega_1)\\ \chi_2(\Omega_2)\end{pmatrix}.$$

As a consequence of Theorem 2.13, we deduce that Υ has a fixed point (ξ,φ) in $X\times X$, which is a mild solution of problem (1). □

The next result is concerned with generalized Darbo's fixed point theorem.

Theorem 3.3. *Assume that the conditions $(H1)-(H4)$ are satisfied, and we assume that the functions p_i, q_i in $(H1)$ they belong to the space $C(\Theta, \mathbb{R}^+)$, such that*

$$p_i^* = \max\{\sup_{\theta\in\Theta}(p_i(\theta)), \sup_{\theta\in\Theta}(q_i(\theta))\}.$$

Also we assume that $\aleph_k$, $\widetilde{\aleph}_k$ are compact and $\widehat{M}$ converges to zero. Then, the system (1) has at least one mild solution.

Proof. From the Step of the Proof of Theorem 3.2 , the operator Υ is continuous and we have $\Upsilon(B_\delta) \subset (B_\delta)$. Then we prove that there exist an iteration of order p_0 of operator Υ such that the operator Υ^{p_0} be M-contractivity.

Let $\Omega_i = \overline{co}\,(\Upsilon_i\,(B_\delta))\; i = 1, 2$. Lemma 2.10 implies that $\Omega_i \subset \overline{B_\delta}, i = 1, 2$ are bounded and equicontinuous, and $\Upsilon : \Omega_1 \times \Omega_2 \to \Omega_1 \times \Omega_2$ is a continuous and bounded operator. Similarly to the Step 3 of the Proof of Theorem 3.2, $\Upsilon_i\,(\Omega)$, $i = 1, 2$ are bounded and equicontinuous on Θ. Now, using Lemma 2.10 and equation 3, we conclude that $\Upsilon_i^p\Omega, i = 1, 2$ are bounded and equicontinuous. For each $p = 1, 2, \ldots$, we have

$$\widehat{\wp}_i(\Upsilon_i^p\Omega) = \sup_{\theta\in\Theta} \wp_i(\Upsilon_i^p\Omega(\theta)).$$

Then, we have

$$\begin{aligned}\wp_i(\Upsilon_i^1\Omega(\theta)) &\le M_{R_i}\int_0^\theta p_i(s)\left(\wp_1\{\Omega_1(s)\} + q_i(s)\wp_2\{\Omega_2(s)\}\right) ds\\ &\le M_{R_i}\,(p_i^*\theta)\left(\wp_1\{\Omega_1(\theta)\} + \wp_2\{\Omega_2(\theta)\}\right).\end{aligned}$$

For $\Upsilon_i^2(\Omega) = \Upsilon_i\overline{co}(\Upsilon_i^1(\Omega))$, we get

$$\begin{aligned}\wp_i(\Upsilon_i^2\Omega(\theta)) \le{}& M_{R_i}\int_0^\theta \wp_i\left\{f_i(s, \overline{co}(\Upsilon_i^1(\Omega))(s), H_i(\overline{co}(\Upsilon_i^2(\Omega))(s)))\right\} ds\\ &+ M_{R_i}\sum_{0<\theta_k<\theta}\wp_i\left\{\widehat{\aleph}_i^k(\overline{co}(\Upsilon_i^1(\Omega)))\right\}\\ \le{}& M_{R_i}\int_0^\theta p_i(s)M_{R_i}\,(p_i^* s)\Big(\wp_1\{\Omega_1(s)\} + \wp_2\{\Omega_2(s)\}\Big) ds\\ \le{}& \left((M_{R_i}p_i^*)^2\frac{\theta^2}{2}\right)\Big(\wp_1\{\Omega_1(\theta)\} + \wp_2\{\Omega_2(\theta)\}\Big).\end{aligned}$$

Also, for $\Upsilon_i^3(\Omega) = \Upsilon_i\left(\overline{co}(\Upsilon_i^2(\Omega))\right)$, we obtain

$$\begin{aligned}\wp_i(\Upsilon_i^3\Omega(\theta)) \le{}& M_{R_i}\int_0^\theta \wp_i\left\{f_i(s, \overline{co}(\Upsilon_i^2(\Omega))(s), H_i(\overline{co}(\Upsilon_i^2(\Omega))(s)))\right\} ds\\ &+ M_{R_i}\sum_{0<\theta_k<\theta}\wp_i\left\{\widehat{\aleph}_i^k(\overline{co}(\Upsilon_i^2(\Omega)))\right\}\\ \le{}& M_{R_i}\int_0^\theta p_i^* M_{R_i}\left((p_i^*)^2\frac{s^2}{2}\right)\Big(\wp_1\{\Omega_1(s)\} + \wp_2\{\Omega_2(s)\}\Big) ds\\ \le{}& \left((M_{R_i}p_i^*)^3\frac{\theta^3}{3!}\right)\Big(\wp_1\{\Omega_1(\theta)\} + \wp_2\{\Omega_2(\theta)\}\Big).\end{aligned}$$

Suppose that

$$\wp_i(\Upsilon_i^p\Omega(\theta)) \le \left((M_{R_i}p_i^*)^p\frac{\theta^p}{p!}\right)\Big(\wp_1\{\Omega_1(\theta)\} + \wp_2\{\Omega_2(\theta)\}\Big).$$

Hence, for any $\theta \in [0,T]$, we obtain

$$\begin{aligned}\wp_i(\Upsilon_i^{p+1}\Omega(\theta)) &\leq M_{R_i}\int_0^\theta \wp_i\left\{f_i(s,\overline{co}(\Upsilon_i^p(\Omega))(s),H_i(\overline{co}(\Upsilon_i^p(\Omega))(s)))\right\}ds \\ &\quad + M_{R_i}\sum_{0<\theta_k<\theta}\wp_i\left\{\widehat{\aleph}_i^k(\overline{co}(\Upsilon_i^p(\Omega)))\right\} \\ &\leq M_{R_i}\int_0^\theta p_i^*\left((M_{R_i}p_i^*)^p\frac{s^p}{p!}\right)\Big(\wp_1\{\Omega_1(s)\}+\wp_2\{\Omega_2(s)\}\Big)ds \\ &\leq \left((M_{R_i}p_i^*)^{p+1}\frac{\theta^{p+1}}{(p+1)!}\right)\Big(\wp_1\{\Omega_1(\theta)\}+\wp_2\{\Omega_2(\theta)\}\Big).\end{aligned}$$

Then, if we put

$$\sigma_p^i = \frac{(M_{R_i}p_i^*T)^p}{p!},$$

and

$$M_p = \begin{pmatrix}\sigma_p^1 & \sigma_p^1\\ \sigma_p^2 & \sigma_p^2\end{pmatrix},$$

it follows that

$$\chi_{PC}(\Upsilon^p\Omega) \leq M_p\chi_{PC}(\Omega).$$

Its clear that

$$\lim_{p\to+\infty}\sigma_p^i = 0.$$

Then, there exist $p_1, p_2 \in \mathbb{N}$, such that for all $p > \max(p_1,p_2)$, we have

$$(\sigma_p^1,\sigma_p^2) < (1,1).$$

Thus, for

$$p_0 = \min\{p\in\mathbb{N} : \|M_p\|_{M_{2\times 2}(\mathbb{R}^+)} < 1\},$$

we get

$$\rho(M_{p_0}) \leq \|M_{p_0}\|_{M_{2\times 2}(\mathbb{R}^+)} < 1.$$

Thus, from Lemma 2.5, we deduce that M_{p_0} converges to zero. Applying now Theorem 2.14, we conclude that Υ has at least one fixed point, which is a mild solution of problem (1). □

Remark 3.4. If we don't assume that $\aleph_k$, $\widetilde{\aleph}_k$ are compact, we can assume in Theorem 3.3 that

$$\left(M_{R_i}\max\left\{\sum_{k=0}^m m_k^i, \sum_{k=0}^m \widetilde{m}_k^i\right\}\right) < 1.$$

Thus, we get

$$\wp_i(\Upsilon_i^p\Omega(\theta)) \leq (M_{R_i})^p\left(\sum_{j=0}^p \xi_j(p_i^*)^j(l_i^*)^{p-j}\frac{\theta^j}{j!}\right)\Big(\wp_1\{\Omega_1(\theta)\}+\wp_2\{\Omega_2(\theta)\}\Big),$$

where our sequence (ξ_j) is defined by

$$\xi_j = \begin{cases}\xi_{j-1}\binom{p-j+1}{n}; & j\in\{1,\dots,p\},\\ 1; \quad j=0,\end{cases}$$

and

$$l_i^* = \max\left\{\sum_{k=0}^{m} m_k^i, \sum_{k=0}^{m} \widetilde{m}_k^i\right\}.$$

Then, if we put

$$\widehat{\sigma}_p^i = (M_{R_i})^p \sum_{j=0}^{p} \frac{\xi_j (p_i^*)^j (l_i^*)^{p-j} T^j}{j!}$$

and

$$\widehat{M}_p = \begin{pmatrix} \widehat{\sigma}_p^1 & \widehat{\sigma}_p^1 \\ \\ \widehat{\sigma}_p^2 & \widehat{\sigma}_p^2 \end{pmatrix},$$

it follows that

$$\chi_{PC}(\Upsilon^p \Omega) \quad \leq \quad \widehat{M}_p \chi_{PC}(\Omega).$$

Consequently, we have that

$$\xi_n = \begin{cases} \xi_{n-1}\left(\frac{m-n+1}{n}\right); & n \in \{1, \dots, m\}, \\ 1; \quad n = 0, \end{cases}$$

is a novel representation of binomial coefficients in a recurrence relation, and it gives all Pascal's triangle lines as well as the binomial formula. Then, using this sequence, all of the results obtained using Newton's binomial formula or Pascal's triangle may be proven.

3.2. **Results of Ulam's type stability.** In this section, we introduce Ulam's type stability concepts for the equation (1).

Let $\widehat{\Delta}_1, \widehat{\Delta}_2 \geq 0$; $k = 1, \dots, m$, and $\Delta_i \in PC(\Theta, \mathbb{R}_+)$ be nondecreasing. We consider the following inequalities:

$$\begin{cases} \|\xi'(\theta) - A_1\xi(\theta) - f_1(\theta, \xi(\theta), \varphi(\theta), H_1(\xi(\theta), \varphi(\theta))) - \int_0^\theta B_1(\theta - s)\xi(s)ds\| \\ \quad \leq \Delta_1(\theta), \text{ for } \theta \in \widetilde{\Theta}, \\ \\ \|\varphi'(\theta) - A_2\varphi(\theta) - f_2(\theta, \xi(\theta), \varphi(\theta), H_1(\xi(\theta), \varphi(\theta))) - \int_0^\theta B_2(\theta - s)\varphi(s)ds\| \\ \quad \leq \Delta_2(\theta), \text{ for } \theta \in \widetilde{\Theta}, \\ \\ \|\xi(\theta_k^+) - \xi(\theta_k^-) - \aleph_k(\xi(\theta_k^-), \varphi(\theta_k^-))\| \leq \widehat{\Delta}_1, \ k = 1, \dots, m, \\ \\ \|\varphi(\theta_k^+) - \varphi(\theta_k^-) + \widetilde{\aleph}_k(\xi(\theta_k^-), \varphi(\theta_k^-))\| \leq \widehat{\Delta}_2, \ k = 1, \dots, m. \end{cases} \tag{5}$$

Consider the space

$$\widetilde{X_i} = \{\xi \in PC^1(\Theta, E) \ : \ \xi(\theta) \in D(A_i)\}.$$

The following concepts are inspired by the papers [39, 42, 41] and the references therein.

Definition 3.5. The system (1) is generalized Ulam-Hyers-Rassias stable with respect to $(\Delta_1, \Delta_2, \widehat{\Delta}_1, \widehat{\Delta}_2)$ if there exists $c_{f_1,\aleph_k,\Delta_1}, c_{f_2,\widetilde{\aleph}_k,\Delta_2} > 0$ such that for each solution (ξ, φ) of the inequalities system (5) there exists a mild solution $(\widehat{\xi}, \widehat{\varphi}) \in \widetilde{X_1} \times \widetilde{X_2}$ of the system (1) with

$$\|\xi(\theta) - \widehat{\xi}(\theta)\| \leq c_{f_1,\aleph_k,\widehat{\Delta}}(\Delta_1(\theta) + \Delta_1), \quad \theta \in \Theta,$$

and

$$\|\varphi(\theta)-\widehat{\varphi}(\theta)\| \leq c_{f_2,\widetilde{\aleph}_k,\widehat{\Delta}}(\Delta_2(\theta))+\Delta_2), \quad \theta\in\Theta.$$

In another sense

$$\begin{aligned}&\|(\xi(\theta)-\widehat{\xi}(\theta),\varphi(\theta)-\widehat{\varphi}(\theta))\|\\ &\leq \max\left(c_{f_1,\aleph_k,\widehat{\Delta}},c_{f_2,\widetilde{\aleph}_k,\widetilde{\Delta}}\right)\left(\Delta_1(\theta)+\Delta_2(\theta)+\widehat{\Delta}_1+\widehat{\Delta}_2\right),\ \theta\in\Theta.\end{aligned}$$

Remark 3.6. A function $(\xi,\varphi)\in\widetilde{X_1}\times\widetilde{X_2}$ is a solution of inequalities (5) if and only if there exist $G_{1,2}\in PC(\Theta,R)$, $g_k,\widehat{g}_k\in\mathbb{R}$ such that:

- $\|G_i(\theta)\|\leq\Delta_i(\theta);\ \theta\in\aleph_k,\ \|g_k\|\leq\widehat{\Delta}_1,\ \|\widehat{g}_k\|\leq\widehat{\Delta}_2,$
- $\xi'(\theta)=A_1\xi(\theta)+f_1\left(\theta,\xi(\theta),\varphi(\theta),H_1(\xi(\theta),\varphi(\theta))\right)+\int_0^\theta B_1(\theta-s)\xi(s)ds+G_1(\theta),$ $\theta\in\widetilde{\Theta},$
- $\varphi'(\theta)=A_2\varphi(\theta)+f_2\left(\theta,\xi(\theta),\varphi(\theta),H_1(\xi(\theta),\varphi(\theta))\right)+\int_0^\theta B_2(\theta-s)\varphi(s)ds+G_2(\theta),\theta\in\widetilde{\Theta},$
- $\aleph_k(\xi(\theta_k^-),\varphi(\theta_k^-))=\xi(\theta_k^+)-\xi(\theta_k^-)+g_k,\ k=1,\ldots,m,$
- $\widetilde{\aleph}_k(\xi(\theta_k^-),\varphi(\theta_k^-))=y(\theta_k^+)-\varphi(\theta_k^-)+\widehat{g}_k,\ k=1,\ldots,m.$

Remark 3.7. If $(\xi,\varphi)\in\widetilde{X_1}\times\widetilde{X_2}$ is a solution of inequalities (5), then $(\xi,\varphi)\in\mathcal{X}\times\mathcal{X}$ is a solution of the following integral inequalities

$$\begin{aligned}&\|\xi(\theta)-R_1(\theta)\xi_0-\int_0^\theta R_1(\theta-s)f_1\left(s,\xi(s),\varphi(s),H_1(\xi(s),\varphi(s))\right)ds\\ &-\sum_{0<\theta_k<\theta}R_1(\theta-\theta_k)\aleph_k(\xi(\theta_k^-),\varphi(\theta_k^-))\|\\ &\leq M_{R_1}\int_0^\theta e^{-\beta_1(\theta-s)}\Delta_1(s)ds+M_{R_1}m\widehat{\Delta}_1;\ \theta\in\Theta,\end{aligned}$$

$$\begin{aligned}&\|\varphi(\theta)-R_2(\theta)\varphi_0-\int_0^\theta R_2(\theta-s)f_2\left(s,\xi(s),\varphi(s),H_2(\xi(s),\varphi(s))\right)ds\\ &-\sum_{0<\theta_k<\theta}R_2(\theta-\theta_k)\widetilde{\aleph}_k(\xi(\theta_k^-),\varphi(\theta_k^-))\|\\ &\leq M_{R_2}\int_0^\theta e^{-\beta_2(\theta-s)}\Delta_2(s)ds+M_{R_2}m\widehat{\Delta}_2;\ \theta\in\Theta.\end{aligned}$$

To discuss stability, we need the following additional assumption:

(H_Δ) We assume that for $\Delta_i\in PC(\Theta,\mathbb{R}^+)$ a nondecreasing function there exists $c_{\Delta_i}>0$, such that

$$\int_0^\theta\Delta_i(s)ds\leq c_{\Delta_i}\Delta_i(\theta).$$

Theorem 3.8. *Assume that $(H1)-(H4)$ and (H_Δ) are satisfied and and $\widehat{M}$ converges to zero. Then the system (1) is generalized Ulam–Hyers–Rassias stable with respect to $(\Delta_1,\Delta_2,\widehat{\Delta}_k,\widetilde{\Delta}_k)$.*

Proof. Let $(v,\widehat{v})$ be a solution of the system of inequalities (5) and $(\xi,\widehat{\xi})\in\mathcal{X}\times\mathcal{X}$ be the mild solution of the system (1) with, $(\xi(0),\widehat{\xi}(0))=(v(0),\widehat{v}(0))=(\xi_0,\varphi_0)$.

Then we get

$$\xi(\theta) = R_1(\theta)\xi_0 + \int_0^\theta R_1(\theta-s) f_1\left(s, \xi(s), \widehat{\xi}(s), H_1(\xi(s), \widehat{\xi}(s))\right) ds$$
$$+ \sum_{0<\theta_k<\theta} R_1(\theta-\theta_k)\aleph_k(\xi(\theta_k^-), \widehat{\xi}(\theta_k^-)); \quad \theta \in \Theta,$$

$$\widehat{\xi}(\theta) = R_2(\theta)\varphi_0 + \int_0^\theta R_2(\theta-s) f_2\left(s, \xi(s), \widehat{\xi}(s), H_2(\xi(s), \widehat{\xi}(s))\right) ds$$
$$+ \sum_{0<\theta_k<\theta} R_2(\theta-\theta_k)\widetilde{\aleph}_k(\xi(\theta_k^-), \widehat{\xi}(\theta_k^-)); \quad \theta \in \Theta.$$

In other hand, we get

$$\Bigg\| v(\theta) - R_1(\theta)\xi_0 - \int_0^\theta R_1(\theta-s) f_1\left(s, v(s), \widehat{v}(s), H_1(v(s), \widehat{v}(s))\right) ds$$
$$- \sum_{0<\theta_k<\theta} R_1(\theta-\theta_k)\aleph_k(v(\theta_k^-), \widehat{v}(\theta_k^-)) \Bigg\|$$
$$\leq M_{R_1}\int_0^\theta e^{-\beta_1(\theta-s)}\Delta_1(s)ds + M_{R_1} m\widehat{\Delta}_1; \ \theta \in \Theta,$$
$$\Bigg\| \widehat{v}(\theta) - R_2(\theta)\varphi_0 - \int_0^\theta R_2(\theta-s) f_2\left(s, v(s), \widehat{v}(s), H_2(v(s), \widehat{v}(s))\right) ds$$
$$- \sum_{0<\theta_k<\theta} R_2(\theta-\theta_k)\widetilde{\aleph}_k(v(\theta_k^-), \widehat{v}(\theta_k^-)) \Bigg\|$$
$$\leq M_{R_2}\int_0^\theta e^{-\beta_2(\theta-s)}\Delta_2(s)ds + M_{R_2} m\widehat{\Delta}_2; \ \theta \in \Theta.$$

Hence, for $\theta \in \Theta$, we have

$$\|v(\theta) - \xi(\theta)\| = \Bigg\| v(\theta) - R_1(\theta)\xi_0 - \int_0^\theta R_1(\theta-s) f_1\left(s, \xi(s), \widehat{\xi}(s), H_1(\xi(s), \widehat{\xi}(s))\right) ds$$
$$- \sum_{0<\theta_k<\theta} R_1(\theta-\theta_k)\aleph_k(\xi(\theta_k^-), \widehat{\xi}(\theta_k^-))$$
$$- \int_0^\theta R_1(\theta-s) f_1\left(s, v(s), \widehat{v}(s), H_1(v(s), \widehat{v}(s))\right) ds$$
$$+ \int_0^\theta R_1(\theta-s) f_1\left(s, v(s), \widehat{v}(s), H_1(v(s), \widehat{v}(s))\right) ds$$
$$- \sum_{0<\theta_k<\theta} R_1(\theta-\theta_k)\aleph_k(v(\theta_k^-), \widehat{v}(\theta_k^-))$$
$$+ \sum_{0<\theta_k<\theta} R_1(\theta-\theta_k)\aleph_k(v(\theta_k^-), \widehat{v}(\theta_k^-)) \Bigg\|$$
$$\leq M_{R_1} c_{\Delta_1}\Delta_1(\theta) + M_{R_1} m\widehat{\Delta}_1$$

$$+ M_{R_1}\int_0^\theta p_1(s)\psi_1(\|v(s)-\xi(s)\|)+q_1(s)\phi_1(\|\widehat{v}(s)-\widehat{\xi}(s)\|)ds,$$
$$+\sum_{k=0}^{m}(m_k^1(\|v(\theta_k^-)-\xi(\theta_k^-)\|)+\widetilde{m}_k^1(\|\widehat{v}(\theta_k^-)-\widehat{\xi}(\theta_k^-)\|)).$$

Also we get

$$\begin{aligned}\|\widehat{v}(\theta)-\widehat{\xi}(\theta)\| &= \Bigg\|\widehat{v}(\theta)-R_2(\theta)\xi_0-\int_0^\theta R_2(\theta-s)f_2\left(s,\xi(s),\widehat{\xi}(s),H_2(\xi(s),\widehat{\xi}(s))\right)ds\\
&\quad-\sum_{0<\theta_k<\theta}R_2(\theta-\theta_k)\aleph_k(\xi(\theta_k^-),\widehat{\xi}(\theta_k^-))\\
&\quad-\int_0^\theta R_2(\theta-s)f_2\left(s,v(s),\widehat{v}(s),H_2(v(s),\widehat{v}(s))\right)ds\\
&\quad+\int_0^\theta R_2(\theta-s)f_2\left(s,v(s),\widehat{v}(s),H_2(v(s),\widehat{v}(s))\right)ds\\
&\quad-\sum_{0<\theta_k<\theta}R_2(\theta-\theta_k)\widetilde{\aleph}_k(v(\theta_k^-),\widehat{v}(\theta_k^-))\\
&\quad+\sum_{0<\theta_k<\theta}R_1(\theta-\theta_k)\widetilde{\aleph}_k(v(\theta_k^-),\widehat{v}(\theta_k^-))\Bigg\|\\
&\le M_{R_2}c_{\Delta_2}\Delta_2(\theta)+M_{R_2}m\widehat{\Delta}_2\\
&\quad+M_{R_2}\int_0^\theta p_2(s)\psi_2(\|v(s)-\xi(s)\|)+q_2(s)\phi_2(\|\widehat{v}(s)-\widehat{\xi}(s)\|)ds,\\
&\quad+\sum_{k=0}^{m}(m_k^2(\|v(\theta_k^-)-\xi(\theta_k^-)\|)+\widetilde{m}_k^2(\|\widehat{v}(\theta_k^-)-\widehat{\xi}(\theta_k^-)\|)).\end{aligned}$$

Thus,

$$\begin{aligned}&\|v(\theta)-\xi(\theta)\|+\|\widehat{v}(\theta)-\widehat{\xi}(\theta)\|\\
&\le M_R(c_{\max}+m)(\Delta_1(\theta)+\Delta_2(\theta)+\widehat{\Delta}_1+\widehat{\Delta}_2)\\
&\quad+M_R\int_0^\theta(p_1(s)+p_2(s))(\|v(s)-\xi(s)\|+\|\widehat{v}(s)-\widehat{\xi}(s)\|)ds,\\
&\quad+4M_R\sum_{k=0}^{m}\widetilde{m_k}\left(\|v(\theta_k^-)-\xi(\theta_k^-)\|+\|\widehat{v}(\theta_k^-)-\widehat{\xi}(\theta_k^-)\|\right),\end{aligned}$$

where

$$c_{\max}=\max(c_{\Delta_1},c_{\Delta_2}),\ \widetilde{m_k}=\max\left\{m_k^i,\widetilde{m}_k^i\right\}.$$

From Lemma 2.16, we get

$$\begin{aligned}&\|v(\theta)-\xi(\theta)\|+\|\widehat{v}(\theta)-\widehat{\xi}(\theta)\|\\
&\le M_R(c_{\max}+m)(\Delta_1(\theta)+\Delta_2(\theta)+\widehat{\Delta}_1+\widehat{\Delta}_2)\\
&\times\prod_{0<\theta_k<\theta}\left(1+4M_R\widetilde{m^*}\right)^k e^{(M_R\|p_1+p_2\|)}.\end{aligned}$$

Now, if we put

$$\Xi_{f_i,\aleph_k,\widetilde{\aleph}_k,c_{\Delta_i}} = M_R(c_{\max}+m)e^{(\|p_1+p_2\|)}\prod_{0<\theta_k<\theta}\left(1+4M_R\widetilde{m^*}\right)^k,$$

then we have for all $\theta\in\Theta$

$$\|v(\theta)-\xi(\theta)\|+\|\widehat{v}(\theta)-\widehat{\xi}(\theta)\|\leq \Xi_{f_i,\aleph_k,\widetilde{\aleph}_k,c_{\Delta_i}}\big(\widehat{\Delta}_1+\widehat{\Delta}_2+\Delta_1(\theta)+\Delta_2(\theta)\big),$$

which implies that the system (1) is generalized Ulam–Hyers–Rassias stable with respect to $(\Delta_1,\Delta_2,\widehat{\Delta}_k,\widetilde{\Delta}_k)$. □

3.3. **Data dependance.** Now, we prove the continuous dependence of solutions on initial conditions. For every $(\xi_0,\varphi_0)\in E\times E$, we denote by $(\xi(\cdot,\xi_0),\varphi(\cdot,\varphi_0))$ the solution of (1).

We need the following condition:

$(H5)$ $p_1=q_1$, $p_2=q_2$, and $M_R(l_1^*+l_2^*)=\max(M_{R_1},M_{R_2})(l_1^*+l_2^*)<1$.

Theorem 3.9. *Assume that conditions $(H_1)-(H_5)$ hold, with matrix $\widehat{M}$ defined in (4) converges to zero. Then $(\xi_0,\varphi_0)\to(\xi(\cdot,\xi_0),\varphi(\cdot,\varphi_0))$ is continuous.*

Proof. Let $(\xi_0,\varphi_0),(\hat{x}_0,\hat{\varphi}_0)\in E\times E$. From Theorem 3.3, we see that there exist $(\xi(\cdot,\xi_0),\varphi(\cdot,\varphi_0))$, $\left(\hat{\xi}\left(\cdot,\hat{\xi}_0\right),\hat{\varphi}(\cdot,\hat{\varphi}_0)\right)\in\mathcal{X}\times\mathcal{X}$ such that

$$\begin{aligned}\xi(\theta,\xi_0)&=R_1(\theta)\xi_0+\int_0^\theta R_1(\theta-s)f_1\left(s,\xi(s,\xi_0),\varphi(s,\varphi_0),H_1(\xi(s,\xi_0),\varphi(s,\varphi_0))\right)ds\\&\quad+\sum_{0<\theta_k<\theta}R_1(\theta-\theta_k)I_k(\xi(\theta_k^-,\xi_0),\varphi(\theta_k^-,\varphi_0));\quad\theta\in\Theta,\\\varphi(\theta,\varphi_0)&=R_2(\theta)\varphi_0+\int_0^\theta R_2(\theta-s)f_2\left(s,\xi(s,\xi_0),\varphi(s,\varphi_0),H_2(\xi(s,\xi_0),\varphi(s,\varphi_0))\right)ds\\&\quad+\sum_{0<\theta_k<\theta}R_2(\theta-\theta_k)\bar{I}_k(\xi(\theta_k^-,\xi_0),\varphi(\theta_k^-,\varphi_0));\quad\theta\in\Theta,\end{aligned}$$

and

$$\begin{aligned}\hat{\xi}(\theta,\hat{\xi}_0)&=R_1(\theta)\hat{\xi}_0+\int_0^\theta R_1(\theta-s)f_1\left(s,\hat{\xi}(s,\hat{\xi}_0),\hat{\varphi}(s,\hat{\varphi}_0),H_1(\hat{\xi}(s,\hat{\xi}_0),\hat{\varphi}(s,\hat{\varphi}_0))\right)ds\\&\quad+\sum_{0<\theta_k<\theta}R_1(\theta-\theta_k)I_k(\hat{\xi}(\theta_k^-,\hat{\xi}_0),\hat{\varphi}(\theta_k^-,\hat{\varphi}_0));\quad\theta\in\Theta,\\\hat{\varphi}(\theta,\hat{\varphi}_0)&=R_2(\theta)\hat{\varphi}_0+\int_0^\theta R_2(\theta-s)f_2\left(s,\hat{\xi}(s,\hat{\xi}_0),\hat{\varphi}(s,\hat{\varphi}_0),H_2(\hat{\xi}(s,\hat{\xi}_0),\hat{\varphi}(s,\hat{\varphi}_0))\right)ds\\&\quad+\sum_{0<\theta_k<\theta}R_2(\theta-\theta_k)\bar{I}_k(\hat{\xi}(\theta_k^-,\hat{\xi}_0),\hat{\varphi}(\theta_k^-,\hat{\varphi}_0));\quad\theta\in\Theta.\end{aligned}$$

We put $Q_\xi(\theta)=\xi(\theta,\xi_0)$, then we get

$$\begin{aligned}\|Q_\xi(\theta)-Q_{\hat{\xi}}(\theta)\|\leq M_{R_1}\|\xi_0-\hat{\xi}_0\|+M_{R_1}\int_0^\theta\Big(&p_1(s)\psi_1(\|Q_\xi(s)-Q_{\hat{\xi}}(s)\|)\\&+q_1(s)\phi_1(\|Q_\varphi(s)-Q_{\hat{\varphi}}(s)\|)\Big)ds\end{aligned}$$

$$+ M_{R_1} \sum_{k=0}^{m} \left(m_k^1 \|Q_\xi(\theta_k^-) - Q_{\hat{\xi}}(\theta_k^-)\| + \widetilde{m}_k^1 \|Q_\varphi(\theta_k^-) - Q_{\hat{\varphi}}(\theta_k^-)\| \right).$$

Similarly, we obtain

$$\|Q_\varphi(\theta) - Q_{\hat{\varphi}}(\theta)\| \leq M_{R_2} \|\varphi_0 - \hat{\varphi}_0\| + M_{R_2} \int_0^\theta \Big(p_2(s)\psi_2(\|Q_\xi(s) - Q_{\hat{\xi}}(s)\|) + q_2(s)\phi_2(\|Q_\varphi(s) - Q_{\hat{\varphi}}(s)\|) \Big) ds + M_{R_2} \sum_{k=0}^{m} (m_k^2 \left(\|Q_\xi(\theta_k^-) - Q_{\hat{\xi}}(\theta_k^-)\| + \widetilde{m}_k^2 \|Q_\varphi(\theta_k^-) - Q_{\hat{\varphi}}(\theta_k^-)\| \right).$$

Then,

$$\left\| Q_\xi(\cdot) - Q_{\hat{\xi}}(\cdot) \right\|_{\mathcal{X}} + \|Q_\varphi(\cdot) - Q_{\hat{\varphi}}(\cdot)\|_{\mathcal{X}} \leq \frac{\left\|\xi_0 - \hat{\xi}_0\right\| + \|\varphi_0 - \hat{\varphi}_0\|}{1 - M_R(l_1^* + l_2^*)} + \frac{M_R}{1 - M_R(l_1^* + l_2^*)} \int_0^\theta (p_1 + p_2)(s) \left(\left\| Q_\xi(s) - Q_{\hat{\xi}}(s) \right\| + \|Q_\varphi(s) - Q_{\hat{\varphi}}(s)\| \right) ds.$$

Then, by Lemma 2.15, we get

$$\left\| Q_\xi(\cdot) - Q_{\hat{\xi}}(\cdot) \right\|_{\mathcal{X}} + \|Q_\varphi(\cdot) - Q_{\hat{\varphi}}(\cdot)\|_{\mathcal{X}} \leq \frac{\left\|\xi_0 - \hat{\xi}_0\right\| + \|\varphi_0 - \hat{\varphi}_0\|}{1 - M_R(l_1^* + l_2^*)} \exp\left(\frac{M_R T \|p_1 + p_2\|_{L^1}}{1 - M_R(l_1^* + l_2^*)} \right).$$

Thus

$$\left\| Q_\xi(\cdot) - Q_{\hat{\xi}}(\cdot) \right\|_{\mathcal{X}} + \|Q_\varphi(\cdot) - Q_{\hat{\varphi}}(\cdot)\|_{\mathcal{X}} \to 0, \text{ as } (\xi_0, \varphi_0) \to \left(\hat{\xi}_0, \hat{\varphi}_0 \right).$$

□

4. **An example.** In order to give verification of the existence, data dependance and Ulam-Hyers-Rassias stability of solutions, we consider the following class of partial integro-differential system:

$$\begin{cases} \frac{\partial}{\partial\theta}\zeta_1(\theta,x) = \frac{\partial^2\zeta_1(\theta,x)}{\partial^2 x} + \int_\theta^0 \Gamma(\theta-s)\frac{\partial^2\zeta_1(s,x)}{\partial^2 x}ds + \frac{\eta(\theta+1)\ \mathrm{sect}(\sqrt[7]{\theta^9+1})}{(1+|\zeta_1(\theta,x)|)} \\ \qquad \times\left(\int_0^1 \frac{e^{-st-s}}{113}|\zeta_1(\theta,x)| + |\zeta_2(\theta,x)|ds \right),\ \theta\in I \text{ and } x\in(0,\pi), \\ \frac{\partial}{\partial\theta}\zeta_2(\theta,x) - \frac{\partial^2\zeta_2(\theta,x)}{\partial^2 x} - \int_0^\theta \Gamma(\theta-s)\frac{\partial^2\zeta_2(s,x)}{\partial^2 x}ds \\ \quad = \frac{\tan\left((\theta^2+1)^{-2}\right)\int_0^1 \frac{\cos^2(s+\theta)}{55}|\zeta_1(\theta,x)| + |\zeta_2(\theta,x)|ds}{2-\sin(2t)+\sin(2(\theta+1))},\ \theta\in I,\ x\in(0,\pi), \\ \Delta\zeta_1(\theta_k,x) = \alpha_k\frac{\zeta_1(\theta_k^-,x)}{\sqrt{1+|\zeta_1(\theta_k^-,x)|}},\ \text{for } \theta_k = \frac{1}{2^k}, k=\overline{1:8},\ \textit{and } x\in(0,\pi), \\ \Delta\zeta_2(\theta_k,x) = \beta_k\frac{\zeta_2(\theta_k^-,x)}{\sqrt{1+|\zeta_2(\theta_k^-,x)|}},\ \text{for } \theta_k = \frac{1}{2^k}, k=\overline{1:8},\ \textit{and } x\in(0,\pi), \\ \zeta_1(\theta,0)=\zeta_1(\theta,1)=\zeta_2(\theta,0)=\zeta_2(\theta,1)=0, \quad \text{for } \theta\geq 0, \\ \zeta_1(0,x)=\zeta_1^0(x),\ \zeta_2(0,x)=\zeta_2^0(x), \text{if } \theta\in\Theta,\ \textit{and } x\in(0,\pi), \end{cases} \tag{6}$$

where $I = [0,\pi]$, $\Gamma_i : \mathbb{R}^+ \mapsto \mathbb{R}$ are continuous, $\sigma_1, \sigma_2 \in \mathbb{R}$, $\eta \in \left(0, \pi^{-1}\right)$, $\widehat{\lambda} \in \{0,1\}$, $\alpha_k,\ \beta_k \in (0, e^{-\pi})$. To rewrite system (6) in the abstract form, we introduce the space $X = L^2(0,\pi)$ and we define the operator A_i as follows:

$$\begin{cases} D(A_i) = \{\varpi \in L^2(0,\pi) \ /\ \varpi,\ \varpi'' \in L^2(0,\pi)\ ,\ \varpi(0) = \varpi(\pi) = 0\}, \\ (A_i\varpi)(x) = \frac{\partial^2 \varpi(\theta,x)}{\partial x^2}. \end{cases}$$

It is well known that A_i generates a strongly continuous semigroup $(S_i(\theta))_{\theta\geq 0}$, which are dissipative and compact with $\|S_i(\theta)\| \leq e^{-\varepsilon_i^2\theta}$, and for some $b_i > \frac{1}{\varepsilon_i^2}$, we assume that $\|\Gamma_i(\theta)\| \leq \frac{e^{-\varepsilon_i^2\theta}}{b_i}$, and $\|\Gamma_i'(\theta)\| \leq \frac{e^{-\varepsilon_i^2\theta}}{b_i^2}$. It follows from [16] that $\|R_i(\theta)\| \leq e^{-\varkappa_i\theta}$, where $\varkappa_i = 1 - b_i^{-1}$.

We define also the operators $B_i(\theta) : H \mapsto H$ as follows:

$$B_i(\theta)z = \Gamma_i(\theta)A_i z,\ \ for\ \theta \geq 0,\ z \in D(\mathrm{A_i}).$$

More appropriate conditions on operators B_i, $(H4)$ hold with $M_{R_i} = 1$ and $\beta_i = 1 - b_i^{-1}$.

We put $\zeta(\theta)(x) = \zeta(\theta,x)$, for $\theta \in [0,+\infty)$, and define

$$\begin{aligned} \mathrm{f}_1(\theta,\phi_1,\phi_2,H_1(\phi_1,\phi_2)(x)) &= \frac{\eta(\theta+1)\ \mathrm{sect}(\sqrt[7]{\theta^9+1})}{(1+|\phi_1(\theta,x)|)} \\ &\quad\times\left(\int_0^1 \frac{e^{-st-s}}{113}|\phi_1(\theta,x)| + |\phi_2(\theta,x)|ds\right), \\ \mathrm{f}_2(\theta,\phi_1,\phi_2,H_2(\phi_1,\phi_2)(x)) &= \frac{\tan\left(\left(\theta^2+1\right)^{-2}\right)}{2-\sin(2t)+\sin(2(\theta+1))} \\ &\quad\times\int_0^1 \frac{\cos^2(s+\theta)}{55}|\phi_1(\theta,x)| + |\phi_2(\theta,x)|ds, \\ \aleph_k(x(\theta_k^-),\varphi(\theta_k^-)) &= \alpha_k\frac{\phi_1(\theta_k^-,x)}{\sqrt{1+|\phi_1(\theta_k^-,x)|}}, \\ \widetilde{\aleph}_k(x(\theta_k^-),\varphi(\theta_k^-)) &= \beta_k\frac{\phi_2(\theta_k^-,x)}{\sqrt{1+|\phi_2(\theta_k^-,x)|}}. \end{aligned}$$

Using these definitions we can represent the system (6) in the following abstract form

$$\begin{cases} x'(\theta) = A_1x(\theta) + f_1\left(\theta, x(\theta), \varphi(\theta), H_1(x(\theta),\varphi(\theta))\right) \\ \qquad\qquad + \displaystyle\int_0^\theta B_1(\theta-s)x(s)ds,\ \text{for}\ \theta\in\Theta\backslash\widehat{\Theta}_m, \\ \varphi'(\theta) = A_2\varphi(\theta) + f_2\left(\theta, x(\theta), \varphi(\theta), H_1(x(\theta),\varphi(\theta))\right) \\ \qquad\qquad + \displaystyle\int_0^\theta B_2(\theta-s)x(s)ds,\ \text{for}\ \theta\in\Theta\backslash\widehat{\Theta}_m, \\ \aleph_k(x(\theta_k^-),\varphi(\theta_k^-)) = x(\theta_k^+) - x(\theta_k^-),\ k = 1,\ldots,m, \\ \widetilde{\aleph}_k(x(\theta_k^-),\varphi(\theta_k^-)) = \varphi(\theta_k^+) - \varphi(\theta_k^-),\ k = 1,\ldots,m, \\ (x(0),y(0)) = (\xi_0,\varphi_0). \end{cases}$$

For $\theta \in \Theta$, we have

$$\begin{aligned}&|f_1(\theta,\varkappa_1(\theta),\varkappa_2(\theta),H_1(\varkappa_1(\theta),\varkappa_2(\theta))) - f_1(\theta,\widehat{\varkappa}_1(\theta),\widehat{\varkappa}_2(\theta),H_1(\widehat{\varkappa}_1(\theta),\widehat{\varkappa}_2(\theta)))|\\ &\quad\le \frac{\operatorname{sect}(\sqrt[7]{\theta^9+1})}{113}\left(|\varkappa_1(\theta)-\varkappa_2(\theta)|+|\widehat{\varkappa}_1(\theta)-\widehat{\varkappa}_2(\theta)|\right),\end{aligned}$$

and

$$\begin{aligned}&|f_2(\theta,\varkappa_1(\theta),\varkappa_2(\theta),H_2(\varkappa_1(\theta),\varkappa_2(\theta))) - f_2(\theta,\widehat{\varkappa}_1(\theta),\widehat{\varkappa}_2(\theta),H_2(\widehat{\varkappa}_1(\theta),\widehat{\varkappa}_2(\theta)))|\\ &\quad\le \frac{1}{220}\tan\left(\left(\theta^2+1\right)^{-2}\right)\left(|\varkappa_1(\theta)-\varkappa_2(\theta)|+|\widehat{\varkappa}_1(\theta)-\widehat{\varkappa}_2(\theta)|\right).\end{aligned}$$

We have that $\psi_i(\theta) = \phi_i(\theta) = \theta$, are continuous nondecreasing functions from $(0,+\infty)$ to $(0,+\infty)$, $i = 1,2$. And, we have

$$p_1(\theta) = q_1(\theta) = \frac{\operatorname{sect}(\sqrt[7]{\theta^9+1})}{113} \in L^1(\Theta,\mathbb{R}^+),$$

$$p_2(\theta) = q_2(\theta) = \frac{1}{220}\tan\left(\left(\theta^2+1\right)^{-2}\right) \in L^1(\Theta,\mathbb{R}^+).$$

Now, about $h_1, h_2, \aleph_k$ and $\widetilde{\aleph}_k$, we have

$$\begin{aligned}|h_1(\theta,s,\varkappa_1,\varkappa_2)-h_1(\theta,s,\widehat{\varkappa}_1,\widehat{\varkappa}_2)| &\le \frac{e^{-st-s}}{113}\left(|\varkappa_1(\theta)-\varkappa_2(\theta)|+|\widehat{\varkappa}_1(\theta)-\widehat{\varkappa}_2(\theta)|\right),\\ |h_2(\theta,s,\varkappa_1,\varkappa_2)-h_2(\theta,s,\widehat{\varkappa}_1,\widehat{\varkappa}_2)| &\le \frac{\cos^2(s+\theta)}{55}\left(|\varkappa_1(\theta)-\varkappa_2(\theta)|+|\widehat{\varkappa}_1(\theta)-\widehat{\varkappa}_2(\theta)|\right),\\ |\aleph_k(\varkappa_1,\varkappa_2)-\aleph_k(\widehat{\varkappa}_1,\widehat{\varkappa}_2)| &\le \alpha_k\left(|\varkappa_1(\theta)-\varkappa_2(\theta)|+|\widehat{\varkappa}_1(\theta)-\widehat{\varkappa}_2(\theta)|\right),\\ |\widetilde{\aleph}_k(\varkappa_1,\varkappa_2)-\widetilde{\aleph}_k(\widehat{\varkappa}_1,\widehat{\varkappa}_2)| &\le \beta_k\left(|\varkappa_1(\theta)-\varkappa_2(\theta)|+|\widehat{\varkappa}_1(\theta)-\widehat{\varkappa}_2(\theta)|\right),\end{aligned}$$

$$h_{c_1} = \bar{h}_{c_1} = \frac{1}{113},\ h_{c_2} = \bar{h}_{c_2} = \frac{1}{55},\ m_k^1 = \widetilde{m}_k^1 = \alpha_k,\ m_k^2 = \widetilde{m}_k^2 = \beta_k,$$

$$\|p_1\|_{L^1} = \|q_1\|_{L^1} \simeq 9.7\times 10^{-3},\ \|p_2\|_{L^1} = \|q_2\|_{L^1} \simeq 7\times 10^{-3},$$

$$\left(\|p_1\|_{L^1} + \sum_{k=1}^{8} m_k^1, \|p_2\|_{L^1} + \sum_{k=1}^{8} m_k^2\right) \simeq \left(3.56\times 10^{-1}, 3.53\times 10^{-1}\right),$$

$$(p_1^*, p_2^*) = \left(5.7\times 10^{-3}, 7.1\times 10^{-3}\right),\ M_R(l_1^*+l_2^*) \simeq 0.69 < 1.$$

Then, we get $\rho(\widehat{M}) \simeq 0,709$, hence the matrix $\widehat{M}$ converge to zero.
Also, we get

$$I - \widehat{M} = \begin{pmatrix} 0,644 & -0,356\\ -0,353 & 0,647\end{pmatrix}.$$

Therefore, we can take

$$\delta_1 \ge 2.22337\|\zeta_1^0\|_H + 1.22337\|\zeta_2^0\|_H,$$

and

$$\delta_2 \ge 1.21306\|\zeta_1^0\|_H + 2.21306\|\zeta_2^0\|_H.$$

Also, for all $p \ge 2$, we obtain

$$\sigma_p^1 \le 0,018;\ \sigma_p^2 \le 0,022.$$

Then, for all $p \ge 2$, we have

$$\rho(M_p) \le \|M_p\|_{M_{2\times 2}(\mathbb{R}^+)} < 1.$$

Thus, all conditions of Theorem 3.2, Theorem 3.8 and Theorem 3.9 are verified. Then, the problem (6) has at least one mild solution, which is generalized Ulam-Hyers-Rassias stable.

5. **Conclusions.** In the present study, we explore the existence of mild solutions, Ulam-Hyers-Rassias stability, and continuous dependence of instantaneous impulsive integro-differential systems through the utilization of resolvent operators in the sense of Grimmer. To attain the desired outcomes for the specified problem, we employed a fixed-point approach in conjunction with techniques involving measures of noncompactness and convergence to zero matrices within generalized Banach spaces. Additionally, we present an illustrative example showcasing the practical application of our key findings. Our results make a significant contribution to the literature in this field and are novel in the given configuration. We anticipate that this research could pave the way for diverse avenues of exploration, including but not limited to hybrid systems, problems incorporating infinite delays, and potential extensions to the fractional case. It is our hope that this article will serve as a starting point for future endeavors in these areas.

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