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Existence and Hyers–Ulam stability of stochastic integrodifferential equations with a random impulse

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Abstract

The theoretical approach of random impulsive stochastic integrodifferential equations (RISIDEs) with finite delay, noncompact semigroups, and resolvent operators in Hilbert space is investigated in this article. Initially, a random impulsive stochastic integrodifferential system is proposed and the existence of a mild solution for the system is established using the Mönch fixed-point theorem and contemplating Hausdorff measures of noncompactness. Then, the stability results including a continuous dependence of solutions on initial conditions, exponential stability, and Hyers–Ulam stability for the aforementioned system are investigated. Finally, an example is proposed to validate the obtained results.

Mathematics Subject Classification: 35R12; 60H99; 35B40; 34G20

Keywords: Stochastic differential equations; Random impulse; Noncompact semigroup; Hyers–Ulam stability; Mean-square exponential stability

1 Introduction

The study of impulsive dynamical systems is an emerging area that is attracting interest from both theoretical as well as practical disciplines. Additionally, the impulsive differential equations act as essential models for the investigation of the dynamical processes that are subject to abrupt changes in their states. The study of impulsive systems, especially the impulsive differential, is of great importance because many evolution processes, optimal control models in economics, mechanics, electricity, several fields of engineering, stimulated neural networks, frequency-modulated systems, and some motions of missiles or aircrafts are characterized by the impulsive dynamical behavior. For more information, see [1–3]. In the past several decades, differential equations with impulses have been utilized to model the processes subjected to abrupt changes at discrete moments and the dynamics of impulsive differential equations have attracted the attention of a large number of scholars, see [4–6]. Furthermore, since real-world systems and unpredictable events are almost inevitably affected by stochastic perturbations, mathematical models cannot ignore the stochastic factors due to a combination of uncertainties and complexities. In order to take them into account, differential equations driven by stochastic processes or equations with random impulses offer a natural and practical approach to describe various impulsive phenomena. It is also possible to successfully apply the theory of stochastic differential systems to a variety of nonmathematical issues, such as those in science, economics, epidemiology, mechanics, and finance. For more details, we refer the reader to books [7–9] and the articles therein [10–13].

Many evolution processes involve stochastic functional differential equations with an impulse. This is widely used in modeling systems in medicine and biology, mechanics, economics, telecommunications, and electronics (see [1, 14, 15]). Impulses can occur at random points, for example, the impulse time t_k is a random variable for $k = 1, 2, \dots$ and the impulsive function $\mathfrak{b}_i(.)$ is a random variable. Recently, there have been massive studies covering the existence and stability of solutions of stochastic differential systems and stochastic functional differential systems with impulses or randomness. Hu and Zhu [16] have used the Lyaponov method to investigate the exponential stability of stochastic differential equations with impulse effects at random effects. In addition, Hu and Zhu established stability analysis by considering impulsive stochastic differential systems using the Lyapunov and Razhumikhin technique [17, 18]. Sakthivel and Luo [19] investigated the existence and asymptotical stability of mild solutions containing impulsive stochastic differential systems. Zihan Li et al. [20] established the existence of solutions to the Sturm–Liouville differential equation with random impulses and boundary value problems via Dhage's fixed-point theorem. Yu Guo et al. [21] solved the viscosity solution of the HJB equation for an optimal control system with random impulsive differential equations. Recently, many researchers have discussed the existence and stability of stochastic differential equations with a random impulse, see [22–24]. However, no papers have been published that investigate stochastic differential equations with random impulses involving a resolvent operator. As a result of the above, we investigate the existence, continuous dependence of solutions on initial conditions, Hyers–Ulam stability, and mean-square exponential stability results for the proposed random impulsive stochastic differential equations.

Let us take into consideration the following stochastic differential equations with a random impulse of the form:

$$
\begin{aligned}
&d\big[\vartheta(\mathrm{t})\big] = \left[\mathfrak{A}\vartheta(\mathrm{t}) + \int_0^{\mathrm{t}} \mathfrak{B}(\mathrm{t}-s)\vartheta(s)\,ds + \mathfrak{f}(\mathrm{t},\vartheta_{\mathrm{t}})\right]d\mathrm{t} + \mathfrak{g}(\mathrm{t},\vartheta_{\mathrm{t}})\,d\omega(\mathrm{t}), \quad \mathrm{t}\geq \mathrm{t}_0, \mathrm{t}\neq \varsigma_k,\\
&\vartheta(\varsigma_k) = \mathfrak{b}_k(\delta_k)\vartheta\big(\varsigma_k^-\big), \quad k = 1,2,\dots, \qquad (1.1)\\
&\vartheta_{\mathrm{t}_0} = \eta = \big\{\eta(\theta) \leq \theta < 0\big\},
\end{aligned}
$$

where $\mathfrak{A}$ is the infinitesimal generator of an analytic semigroup $(\mathfrak{R}(\mathrm{t}))_{\mathrm{t}\geq 0}$ of bounded linear operators in a real separable Hilbert space $\mathbb{X}$, $\mathfrak{A}$ is a closed linear operator with dense domain $\mathfrak{D}(\mathfrak{A})$ that is independent of t, $\mathfrak{B}$ is a closed linear operator with domain $\mathfrak{D}(\mathfrak{B}) \supset \mathfrak{D}(\mathfrak{A})$, and $\omega(\mathrm{t})$ is the standard Weiner process on $\mathbb{X}$. The maps $\mathfrak{f} : [\mathrm{t}_0, +\infty] \times \mathscr{X} \to \mathbb{X}$, $\mathfrak{g} : [\mathrm{t}_0, +\infty] \times \mathscr{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ are Borel measurable functions. Let δ_k be a random variable from Ω to $\mathscr{D}_k := (0, \mathfrak{d}_k)$ with $0 < \mathfrak{d}_k < +\infty$ for $k = 1, 2, \dots$, with δ_i, δ_j being independent of each other as $i \neq j$ for $i, j = 1, 2, \dots$. Here, $\mathfrak{b}_k : \mathscr{D}_k \to \mathbb{X}$, ϑ_{t} is an $\mathbb{X}$-valued stochastic process $\ni \vartheta_{\mathrm{t}} \in \mathbb{X}$, $\vartheta_{\mathrm{t}} = \{\vartheta(\mathrm{t}+\theta) : -\delta \leq \theta \leq 0\}$ and $\varsigma_0 = \mathrm{t}_0$ and $\varsigma_k = \varsigma_{k-1} + \delta_k$ for $k = 1, 2, \dots$, where

$\mathrm{t}_0 \in [\delta, +\infty]$ is an arbitrary given nonnegative number. It is obvious that

$$\mathrm{t}_0 = \varsigma_0 < \varsigma_1 < \cdots < \lim_{\mathrm{k}\to\infty} \varsigma_{\mathrm{k}} = +\infty,$$

then, $\{\varsigma_{\mathrm{k}}\}$ is a process with independent increments. Denoting $\vartheta(\varsigma_{\mathrm{k}}^-) := \lim_{\vartheta\to\varsigma_{\mathrm{k}}^-} \vartheta(\mathrm{t})$, the norm

$$\|\vartheta\|_{\mathrm{t}} := \sup_{\mathrm{t}-\delta\le s\le \mathrm{t}} \|\vartheta\|_{\mathbb{X}},$$

with the jump

$$\Delta\vartheta(\varsigma_{\mathrm{k}}) := \bigl[\mathfrak{b}_{\mathrm{k}}(\delta_{\mathrm{k}}) - 1\bigr]\vartheta\bigl(\varsigma_{\mathrm{k}}^-\bigr)$$

represents the random impulsive effect in the state ϑ at time ς_{k}. The initial data $\eta : [-\delta, 0] \to \mathbb{X}$ is a function with respect to ϑ when $\mathrm{t} = \mathrm{t}_0$. We may assume that $\{\mathscr{N}(\mathrm{t}), \mathrm{t} \ge 0\}$ is a simple counting process generated by $\{\varsigma_k\}$, $\Im_{\mathrm{t}}^{(1)}$ is the σ-algebra generated by $\{\mathscr{N}(\mathrm{t}), \mathrm{t} \ge 0\}$, and $\Im_{\mathrm{t}}^{(2)}$ indicates the σ-algebra generated by $\{\omega(\mathrm{t}) : \mathrm{t} \ge 0\}$, where $\Im_{\infty}^{(1)}, \Im_{\infty}^{(2)}$, and ς are mutually independent.

2 Preliminaries and notations

Let $\mathbb{X}$ and $\mathbb{Y}$ be real separable Hilbert spaces with norm $\|\cdot\|$ and $\|\cdot\|_{\mathbb{Y}}$ and $\mathcal{L}(\mathbb{Y}, \mathbb{X})$ denotes the space of bounded linear operators from $\mathbb{Y}$ to $\mathbb{X}$. Let $(\Omega, \mathfrak{F}, \mathcal{P})$ be a complete filtered probability space provided the filtration $\Im_{\mathrm{t}}^{(1)} \vee \Im_{\mathrm{t}}^{(2)}(\mathrm{t} \ge 0)$ satisfies the usual notation. Let $\{\beta_n(\mathrm{t}), \mathrm{t} \ge 0\}$ be a real-valued one-dimensional standard Brownian motion mutually independent over probability space $(\Omega, \mathfrak{F}, \mathcal{P})$. Let $\mathcal{L}^2(\Omega)$ denote the space of square-integrable random variables for the probability measure $\mathcal{P}$. Let $\mathcal{Q} \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$ be a positive trace class operator on $\mathcal{L}^2(\mathbb{X})$ and $(\lambda_n, e_n)_n$ symbolizes its spectral elements. The Weiner process $\omega(\mathrm{t})$ is expressed as follows:

$$\omega(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda_n}\beta_n(\mathrm{t})e_n \quad \text{with } \operatorname{tr}\mathcal{Q} = \sum_{n=1}^{+\infty} \lambda_n < +\infty.$$

Then, the $\mathbb{Y}$-valued stochastic process $\omega(\mathrm{t})$ is called a $\mathcal{Q}$-Weiner process.

Definition 2.1 Letting $\varsigma \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$, we define

$$\|\varsigma\|_{\mathcal{L}_2^0}^2 := \operatorname{tr}\bigl(\varsigma\mathcal{Q}\varsigma^*\bigr) = \left\{\sum_{n=1}^{+\infty} \|\sqrt{\lambda_n}\varsigma e_n\|^2\right\}.$$

If $\|\varsigma\|_{\mathcal{L}_2^0}^2 < +\infty$, then ς is called a $\mathcal{Q}$-Hilbert–Schmidt operator and $\mathcal{L}_2^0$ is the space of all $\mathcal{Q}$-Schmidt operators $\varsigma : \mathbb{Y} \to \mathbb{X}$.

Partial integrodifferential equations Let $\mathfrak{A}$ and $\Upsilon(t)$ be closed linear operators on a Banach space denoted by $\mathbb{X}$, and $\mathbb{Y}$ is the Banach space $\mathscr{D}(\mathfrak{A})$ endowed with the norm

$$|y|_{\mathbb{Y}} := |\mathfrak{A}y| + |y| \quad \text{for } y \in \mathbb{Y}.$$

The notations $\mathscr{C}([0,+\infty);\mathbb{Y})$, $\mathscr{C}^1([0,+\infty);\mathbb{X})$, and $\mathcal{L}(\mathbb{Y},\mathbb{X})$ denote the space of continuous functions from $[0,+\infty)$ into $\mathbb{Y}$, the space of continuously differentiable functions from $[0,+\infty)$ into $\mathbb{X}$ and the set of bounded linear operators from $\mathbb{Y}$ into $\mathbb{X}$, respectively.

Let us consider the problem

$$d\nu(t) = \left(\mathfrak{A}\nu(t) + \int_0^t \Upsilon(t-s)\nu(s)\,ds\right)dt, \quad t \geq 0, \tag{2.1}$$

with $\nu(0) = \nu_0 \in \mathbb{X}$.

Definition 2.2 [25] If the following conditions are met, a bounded linear operator-valued function $\mathscr{R}(t) \in \mathfrak{L}(\mathrm{T}), t \geq 0$ is called a resolvent operator for (2.1):

(i) $\mathfrak{R}(0) = \mathcal{I}$ and $\exists$ two constants $\alpha \geq 1$ and $\delta \ni |\mathfrak{R}(t)| \leq \alpha \exp(\sigma t)\ \forall t \geq 0$.

(ii) For each element x in $\mathbb{X}$, the function $t \mapsto \mathfrak{R}(t)x$ is strongly continuous for each $t \geq 0$ and for x in $\mathbb{Y}$, $\mathfrak{R}(.)x \in \mathscr{C}^1([0,+\infty);\mathbb{X}) \cap \mathscr{C}([0,+\infty);\mathbb{Y})$ and satisfies

$$\begin{aligned} d\mathfrak{R}(t)x &= \left(\mathfrak{A}\mathfrak{R}(t)x + \int_0^t \Upsilon(t-s)\mathfrak{R}(s)x\,ds\right)dt \\ &= \left(\mathfrak{R}(t)\mathfrak{A}x + \int_0^t \mathfrak{R}(t-s)\Upsilon(s)x\,ds\right)dt. \end{aligned}$$

When Definition 2.1(i) holds with $\delta < 0$, the resolvent operator is said to be exponentially stable. The two conditions derived from Grimmer [25] are sufficient to guarantee the existence of solutions for (2.1).

(H1) The operator $\mathfrak{A}$ is an infinitesimal generator of a C_0-semigroup on $\mathbb{X}$.

(H2) $\forall t \geq 0$, $\Upsilon(t)$ denotes a closed continuous linear operator from $\mathscr{D}(\mathfrak{A})$ to $\mathbb{X}$ and $\Upsilon(t) \in \mathcal{L}(\mathbb{Y},\mathbb{X})$. For any $\mathfrak{y} \in \mathbb{Y}$, the map $t \mapsto \Upsilon(t)\mathfrak{y}$ is bounded, differentiable, and its derivative $d\Upsilon(t)\mathfrak{y}/dt$ is bounded and uniformly continuous on $[0,\infty)$.

Now, consider the conditions that ensure the existence of solutions to the deterministic integrodifferential equation:

$$d\nu(t) = \left(\mathfrak{A}\nu(t) + \int_0^t \Upsilon(t-s)\nu(s)\,ds + m(t)\right), \quad t \geq 0, \tag{2.2}$$

with $\nu(0) = \nu_0 \in \mathbb{X}$ and $m : [0,+\infty) \to \mathbb{X}$ is a continuous function.

Lemma 2.1 ([25]) *Suppose the assumptions* $(H1)$ *and* $(H2)$ *hold and if* ν *is a strict solution of* (2.2), *then*

$$\nu(t) = \mathfrak{R}(t)\nu_0 + \int_0^t \mathfrak{R}(t-s)m(s)\,ds, \quad t \geq 0. \tag{2.3}$$

Lemma 2.2 ([25]) *Assuming* $(H1)$, $(H2)$ *holds, the resolvent operator* $\mathfrak{R}(t)$ *is continuous for* $t \geq 0$ *on the operator norm, namely for* $t_0 \geq 0$,

$$\lim_{\tau\to 0}\left\|\mathfrak{R}(t_0-\tau) - \mathfrak{R}(t_0)\right\| = 0.$$

Lemma 2.3 ([25]) *Assume* $(H1)$, $(H2)$ *are satisfied, then* $\exists \mathscr{G} > 0 \ni$

$$\left\|\mathfrak{R}(t+\epsilon) - \mathfrak{R}(\epsilon)\mathfrak{R}(t)\right\| \leq \mathscr{G}\epsilon.$$

Lemma 2.4 ([26]) *If* $\Psi(s)$ *is a* $\mathcal{L}_2^0(\mathbb{Y},\mathbb{X})$*-valued stochastically integrable process in* $[0,\mathtt{T}]$, *then for* $\mathfrak{p}\geq 2$, $\exists\mathscr{C}'_{\mathfrak{p}}=(\mathfrak{p}(\mathfrak{p}-1)/2)^{\mathfrak{p}/2}\ni$, *for every* $\mathsf{t}\geq 0$:

$$\sup_{s\in[0,\mathsf{t}]}\mathbb{E}\left\|\int_0^s\Psi(m)\,d\omega(m)\right\|^{\mathfrak{p}}\leq\left(\frac{\mathfrak{p}(\mathfrak{p}-1)}{2}\right)^{\frac{\mathfrak{p}}{2}}\left(\int_0^{\mathsf{t}}\left(\mathbb{E}\left\|\Psi(s)\right\|_{\mathcal{L}_2^0}^{\mathfrak{p}}\right)^{\frac{2}{\mathfrak{p}}}ds\right)^{\frac{2}{\mathfrak{p}}}.$$

The Hausdorff measure of noncompactness $\alpha(.)$ defined on a bounded subset $\mathscr{E}$ of a Banach space $\mathbb{X}$ is

$$\alpha(\mathscr{E})=\inf\{\varepsilon>0:\mathscr{E}\text{ has a finite }\varepsilon-\text{ net in }\mathbb{X}\}.$$

Lemma 2.5 ([26]) *Let* $\mathbb{X}$ *be a real Banach space and* $\mathscr{M},\mathscr{N}\subset\mathbb{X}$ *be bounded. Then, we have the following properties*:

(1) $\mathscr{M}$ *is precompact if and only if* $\alpha(\mathscr{M})=0$;

(2) $\alpha(\mathscr{M})=\alpha(\overline{\mathscr{M}})=\alpha(\operatorname{conv}\mathscr{M})$, *where* $\overline{\mathscr{M}}$ *and* $\operatorname{conv}\mathscr{M}$ *are the closure and the convex hull of* $\mathscr{M}$, *respectively*;

(3) $\alpha(\mathscr{M})\leq\alpha(\mathscr{N})$ *when* $\mathscr{M}\subset\mathscr{N}$;

(4) $\alpha(\mathscr{M}+\mathscr{N})\leq\alpha(\mathscr{M})+\alpha(\mathscr{N})$, *where* $\mathscr{M}+\mathscr{N}=\{\vartheta+\varpi:\vartheta\in\mathscr{M},\varpi\in\mathscr{N}\}$;

(5) $\alpha(\mathscr{M}\cup\mathscr{N})\leq\max\{\alpha(\mathscr{M}),\alpha(\mathscr{N})\}$;

(6) $\alpha(\lambda\mathscr{M})\leq|\lambda|\alpha(\mathscr{N})$ *for any* $\lambda\in\mathfrak{R}$;

(7) *If* $\mathscr{K}\subset\mathscr{C}([0,\mathtt{T}])$ *is bounded, then*

$$\alpha\bigl(\mathscr{K}(\mathsf{t})\bigr)\leq\alpha(\mathscr{K})\quad\forall\mathsf{t}\in[0,\mathtt{T}],$$

where $\mathscr{K}(\mathsf{t})=\{m(\mathsf{t}):m\in\mathscr{K}\subset\mathbb{X}\}$. *Further, if* $\mathscr{K}$ *is equicontinuous on* $[0,\mathtt{T}]$, *then* $\mathsf{t}\to\mathscr{K}(\mathsf{t})$ *is continuous on* $[0,\mathtt{T}]$, *and* $\alpha(\mathscr{K})=\sup\{\mathscr{K}(\mathsf{t}):\mathsf{t}\in[0,\mathtt{T}]\}$;

(8) *If* $\mathscr{K}\subset\mathscr{C}([0,\mathtt{T}],\mathbb{X})$ *is bounded and equicontinuous, then* $\mathsf{t}\to\alpha(\mathscr{K}(\mathsf{t}))$ *is continuous on* $[0,\mathtt{T}]$ *and* $\alpha(\int_0^{\mathsf{t}}\mathscr{K}(s)\,ds)\leq\int_0^{\mathsf{t}}\alpha(\mathscr{K}(s))\,ds$ $\forall\mathsf{t}\in[0,\mathtt{T}]$, *where* $\int_0^{\mathsf{t}}\mathscr{K}(s)\,ds=\{\int_0^{\mathsf{t}}m(s)\,ds:m\in\mathscr{K}\}$;

(9) *Let* $\{m_n\}_{n=1}^{\infty}$ *be a sequence of Bochner integrable functions from* $[0,\mathtt{T}]$ *to* $\mathbb{X}$ *with* $\|m_n(\mathsf{t})\|\leq\hat{u}(\mathsf{t})$ *for almost all* $\mathsf{t}\in[0,\mathtt{T}]$ *and* $n\geq 1$, *where* $\hat{u}(\mathsf{t})\in\mathcal{L}([0,\mathtt{T}],\mathscr{R}^+)$, *then* $\Psi(\mathsf{t})=\alpha(\{m_n(\mathsf{t})\}_{n=1})\in\mathcal{L}([0,\mathtt{T}],\mathscr{R}^+)$ *and satisfies*

$$\alpha\left(\left\{\int_0^{\mathsf{t}}m_n(s)\,ds:n\geq 1\right\}\right)\leq 2\int_0^{\mathsf{t}}\Psi(s)\,ds.$$

Lemma 2.6 ([26]) *If* $\mathscr{K}\subset\mathscr{C}([0,\mathtt{T}],\mathcal{L}_2^0(\mathbb{Y},\mathbb{X}))$ *and* ω *is a Weiner process,*

$$\alpha\left(\int_0^{\mathsf{t}}\mathscr{K}(s)\,d\omega(s)\right)\leq\sqrt{\mathtt{T}}\alpha\bigl(\mathscr{K}(\mathsf{t})\bigr),$$

where,

$$\int_0^{\mathsf{t}}\mathscr{K}(s)\,d\omega(s)=\left\{\int_0^{\mathsf{t}}m(s)\,d\omega(s):\ \forall m\in\mathscr{K},\mathsf{t}\in[0,\mathtt{T}]\right\}.$$

Lemma 2.7 ([26]) *Let $\mathbb{D}$ be a closed convex subset of $\mathbb{X}$ with $0 \in \mathbb{D}$. Suppose $\Psi : \mathbb{D} \to \mathbb{D}$ is a continuous map of Mönch type that satisfies*:

$$\mathcal{M} \subset \mathbb{D} \textit{ countable and } \mathcal{M} \subset \overline{co}\big(\{0\} \cup \Psi(\mathcal{M})\big) \textit{ implies that } \mathcal{M} \textit{ is relatively compact,}$$

then, Ψ has a fixed point in $\mathbb{D}$.

3 Existence results

Definition 3.1 For a given $\mathtt{T} \in (\mathrm{t}_0, +\infty)$, an $\mathbb{X}$-valued stochastic process $\{\vartheta(\mathrm{t}), \mathrm{t} \in [\mathrm{t}_0, \mathtt{T}]\}$ is said to be a mild solution of (1.1) provided:

(i) $\vartheta(\mathrm{t})$ is an $\Im_{\mathrm{t}}$-adapted process for $\mathrm{t} \geq \mathrm{t}_0$;
(ii) $\vartheta(\mathrm{t}) \in \mathbb{X}$ has a cadlag path on $\mathrm{t} \in [\mathrm{t}_0, \mathtt{T}]$ almost surely,
(iii) $\vartheta(\mathrm{t}) = \eta$ if $\mathrm{t} \in [-\delta, 0]$ and for each $\mathrm{t} \in [\mathrm{t}_0, \mathtt{T}]$, we have

$$\begin{aligned}\vartheta(\mathrm{t}) = \sum_{\mathrm{k}=0}^{+\infty}\Bigg[&\prod_{i=1}^{\mathrm{k}} \mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\eta(0) + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta_s)\,ds \\ &+ \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta_s)\,ds + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s) \\ &+ \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}} \mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t}),\end{aligned}$$

where $\prod_{j=i}^{\mathrm{k}}(.) = 1$ as $i > \mathrm{k}$, $\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j) = \mathfrak{b}_k(\delta_k)\mathfrak{b}_{\mathrm{k}-1}(\delta_{\mathrm{k}-1})\cdots\mathfrak{b}_i(\delta_i)$, $\mathcal{I}_{\mathfrak{A}}(.)$ is the indicator function expressed as,

$$\mathcal{I}_{\mathfrak{A}}(\mathrm{t}) = \begin{cases} 1 & \text{if } \mathrm{t} \in \mathfrak{A}, \\ 0 & \text{if } \mathrm{t} \notin \mathfrak{A}. \end{cases}$$

We may take into consideration the following hypotheses:

(A1) The map $\mathfrak{f} : [\mathrm{t}_0, \mathtt{T}] \times \mathbb{X} \to \mathbb{X}$ satisfies
  (i) $\mathfrak{f}(.,\vartheta) : [\mathrm{t}_0,, \mathtt{T}] \to \mathbb{X}$ is measurable for each $\vartheta \in \mathbb{X}$ and $\mathfrak{f}(\mathrm{t},.) : \mathbb{X} \to \mathbb{X}$ is continuous for each $\mathrm{t} \in [\mathrm{t}_0, \mathtt{T}]$.
  (ii) There occurs a continuous function $\nu_{\mathfrak{f}}(\mathrm{t}) : [\mathrm{t}_0, \mathtt{T}] \to \mathscr{R}^+$ and a continuous nondecreasing function $\Gamma_{\mathfrak{f}} : \mathscr{R}^+ \to \mathscr{R}^+$ and $\|\vartheta\|^2 \leq \mathfrak{r} \ni$

$$\big\|\mathfrak{f}(\mathrm{t},\vartheta)\big\|^2 \leq \nu_{\mathfrak{f}}(\mathrm{t})\Gamma_{\mathfrak{f}}\big(\|\vartheta\|^2\big) \leq \nu_{\mathfrak{f}}(\mathrm{t})\Gamma_{\mathfrak{f}}(\mathfrak{r}).$$

  (iii) $\exists$ a positive function $\mathscr{C}_{\mathfrak{f}}(\mathrm{t}) \in \mathcal{L}^1([\mathrm{t}_0, \mathtt{T}]), \mathscr{R}^+ \ni$ for any bounded subsets $\beta_1 \subset \mathbb{X}$, we have

$$\alpha\big(\mathfrak{f}(\mathrm{t},\vartheta)\big) \leq \mathscr{C}_{\mathfrak{f}}(\mathrm{t}) \sup_{\theta\in(-\delta,0]} \alpha\big(\beta_1(\theta)\big).$$

(A2) The function $\mathfrak{g} : [\mathrm{t}_0, \mathtt{T}] \times \mathbb{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ satisfies
  (i) $\mathfrak{g}(.,\vartheta) : [\mathrm{t}_0,, \mathtt{T}] \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ is measurable for each $\vartheta \in \mathbb{X}$ and $\mathfrak{f}(\mathrm{t},.) : \mathbb{X} \to \mathcal{L}_2^0(\mathbb{Y}, \mathbb{X})$ is continuous for each $\mathrm{t} \in [\mathrm{t}_0, \mathtt{T}]$.

(ii) There occurs a continuous function $\nu_{\mathfrak{g}}(\mathrm{t}) : [\mathrm{t}_0, \mathrm{T}] \to \mathscr{R}^+$ and a continuous nondecreasing function $\Gamma_{\mathfrak{g}} : \mathscr{R}^+ \to \mathscr{R}^+$ and $\|\vartheta\|^2 \le \mathfrak{r} \ni$

$$\|\mathfrak{g}(\mathrm{t}, \vartheta)\|^2 \le \nu_{\mathfrak{g}}(\mathrm{t})\Gamma_{\mathfrak{g}}(\|\vartheta\|^2) \le \nu_{\mathfrak{g}}(\mathrm{t})\Gamma_{\mathfrak{g}}(\mathfrak{r}).$$

(iii) $\exists$ a positive function $\mathscr{C}_{\mathfrak{g}}(\mathrm{t}) \in \mathcal{L}^1([\mathrm{t}_0, \mathrm{T}]), \mathscr{R}^+ \ni$ for any bounded subsets $\beta_2 \subset \mathbb{X}$, we have

$$\alpha(\mathfrak{g}(\mathrm{t}, \vartheta)) \le \mathscr{C}_{\mathfrak{g}}(\mathrm{t}) \sup_{\theta \in (-\delta, 0]} \alpha(\beta_2(\theta)).$$

(A3) $\mathbb{E}[\max_{i,k}\{\prod_{j=i}^{k} \|\mathfrak{b}_j(\delta_j)\|\}] < +\infty \ \exists \mathscr{B} > 0 \ni$

$$\mathbb{E}\left(\max_{i,k}\left\{\prod_{j=i}^{k}\|\mathfrak{b}_j(\delta_j)\|\right\}\right) \le \mathscr{B} \quad \text{for all } \delta_j \in \mathscr{D}_j, j \in \mathbb{N}.$$

(A4) $3\max\{1, \mathscr{B}^2\}(\mathrm{T} - \mathrm{t}_0)\mathscr{H}^2[\lim_{\mathfrak{r}\to+\infty} \frac{\Gamma_{\mathfrak{f}}(\mathfrak{r})}{\mathfrak{r}} \int_{\mathrm{t}_0}^{\mathrm{t}} \nu_{\mathfrak{f}}(s)\, ds + \lim_{\mathfrak{r}\to+\infty} \frac{\Gamma_{\mathfrak{g}}(\mathfrak{r})}{\mathfrak{r}} \int_{\mathrm{t}_0}^{\mathrm{t}} \nu_{\mathfrak{g}}(s)\, ds] \le 1.$

Theorem 3.1 *Assume the conditions* (*A*1)–(*A*4) *hold, then there exists at least one mild solution for* (1.1) *provided*:

$$\begin{aligned} &\max\{1, \mathscr{B}^2\}\mathscr{H}^2(\mathrm{T} - \mathrm{t}_0)\|\mathscr{C}_{\mathfrak{f}}\|_{\mathcal{L}^1([\mathrm{t}_0,\mathrm{T}],\mathscr{R}^+)} \\ &\quad + \max\{1, \mathscr{B}^2\}\mathscr{H}^2(\mathrm{T} - \mathrm{t}_0)^{\frac{1}{2}}\|\mathscr{C}_{\mathfrak{g}}\|_{\mathcal{L}^2([\mathrm{t}_0,\mathrm{T}],\mathscr{R}^+)} < 1. \end{aligned} \tag{3.1}$$

Proof Let us introduce the set $\Upsilon_{\mathrm{T}} : PC([\mathrm{t}_0 - \delta, \mathrm{T}], \mathcal{L}^2(\Omega, \mathbb{X}))$ equipped with the norm

$$\|\vartheta\|^2_{\Upsilon_{\mathrm{T}}} = \sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \mathbb{E}\|\vartheta\|^2_{\mathrm{t}} = \sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \mathbb{E}\Big(\sup_{\mathrm{t}-\delta\le s\le \mathrm{t}} \|\vartheta(s)\|^2\Big).$$

It is obvious that Υ_{T} is a Banach space and we may define

$$\overline{\Upsilon}_{\mathrm{T}} = \{\vartheta \in \Upsilon_{\mathrm{T}} : \vartheta(s) = \eta(s), \text{ for } s \in [-\delta, 0]\},$$

with the norm $\|\vartheta\|^2_{\Upsilon_{\mathrm{T}}}$. Thus, (1.1) can be transformed into a fixed-point problem. We may define an operator $\Theta : \overline{\Upsilon}_{\mathrm{T}} \to \overline{\Upsilon}_{\mathrm{T}}$ by

$$\begin{aligned} (\Theta\vartheta)(\mathrm{t}) = \sum_{\mathrm{k}=0}^{+\infty}\Bigg[&\prod_{i=1}^{\mathrm{k}} \mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t} - \mathrm{t}_0)\eta(0) + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j) \int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}} \mathfrak{R}(\mathrm{t} - s)\mathfrak{f}(s, \vartheta_s))\, ds \\ &+ \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}} \mathfrak{R}(\mathrm{t} - s)\mathfrak{f}(s, \vartheta_s))\, ds + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}} \mathfrak{b}_j(\delta_j) \int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_k} \mathfrak{R}(\mathrm{t} - s)\mathfrak{g}(s, \vartheta_s)\, d\omega(s) \\ &+ \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}} \mathfrak{R}(\mathrm{t} - s)\mathfrak{g}(s, \vartheta_s)\, d\omega(s)\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})(\mathrm{t})}, \quad \mathrm{t} \in [\mathrm{t}_0, \mathrm{T}] \end{aligned}$$

and

$$(\Theta\vartheta) = \eta(\theta), \quad \mathrm{t} \in [-\delta, 0].$$

Let us divide our proof into several steps.

Step 1: Initially, we have to compute that Θ satisfies the property $\mathcal{N}(\mathbb{B}_{\mathfrak{r}}) \subset \mathbb{B}_{\mathfrak{r}}$, $\mathbb{B}_{\mathfrak{r}} = \{\vartheta \in \Upsilon_{\mathrm{T}} : \|\vartheta\|^2_{\Upsilon_{\mathrm{T}}} \leq \mathfrak{r}\}$. If the result contradicts, for $\vartheta \in \mathbb{B}_{\mathfrak{r}}$, $\mathcal{N}(\mathbb{B}_{\mathfrak{r}}) \not\subseteq \mathbb{B}_{\mathfrak{r}}$. Thus, we may find $\mathrm{t} \in [\mathrm{t}_0, \mathrm{T}]$ satisfying $\mathbb{E}\|(\Theta\vartheta)(\mathrm{t})\|^2 > \mathfrak{r}$. By the aforementioned assumptions,

$$\begin{aligned}
\mathbb{E}\big\|(\Theta\vartheta)(\mathrm{t})\big\|^2 &= \mathbb{E}\Bigg[\sum_{\mathrm{k}=0}^{+\infty}\Bigg[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\eta(0) + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta_s)\,ds \\
&\quad + \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta_s)\,ds + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s) \\
&\quad + \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\Bigg]\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})(\mathrm{t})}, \\
&\leq 3\mathbb{E}\Bigg(\Bigg(\max_{\mathrm{k}}\Bigg\{\prod_{i=1}^{\mathrm{k}}\big\|\mathfrak{b}_i(\delta_i)\big\|\Bigg\}\Bigg)^2\Bigg)\big\|\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\big\|^2\mathbb{E}\big\|\eta(0)\big\|^2 \\
&\quad + 3\mathbb{E}\Bigg(\max_{i,\mathrm{k}}\Bigg\{\prod_{j=i}^{\mathrm{k}}\big\|\mathfrak{b}_j(\delta_j)\big\|, 1\Bigg\}\Bigg)^2 \\
&\quad \times \mathbb{E}\big(\big\|\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\vartheta_s)\,ds\big\|^2\big) + 3\mathbb{E}\Bigg(\Bigg(\max_{i,\mathrm{k}}\Bigg\{\prod_{j=i}^{\mathrm{k}}\big\|\mathfrak{b}_j(\delta_j)\big\|, 1\Bigg\}\Bigg)^2\Bigg) \\
&\quad \times \mathbb{E}\big(\big\|\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\big\|^2\big) \\
&\leq 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\big\|\eta(0)\big\|^2 \\
&\quad + 3\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{f}}(s)\Gamma_{\mathfrak{f}}(\mathfrak{r})\,ds \\
&\quad + 3\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{g}}(s)\Gamma_{\mathfrak{g}}(\mathfrak{r})\,ds.
\end{aligned}$$

Dividing the above inequality by $\mathfrak{r}$, and letting $\mathfrak{r} \to +\infty$, we have

$$3\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathrm{t}_0)\Bigg(\lim_{\mathfrak{r}\to+\infty}\frac{\Gamma_{\mathfrak{f}}(\mathfrak{r})}{\mathfrak{r}}\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{f}}(s)\,ds + \lim_{\mathfrak{r}\to+\infty}\frac{\Gamma_{\mathfrak{g}}(\mathfrak{r})}{\mathfrak{r}}\int_{\mathrm{t}_0}^{\mathrm{t}}\nu_{\mathfrak{g}}(s)\,ds\Bigg) > 1,$$

which contradicts our assumption (A4). Thus, $\exists$ some $\vartheta \in \mathbb{B}_{\mathfrak{r}} \ni \mathcal{N}(\mathbb{B}_{\mathfrak{r}}) \subset \mathbb{B}_{\mathfrak{r}}$.

Step 2: In order to compute the continuity of the operator Θ in $\mathbb{B}_{\mathfrak{r}}$, let $\vartheta, \vartheta_n \in \mathbb{B}_{\mathfrak{r}}$ and $\vartheta_n \to \vartheta$ as $n \to +\infty$. By condition (ii) of (A1) and (A2), we have

$$\begin{aligned}
&\mathfrak{f}(\mathrm{t},\vartheta_n) \to \mathfrak{f}(\mathrm{t},\vartheta), \quad n \to +\infty, \qquad \big\|\mathfrak{f}(\mathrm{t},\vartheta_n) - \mathfrak{f}(\mathrm{t},\vartheta)\big\|^2 \leq 2\nu_{\mathfrak{f}}(\mathrm{t})\Gamma_{\mathfrak{f}}(\mathrm{t}), \\
&\mathfrak{g}(\mathrm{t},\vartheta_n) \to \mathfrak{g}(\mathrm{t},\vartheta), \quad n \to +\infty, \qquad \big\|\mathfrak{g}(\mathrm{t},\vartheta_n) - \mathfrak{g}(\mathrm{t},\vartheta)\big\|^2 \leq 2\nu_{\mathfrak{g}}(\mathrm{t})\Gamma_{\mathfrak{g}}(\mathrm{t}).
\end{aligned}$$

Using the Dominated Convergence theorem and (A3), we may deduce that

$$\begin{aligned}
&\mathbb{E}\big\|(\Theta\vartheta_n)(\mathrm{t}) - (\Theta\vartheta)(\mathrm{t})\big\|^2 \\
&\quad \leq 3\mathbb{E}\Bigg\|\sum_{\mathrm{k}=0}^{+\infty}\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\big(\vartheta_n(0) - \vartheta(0)\big)\Bigg\|^2
\end{aligned}$$

$$
\begin{aligned}
&+3\mathbb{E}\Bigg\|\sum_{\mathrm{k}=0}^{+\infty}\Bigg(\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\big[\mathfrak{f}\big(s,(\vartheta_s)_n\big)-\mathfrak{f}(s,\vartheta_s)\big]\,ds\\
&+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\big[\mathfrak{f}\big(s,(\vartheta_s)_n\big)-\mathfrak{f}(s,\vartheta_s)\big]\,ds\Bigg)\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}\Bigg\|^2\\
&+3\mathbb{E}\Bigg\|\sum_{\mathrm{k}=0}^{+\infty}\Bigg(\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\big[\mathfrak{g}\big(s,(\vartheta_s)_n\big)-\mathfrak{g}(s,\vartheta_s)\big]\,d\omega(s)\\
&+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\big[\mathfrak{g}\big(s,(\vartheta_s)_n\big)-\mathfrak{g}(s,\vartheta_s)\big]\,d\omega(s)\Bigg)\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}\Bigg\|^2\\
\le{}&3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\big\|\vartheta_n(0)-\vartheta(0)\big\|^2\\
&+3\max\big\{1,\mathscr{B}^2\big\}\mathscr{H}^2(\mathrm{t}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\mathbb{E}\big\|\mathfrak{f}\big(s,(\vartheta_s)_n\big)-\mathfrak{f}(s,\vartheta_s)\big\|^2\,ds\\
&+3\max\big\{1,\mathscr{B}^2\big\}\mathscr{H}^2(\mathrm{t}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\mathbb{E}\big\|\mathfrak{g}\big(s,(\vartheta_s)_n\big)-\mathfrak{g}(s,\vartheta_s)\big\|_{\mathcal{L}_2^0}^2\,ds\\
\to{}&0\quad\text{as } n\to+\infty.
\end{aligned}
$$

Therefore, Θ is continuous on $\mathbb{B}_{\mathfrak{r}}$.

Step 3: To prove Θ is equicontinuous on $[\mathrm{t}_0,\mathtt{T}]$, for $\mathrm{t}_0<\mathrm{t}_1<\mathrm{t}_2<\mathtt{T}$ and $\vartheta\in\mathbb{B}_{\mathfrak{r}}$, we have

$$
\begin{aligned}
&(\Theta\vartheta)(\mathrm{t}_2)-(\Theta\vartheta)(\mathrm{t}_1)\\
&=\sum_{\mathrm{k}=0}^{+\infty}\Bigg[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_0)\eta(0)+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{f}(s,\vartheta_s)\,ds\\
&\quad+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{f}(s,\vartheta_s)\,ds+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\\
&\quad+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t}_2)\\
&\quad-\sum_{\mathrm{k}=0}^{+\infty}[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}_1-\mathrm{t}_0)\eta(0)+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}_1-s)\mathfrak{f}(s,\vartheta_s)\,ds\\
&\quad+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_1}\mathfrak{R}(\mathrm{t}_1-s)\mathfrak{f}(s,\vartheta_s)\,ds+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}_1-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\\
&\quad+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_1}\mathfrak{R}(\mathrm{t}_1-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t}_1)\\
&=\sum_{\mathrm{k}=0}^{+\infty}\Bigg[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_0)\eta(0)+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{f}(s,\vartheta_s)\,ds\\
&\quad+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{f}(s,\vartheta_s)\,ds+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\Bigg]\big(\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t}_2)-\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t}_1)\big)\\
&+\sum_{\mathrm{k}=0}^{+\infty}\Bigg[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\big(\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_0)-\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_1)\big)\eta(0)\\
&+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\big(\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\big)\\
&\times\mathfrak{f}(s,\vartheta_s)\,ds+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\big(\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\big)\mathfrak{f}(s,\vartheta_s)\,ds+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\\
&\times\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\big(\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\big)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\big(\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\big)\\
&\times\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})(\mathrm{t}_1)}\\
=\;&2\mathbb{E}\|\mathscr{J}_1\|^2+2\mathbb{E}\|\mathscr{J}_2\|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{E}\|\mathscr{J}_1\|^2=\mathbb{E}\Bigg\|&\sum_{\mathrm{k}=0}^{+\infty}\Bigg[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_0)\eta(0)+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{f}(s,\vartheta_s)\,ds\\
&+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{f}(s,\vartheta_s)\,ds+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\\
&+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\mathfrak{R}(\mathrm{t}_2-s)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\Bigg]\big(\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t}_2)-\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t}_1)\big)\Bigg\|^2,\\
\mathbb{E}\|\mathscr{J}_2\|^2=\mathbb{E}\Bigg\|&\sum_{\mathrm{k}=0}^{+\infty}\Bigg[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\big(\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_0)-\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_1)\big)\eta(0)\\
&+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\big(\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\big)\\
&\times\mathfrak{f}(s,\vartheta_s)\,ds+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\big(\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\big)\mathfrak{f}(s,\vartheta_s)\,ds+\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\\
&\times\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\big(\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\big)\mathfrak{g}(s,\vartheta_s)\,d\omega(s)+\int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}_2}\big(\mathfrak{R}(\mathrm{t}_2-s)-\mathfrak{R}(\mathrm{t}_1-s)\big)\\
&\times\mathfrak{g}(s,\vartheta_s)\,d\omega(s)\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})(\mathrm{t}_1)}\Bigg\|^2.
\end{aligned}
$$

By treating each term separately,

$$
\mathbb{E}\|\mathscr{J}_1\|^2\leq 3\mathbb{E}\left(\max_{\mathrm{k}}\left\{\prod_{i=1}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|^2\right\}\right)\|\mathfrak{R}(\mathrm{t}_2-\mathrm{t}_0)\|^2
$$

$$
\begin{aligned}
&\times \mathbb{E}\left\|\eta(0)\right\|^2\left(\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t_2)-\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t_1)\right)^2 \\
&+3\mathbb{E}\left(\max_{i,k}\left\{\prod_{j=i}^{k}\left\|\mathfrak{b}_i(\delta_i)\right\|,1\right\}\right)^2\mathbb{E}\left(\sum_{k=0}^{+\infty}\int_{t_0}^{t_2}\left\|\Re(t_2-s)\right\|^2\left\|\mathfrak{f}(s,\vartheta_s)\right\|^2ds\right) \\
&\times\left(\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t_2)-\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t_1)\right)^2+3\mathbb{E}\left(\max_{i,k}\left\{\prod_{j=i}^{k}\left\|\mathfrak{b}_i(\delta_i)\right\|,1\right\}\right)^2 \\
&\times\mathbb{E}\left(\sum_{k=0}^{+\infty}\int_{t_0}^{t_2}\left\|\Re(t_2-s)\right\|^2\left\|\mathfrak{g}(s,\vartheta_s)\right\|^2d\omega(s)\right)\left(\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t_2)-\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t_1)\right)^2 \\
\to{}& 0\quad\text{as } t_2\to t_1.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathbb{E}\|\mathscr{J}_2\|^2 \le{}& 5\mathscr{B}^2\left\|\Re(t_2-t_0)-\Re(t_1-t_0)\right\|^2\mathbb{E}\left\|\eta(0)\right\|^2+5\max\left\{1,\mathscr{B}^2\right\}(t_1-t_0) \\
&\times\int_{t_0}^{t_1}\left\|\Re(t_2-s)-\Re(t_1-s)\right\|^2\mathbb{E}\left\|\mathfrak{f}(s,\vartheta_s)\right\|^2ds+5(t_2-t_1)\int_{t_1}^{t_2}\left\|\Re(t_2-s)\right\|^2 \\
&\times\mathbb{E}\left\|\mathfrak{f}(s,\vartheta_s)\right\|^2ds+5\max\left\{1,\mathscr{B}^2\right\}(t_1-t_0)\int_{t_0}^{t_1}\left\|\Re(t_2-s)-\Re(t_1-s)\right\|^2 \\
&\times\mathbb{E}\left\|\mathfrak{g}(s,\vartheta_s)\right\|^2ds+5(t_2-t_1)\int_{t_1}^{t_2}\left\|\Re(t_2-s)\right\|^2\mathbb{E}\left\|\mathfrak{g}(s,\vartheta_s)\right\|^2ds \\
\to{}& 0\quad\text{as } t_2\to t_1.
\end{aligned}
$$

Thus, we have

$$
\mathbb{E}\left\|(\Theta\vartheta)(t_2)-(\Theta\vartheta)(t_1)\right\|^2\to 0\quad\text{as } t_2\to t_1,
$$

which implies Θ is equicontinuous on $[t_0,T]$.

Step 4: Now, to compute the Mönch condition, let $\gamma\subset\Upsilon_T$ be a nonempty set and $\vartheta_1,\vartheta_2\in\gamma$, by probability 1, we have

$$
d\left(\Theta\vartheta_1(t),\Theta\vartheta_2(t)\right)=d\left(\overline{\Theta}\vartheta_1(t),\overline{\Theta}\vartheta_2(t)\right),
$$

where

$$
\begin{aligned}
&(\overline{\Theta}\vartheta)(t) \\
&\quad=\max\{1,\mathscr{B}\}\sum_{k=0}^{+\infty}\left[\int_{\varsigma_{k-1}}^{\varsigma_k}\Re(t-s)\mathfrak{f}(s,\vartheta_s)\,ds+\int_{\varsigma_k}^{t}\Re(t-s)\mathfrak{f}(s,\vartheta_s)\,ds\right]\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t) \\
&\qquad+\max\{1,\mathscr{B}\}\sum_{k=0}^{+\infty}\left[\int_{\varsigma_{k-1}}^{\varsigma_k}\Re(t-s)\mathfrak{g}(s,\vartheta_s)\,ds+\int_{\varsigma_k}^{t}\Re(t-s)\mathfrak{g}(s,\vartheta_s)\,ds\right]\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t) \\
&\quad=\overline{\Theta}_1+\overline{\Theta}_2.
\end{aligned}
$$

By a similar procedure to that used in Lemma 2.3,

$$\alpha\big((\Theta\vartheta)(\mathrm{t})\big) = \alpha\big((\overline{\Theta})(\mathrm{t})\big).$$

Let $\Delta \subset \mathbb{B}_{\mathfrak{r}}$ be countable and $\Delta \subset \overline{co}(\{0\} \cup \Theta(\Delta))$. By proving $\alpha(\Delta) = 0$ the Mönch condition is verified. Set $\Delta = \{\vartheta^n\}_{n=1}^{\infty}$, then it is well defined that $\Delta \subset \overline{co}(\{0\} \cup \Theta(\Delta))$ is equicontinuous on $[\mathrm{t}_0, \mathrm{T}]$ by step 3.

By Lemma 2.2 and Lemma 2.3,

$$\begin{aligned}\alpha\big(\big\{\overline{\Theta}_1\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big) &\leq \max\{1,\mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}} \mathscr{C}_{\mathfrak{f}}(\mathrm{t}) \sup_{\theta\in(-\delta,0]} \alpha\big(\big\{\vartheta^n\big(\theta-\mu(\theta)\big)\big\}_{n=1}^{\infty}\big)\,ds \\ &\leq \max\{1,\mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathrm{t}_0)\|\mathscr{C}_{\mathfrak{f}}\|_{\mathcal{L}^1([\mathrm{t}_0,\mathrm{T}],\mathfrak{R}^+)} \sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \alpha\big(\big\{\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big), \\ \alpha\big(\big\{\overline{\Theta}_2\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big) &\leq \max\{1,\mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathrm{t}_0)^{\frac{1}{2}}\|\mathscr{C}_{\mathfrak{g}}\|_{\mathcal{L}^2([\mathrm{t}_0,\mathrm{T}],\mathfrak{R}^+)} \sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \alpha\big(\big\{\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big).\end{aligned}$$

By using Lemma 2.3,

$$\begin{aligned}\alpha\big(\big\{\Theta_1\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big) &= \alpha\big(\big\{\overline{\Theta}_1\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big) \\ &\leq \alpha\big(\big\{\overline{\Theta}_1\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big) + \alpha\big(\big\{\overline{\Theta}_2\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big) \\ &\leq \big[\max\{1,\mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathrm{t}_0)\|\mathscr{C}_{\mathfrak{f}}\|_{\mathcal{L}^1([\mathrm{t}_0,\mathrm{T}],\mathfrak{R}^+)} + \max\{1,\mathscr{B}\}\mathscr{H}(\mathrm{T}-\mathrm{t}_0)^{\frac{1}{2}} \\ &\quad \times \|\mathscr{C}_{\mathfrak{g}}\|_{\mathcal{L}^2([\mathrm{t}_0,\mathrm{T}],\mathfrak{R}^+)}\big]\alpha\big(\big\{\vartheta^n(\mathrm{t})\big\}_{n=1}^{\infty}\big).\end{aligned}$$

It follows that

$$\alpha(\Delta) \leq \alpha\big(\overline{co}\big(\{0\} \cup \Theta(\Delta)\big)\big) = \alpha\big(\Theta(\Delta)\big) \leq \alpha(\Delta),$$

implying $\alpha(\Delta) = 0$ and then Δ is a relatively compact set. Thus, Θ has a fixed point in Δ that is the mild solution of (1.1). This completes the proof. □

4 Stability

4.1 Continuous dependence of solutions on initial conditions

(A5) $\exists$ constants $\mathscr{C}_1, \mathscr{C}_2 \ni$

$$\big\|\mathfrak{f}(\mathrm{t},\vartheta) - \mathfrak{f}(\mathrm{t},\varpi)\big\| \leq \mathscr{C}_1\|\vartheta - \varpi\|, \qquad \big\|\mathfrak{g}(\mathrm{t},\vartheta) - \mathfrak{g}(\mathrm{t},\varpi)\big\|_{\mathcal{L}_2^0} \leq \mathscr{C}_2\|\vartheta - \varpi\|_{\mathcal{L}_2^0}.$$

Theorem 4.1 *Let $\vartheta(\mathrm{t})$ and $\overline{\vartheta}(\mathrm{t})$ be mild solutions for (1.1) with initial values $\eta(0)$ and $\overline{\eta}(0)$, respectively. Assuming (A3), (A5) holds, then the mild solution of (1.1) is stable in the mean-square.*

Proof

$$\begin{aligned}&\mathbb{E}\|\vartheta - \overline{\vartheta}\|_{\mathrm{t}}^2 \\ &\quad\leq 3\mathbb{E}\left\|\sum_{k=0}^{+\infty}\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\right\|^2 \big\|\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\big\|^2 + 3\mathbb{E}\left\|\sum_{\mathrm{k}=0}^{+\infty}\left[\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\big(\mathfrak{f}(s,\vartheta_s)\right.\right.\end{aligned}$$

$$-\mathfrak{f}(s,\overline{\vartheta_s})\big)\,ds+\int_{\varsigma_k}^{t}\mathfrak{R}(t-s)\big(\mathfrak{f}(s,\vartheta_s)-\mathfrak{f}(s,\overline{\vartheta_s})\big)\,ds\Bigg]\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t)\Bigg\|^2$$

$$+3\mathbb{E}\Bigg\|\sum_{k=0}^{+\infty}\Bigg[\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{k-1}}^{\varsigma_k}\mathfrak{R}(t-s)\big(\mathfrak{g}(s,\vartheta_s)-\mathfrak{g}(s,\overline{\vartheta_s})\big)\,ds$$

$$+\int_{\varsigma_k}^{t}\mathfrak{R}(t-s)\big(\mathfrak{g}(s,\vartheta_s)-\mathfrak{g}(s,\overline{\vartheta_s})\big)\,ds\Bigg]\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t)\Bigg\|^2$$

$$\le 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\big\|\eta(0)-\overline{\eta}(0)\big\|^2+3\max\big\{1,\mathscr{B}^2\big\}(T-t_0)\Bigg[\int_{t_0}^{t}\mathbb{E}\big\|\mathfrak{f}(s,\vartheta_s)-\mathfrak{f}(s,\overline{\vartheta_s})\big\|^2\,ds$$

$$+\int_{t_0}^{t}\mathbb{E}\big\|\mathfrak{g}(s,\vartheta_s)-\mathfrak{g}(s,\overline{\vartheta_s})\big\|^2\,ds\Bigg],$$

which implies

$$\sup_{t\in[t_0,T]}\mathbb{E}\|\vartheta-\overline{\vartheta}\|_t^2\le 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\big\|\eta(0)-\overline{\eta}(0)\big\|^2$$

$$+3\max\big\{1,\mathscr{B}^2\big\}\mathscr{H}^2(T-t_0)(\mathscr{C}_1+\mathscr{C}_2)\int_{t_0}^{t}\sup_{s\in[t_0,t]}\mathbb{E}\|\vartheta-\overline{\vartheta}\|_s^2\,ds.$$

By Gronwall's inequality

$$\sup_{t\in[t_0,T]}\mathbb{E}\|\vartheta-\overline{\vartheta}\|_t^2\le 3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\big\|\eta(0)-\overline{\eta}(0)\big\|^2\exp\big\{3\mathscr{H}^2\max\big\{1,\mathscr{B}^2\big\}(T-t_0)(\mathscr{C}_1+\mathscr{C}_2)\big\}.$$

For $\epsilon>0$, there exists a positive number

$$\tau=\frac{\epsilon}{3\mathscr{B}^2\mathscr{H}^2\exp\{3\mathscr{H}^2\max\{1,\mathscr{B}^2\}(T-t_0)(\mathscr{C}_1+\mathscr{C}_2)\}}>0.$$

$\ni\mathbb{E}\|\eta(0)-\overline{\eta}(0)\|^2<\tau$, then

$$\sup_{t\in[t_0,T]}\mathbb{E}\|\vartheta-\overline{\vartheta}\|_t^2\le\epsilon.$$

This completes the proof. □

4.2 Hyers–Ulam stability

Definition 4.1 Suppose $\varpi(t)$ is a $\mathbb{Y}$-valued stochastic process and there exists a real number $\mathscr{C}>0\ni$ for arbitrary $\epsilon>0$ satisfying

$$\mathbb{E}\Bigg\|\varpi(t)-\sum_{k=0}^{+\infty}\Bigg[\prod_{i=1}^{k}\mathfrak{b}_i(\delta_i)\mathfrak{R}(t-t_0)\eta(0)+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{k-1}}^{\varsigma_k}\mathfrak{R}(t-s)\mathfrak{f}(s,\varpi_s)\,ds$$

$$+\int_{\varsigma_k}^{t}\mathfrak{R}(t-s)\mathfrak{f}(s,\varpi_s)\,ds+\sum_{i=1}^{k}\prod_{j=i}^{k}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{k-1}}^{\varsigma_k}\mathfrak{R}(t-s)\mathfrak{g}(s,\varpi_s)\,d\omega(s)$$

$$+\int_{\varsigma_k}^{t}\mathfrak{R}(t-s)\mathfrak{g}(s,\varpi_s)\,d\omega(s)\Bigg]\mathcal{I}_{[\varsigma_k,\varsigma_{k+1})}(t)\Bigg\|^2\le\epsilon,\quad\forall t\in[t_0,T].\tag{4.1}$$

For each solution $\varpi(\mathrm{t})$ with the initial value $\varpi_{\mathrm{t}_0} = \vartheta_{\mathrm{t}_0} = \eta$, if $\exists$ a solution $\vartheta(\mathrm{t})$ of (1.1) with $\mathbb{E}\|\varpi(\mathrm{t}) - \vartheta(\mathrm{t})\|^2 \le \mathscr{C}\epsilon$, for $\mathrm{t} \in [\mathrm{t}_0, \mathtt{T}]$. Then, (1.1) has Hyers–Ulam stability.

Theorem 4.2 *Assume conditions* $(A3)$ *and* $(A5)$ *are satisfied, then* (1.1) *has Hyers–Ulam stability.*

Proof Let $\vartheta(\mathrm{t})$ be a mild solution of 1.1 and $\varpi(\mathrm{t})$ a $\mathbb{Y}$-valued stochastic process to satisfy (4.1). Obviously, $\mathbb{E}\|\varpi(\mathrm{t}) - \vartheta(\mathrm{t})\|^2 = 0$ for $\mathrm{t} \in [-\delta, 0]$. Moreover, for $\mathrm{t} \in [\mathrm{t}_0, \mathtt{T}]$, we have

$$\begin{aligned}
&\mathbb{E}\|\varpi - \vartheta\|_{\mathrm{t}}^2 \\
&\quad\le 2\mathbb{E}\Bigg\|\varpi(\mathrm{t}) - \sum_{\mathrm{k}=0}^{+\infty}\Bigg[\prod_{i=1}^{\mathrm{k}}\mathfrak{b}_i(\delta_i)\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\eta(0) + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\varpi_s)\,ds \\
&\qquad + \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{f}(s,\varpi_s)\,ds + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\varpi_s)\,d\omega(s) \\
&\qquad + \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\mathfrak{g}(s,\varpi_s)\,d\omega(s)\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t})\Bigg\|^2 + 2\mathbb{E}\Bigg\|\sum_{\mathrm{k}=0}^{+\infty}\Bigg[\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s) \\
&\qquad \times\big(\mathfrak{f}(s,\varpi_s) - \mathfrak{f}(s,\vartheta_s)\big)\,ds + \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\big(\mathfrak{f}(s,\varpi_s) - \mathfrak{f}(s,\vartheta_s)\big)\,ds \\
&\qquad + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\big(\mathfrak{g}(s,\varpi_s) - \mathfrak{g}(s,\vartheta_s)\big)\,ds + \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s) \\
&\qquad \times\big(\mathfrak{g}(s,\varpi_s) - \mathfrak{g}(s,\vartheta_s)\big)\,ds\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t})\Bigg\|^2 \\
&\quad\le 2\epsilon + 2\mathbb{E}\|\mathscr{J}\|^2.
\end{aligned}$$

Now, we consider

$$\begin{aligned}
\mathbb{E}\|\mathscr{J}\|^2 &= 2\mathbb{E}\Bigg\|\sum_{\mathrm{k}=0}^{+\infty}\Bigg[\sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s)\big(\mathfrak{f}(s,\varpi_s) - \mathfrak{f}(s,\vartheta_s)\big)\,ds \\
&\quad + \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\big(\mathfrak{f}(s,\varpi_s) - \mathfrak{f}(s,\vartheta_s)\big)\,ds + \sum_{i=1}^{\mathrm{k}}\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\int_{\varsigma_{\mathrm{k}-1}}^{\varsigma_{\mathrm{k}}}\mathfrak{R}(\mathrm{t}-s) \\
&\quad \times\big(\mathfrak{g}(s,\varpi_s) - \mathfrak{g}(s,\vartheta_s)\big)\,ds + \int_{\varsigma_{\mathrm{k}}}^{\mathrm{t}}\mathfrak{R}(\mathrm{t}-s)\big(\mathfrak{g}(s,\varpi_s) \\
&\quad - \mathfrak{g}(s,\vartheta_s)\big)\,ds\Bigg]\mathcal{I}_{[\varsigma_{\mathrm{k}},\varsigma_{\mathrm{k}+1})}(\mathrm{t})\Bigg\|^2 \\
&\le 2\max\big\{1,\mathscr{B}^2\big\}\mathscr{H}^2(\mathtt{T}-\mathrm{t}_0)\int_{\mathrm{t}_0}^{\mathrm{t}}\mathbb{E}\big\|\mathfrak{f}(s,\varpi_s) - \mathfrak{f}(s,\vartheta_s)\big\|^2\,ds \\
&\quad + 2\max\big\{1,\mathscr{B}^2\big\}\mathscr{H}^2\int_{\mathrm{t}_0}^{\mathrm{t}}\mathbb{E}\big\|\mathfrak{g}(s,\varpi_s) - \mathfrak{g}(s,\vartheta_s)\big\|^2\,ds.
\end{aligned}$$

Taking the supremum on both sides and using (A5),

$$\sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \mathbb{E}\|\varpi - \vartheta\|_{\mathrm{t}}^2 \le 2\epsilon + 4\max\{1,\mathscr{B}^2\}\mathscr{H}^2(\mathrm{T}-\mathrm{t}_0)\mathscr{C}_1\int_{\mathrm{t}_0}^{\mathrm{t}} \sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \mathbb{E}\|\varpi - \vartheta\|_s^2\,ds$$
$$+ 4\max\{1,\mathscr{B}^2\}\mathscr{H}^2\mathscr{C}_2\int_{\mathrm{t}_0}^{\mathrm{t}} \sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \mathbb{E}\|\varpi - \vartheta\|_s^2\,ds.$$

By following Gronwall's inequality, there occurs a constant

$$\mathscr{C} := 2\exp\{\max\{1,\mathscr{B}^2\}\mathscr{H}^2[(\mathrm{T}-\mathrm{t}_0)\mathscr{C}_1 + \mathscr{C}_2]\} > 0.$$

This implies that

$$\sup_{\mathrm{t}\in[\mathrm{t}_0,\mathrm{T}]} \mathbb{E}\|\varpi - \vartheta\|_{\mathrm{t}}^2 \le \mathscr{C}\epsilon.$$

This implies the Hyers–Ulam stability of (1.1). Thus, the proof is complete. □

4.3 Mean-square exponential stability

In order to prove the theorem we may take into consideration the following lemma

Lemma 4.1 [26] *For $\rho > 0$, $\exists$ some positive constants $\upsilon, \upsilon' > 0 \ni$ if $\upsilon' < \rho$, the following inequality*

$$\varpi(\mathrm{t}) = \begin{cases} \upsilon e^{-\rho(\mathrm{t}-\mathrm{t}_0)}, & \mathrm{t}\in[-\delta,0] \\ \upsilon e^{-\rho(\mathrm{t}-\mathrm{t}_0)} + \upsilon'\int_{\mathrm{t}_0}^{\mathrm{t}} e^{-\rho(\mathrm{t}-s)}\sup_{\theta\in(-\delta,0]}\varpi(s+\theta)\,ds, & \mathrm{t}\ge \mathrm{t}_0 \end{cases}$$

holds. Then, we have $\varpi(\mathrm{t}) \le \mathscr{F}e^{-\tau(\mathrm{t}-\mathrm{t}_0)}$, where $\tau > 0$ satisfying

$$\frac{\upsilon'}{\rho-\tau}e^{\tau(\delta+\mathrm{t}_0)} = 1$$

and

$$\mathscr{F} = \max\left\{\frac{\upsilon}{\upsilon'}(\rho-\tau)e^{-\tau\delta}, \rho\right\}.$$

Theorem 4.3 *Assume (A3), (A5) is satisfied, then the mild solution of (1.1) is mean-square exponentially stable.*

Proof Together with the assumed hypotheses and Holder's inequality,

$$\mathbb{E}\|\vartheta(\mathrm{t})\|^2$$
$$\le 3\mathbb{E}\left(\max_{\mathrm{k}}\left\{\prod_{i=1}^{\mathrm{k}}\|\mathfrak{b}_i(\delta_i)\|^2\right\}\right)^2\|\mathfrak{R}(\mathrm{t}-\mathrm{t}_0)\|^2\mathbb{E}\|\eta(0)\|^2 + 3\mathbb{E}\left(\max_{i,\mathrm{k}}\left\{\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\right\},1\right)^2$$
$$\times\,\mathbb{E}\left(\int_{\mathrm{t}_0}^{\mathrm{t}}\|\mathfrak{R}(\mathrm{t}-s)\|\,\|\mathfrak{f}(s,\vartheta_s)\|\,ds\right)^2 + 3\mathbb{E}\left(\max_{i,\mathrm{k}}\left\{\prod_{j=i}^{\mathrm{k}}\mathfrak{b}_j(\delta_j)\right\},1\right)^2$$

$$
\begin{aligned}
&\times \mathbb{E}\left(\int_{t_0}^{t} \|\mathfrak{R}(t-s)\| \|\mathfrak{g}(s,\vartheta_s)\| \, d\omega(s)\right)^2 \\
&\leq 3\mathscr{B}^2\mathscr{H}^2 e^{-\rho(t-t_0)}\mathbb{E}\|\eta(0)\|^2 + 3\max\{1,\mathscr{B}^2\}\mathscr{H}^2 \int_{t}^{t_0} e^{-\rho(t-t_0)}\mathbb{E}\|\mathfrak{f}(s,\vartheta_s)\|^2 \, ds \\
&\quad \times \int_{t_0}^{t} e^{-\rho(t-t_0)} \, ds + 3\max\{1,\mathscr{B}^2\}\mathscr{H}^2 \int_{t_0}^{t} e^{-\rho(t-t_0)} \, ds \int_{t}^{t_0} e^{-\rho(t-t_0)}\mathbb{E}\|\mathfrak{g}(s,\vartheta_s)\|^2 \, ds \\
&\leq 3\mathscr{B}^2\mathscr{H}^2 e^{-\rho(t-t_0)}\mathbb{E}\|\eta(0)\|^2 \\
&\quad + 3\max\{1,\mathscr{B}^2\}\frac{\mathscr{H}^2(\mathscr{C}_1+\mathscr{C}_2)}{\rho}\int_{t_0}^{t} \sup_{\theta\in[-\delta,0]} \mathbb{E}\|\vartheta(s+\theta)\|^2 \, ds \\
&\leq \mathscr{F} e^{-\rho(t-t_0)}, \quad \forall t \in [-\delta,0],
\end{aligned}
$$

where $\mathscr{F} = \max\{3\mathscr{B}^2\mathscr{H}^2\mathbb{E}\|\eta(0)\|^2, \sup_{\theta\in[-\delta,0]}\mathbb{E}\|\eta\|^2\}$.

Thus, by Lemma 4.1, $\forall t \in [t_0-\delta,+\infty]$,

$$
\mathbb{E}\|\vartheta(t)\|^2 \leq \mathscr{F} e^{-\tau t}.
$$

This completes the proof. □

5 Illustration

In order to validate the abstract theory, let us take into account the system on a bounded domain $\Omega \subset \mathscr{R}^n$ with the boundary $\partial\Omega$:

$$
\begin{aligned}
&\frac{d[z(t,\vartheta)}{\partial t}] = \frac{\partial^2}{\partial\vartheta^2}z(t,\vartheta) + \int_0^t \alpha(t-s)\frac{\partial^2}{\partial\vartheta^2}z(s,\vartheta)\,ds + \int_{-\mathfrak{r}}^{t}\left[\kappa_1(\theta)z(t+\theta)\,d\theta\right] \\
&\qquad + \int_{-\mathfrak{r}}^{t}\left[\kappa_2(\theta)z(t+\theta)\,d\theta\right]d\omega(t), \quad t \geq \delta, t \neq \varsigma_k, \\
&z(\varsigma_k,\vartheta) = \mathfrak{h}(k)\delta_k z(\varsigma_k^-,\vartheta), \quad \vartheta \in \Omega \\
&z(t_0,\vartheta) = \eta(\theta,\vartheta) = \{\eta(\theta) \leq \theta < 0\}, \quad \vartheta \in \Omega, \theta \in [-\delta,0] \\
&z(t,\vartheta) = 0, \quad \vartheta \in \partial\Omega.
\end{aligned}
\tag{5.1}
$$

Let $\mathscr{X} = \mathcal{L}^2(\Omega)$, $\alpha : \mathscr{R}_+ \to \mathscr{R}_+$. κ_1, κ_2 be positive functions from $[-\delta,0]$ to $\mathscr{R}$. Assuming δ_k to be a random variable defined on $\mathscr{D}_k = (0,\mathfrak{d}_k)$ with $0 < \mathfrak{d}_k < +\infty$ for $k = 1,2,\ldots$. Without loss of generality, we may assume that $\{\delta_k\}$ follows an Erlang distribution. δ_i, δ_j are mutually independent with $i \neq j$ for $i,j = 1,2,\ldots$. $\mathfrak{h}$ is a function of k, $\varsigma_k = \varsigma_{k-1} + \delta_k$, where $\{\varsigma_k\}$ forms a strictly increasing process with independent increments and $t_0 \in [0,\mathrm{T}]$ is an arbitrary real number.

Let $\mathfrak{A}$ be an operator on $\mathscr{X}$ by $\mathfrak{A}z = \frac{\partial^2 z}{\partial\vartheta}$,

$$
\mathscr{D}(\mathfrak{A}) = \{z \in \mathscr{X} : z \text{ and } z_\vartheta \text{ are absolutely continuous}, z_{\vartheta\vartheta} \in \mathscr{X}, z = 0 \text{ on } \partial\Omega\}.
$$

Also, let the map $\mathfrak{B} : \mathscr{D}(\mathfrak{A}) \subset \mathscr{X} \to \mathscr{X}$ be the operator defined by

$$
\mathfrak{B}(t)(z) = \alpha(t)\mathfrak{A}z \quad \text{for } t \geq 0 \text{ and } z \in \mathscr{D}\mathfrak{A}.
$$

The operator $\mathfrak{A}$ can be expressed as

$$\mathfrak{A}z = \sum_{n=1}^{\infty} \mathfrak{n}^2 \langle z, z_{\mathfrak{n}} \rangle z_{\mathfrak{n}}, \quad z \in \mathscr{D}\mathfrak{A},$$

where $z_{\mathfrak{n}}(\varpi) = (\frac{2}{\pi})^{\frac{1}{2}}$ are the corresponding eigenvectors of $\mathfrak{A}$. Obviously, $z_{\mathfrak{n}}(\varpi)$ form an orthonormal system in $\mathbb{X}$. Moreover, $\mathfrak{A}$ is the infinitesimal generator of an analytic semigroup $(\mathfrak{R}(\mathrm{t}))_{\mathrm{t}\geq 0}$ in $\mathbb{X}$, satisfying

$$\left\|\mathfrak{R}(\mathrm{t})\right\| \leq \exp\left\{-\pi^2(\mathrm{t} - \mathrm{t}_0)\right\}, \quad \mathrm{t} \geq \mathrm{t}_0.$$

Also, we have the following additional conditions:
(i) $\int_{-\delta}^{0} \kappa_1(\theta)^2 \, d\theta < \infty, \int_{-\delta}^{0} \kappa_2(\theta)^2 \, d\theta < \infty$,
(ii) $\mathbb{E}(\max_{i,k}\{\prod_{j=i}^{k} \|\mathfrak{h}(j)(\delta_j)\|\}^2) < \infty$.
Using the aforementioned conditions, (5.1) can be modeled as the abstract random impulsive stochastic differential equation of the form (1.1),

$$\mathfrak{f}(t, z_{\mathrm{t}}) = \int_{-\mathfrak{r}}^{\mathrm{t}} \kappa_1(\theta) z(\mathrm{t} + \theta) \, d\theta,$$

$$\mathfrak{g}(t, z_{\mathrm{t}}) = \int_{-\mathfrak{r}}^{\mathrm{t}} \kappa_2(\theta) z(\mathrm{t} + \theta) \, d\theta,$$

$$\mathfrak{b}_k(\delta_k) = \mathfrak{h}(k)\delta_k.$$

Condition (i) implies that (A5) holds with

$$\mathscr{C}_i = \int_{\mathfrak{r}}^{0} \kappa_i^2(\theta) \, d\theta, \quad \text{for } i = 1, 2,$$

along with condition (ii), implying (A3). This shows that (5.1) has a mild solution. Moreover, we achieve the stability results [continuous dependence of solution on initial conditions and Hyers–Ulam stability] as in Sect. 4. Finally, if $\lambda' \leq \tau$, i.e.,

$$3 \max\left\{1, \mathscr{B}^2\right\}(\mathscr{C}_1 + \mathscr{C}_2)/\left(\pi^2\right) \leq \pi^2,$$

then (5.1) is mean-square exponentially stable under the assumptions (A3) and (A5).

6 Outlook

In this paper, the random impulsive stochastic delay differential system with resolvent operator (1.1) has been proposed and the existence and various stabilities including the continuous dependence of solution on initial conditions, Hyers–Ulam stability, and mean-square exponential stability results are carried out with the use of stochastic analysis techniques and functional analysis. Significantly, this system can be further extended to neutral problems, fractional stochastic differential systems with random impulses.

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References
1. Lakshmikantham, V., Bianov, D.D., Simenonov, P.S.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
2. Samoilenko, A., Perestyuk, N.: Impulsive Differential Equations. World Scientific, Singapore (1995)
3. Pan, L., Cao, J.: Exponential stability of impulsive stochastic functional differential equations. J. Math. Anal. Appl. **382**, 672–685 (2011)
4. Anguraj, A., Wu, S., Vinodkumar, A.: The existence and exponential stability of semilinear functional differential equations with random impulses under non-uniqueness. Nonlinear Anal., Theory Methods Appl. **74**, 331–342 (2011)
5. Anguraj, A., Karthikeyan, K.: Existence of solutions for impulsive neutral functional differential equations with nonlocal conditions. Nonlinear Anal., Theory Methods Appl. **70**(7), 2717–2721 (2009)
6. Shu, X.B., Lai, Y.Z., Chen, Y.M.: The existence of mild solutions for impulsive fractional partial differential equations. Nonlinear Anal. **74**, 2003–2011 (2011)
7. Mao, X.: Stochastic Differential Equations and Applications. M. Horwood, Chichester (1997)
8. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992)
9. Oksendal, B.: Stochastic Differential Equations: An Introduction with Applications. Springer, Berlin (2013)
10. Yang, X., Zhu, Q.: Pth moment exponential stability of stochastic partial differential equations with Poisson jumps. Asian J. Control **16**, 1482–1491 (2014)
11. Ravikumar, K., Chalishajar, D.N., Ramkumar, K., Anguraj, A.: Controllability of neutral impulsive stochastic functional integro-differential equations driven by a fractional Brownian motion with infinite delay via resolvent operator. J. Nonlinear Sci. Appl. **15**, 172–185 (2022)
12. Ramkumar, K., Ravikumar, K., Elsayed, E.: Well-posedness and stability of time-dependent impulsive neutral stochastic partial imtegro-differential equations with fractional Brownian motion and Poisson jumps. J. Math. Ext. **16**(7), 1–12 (2021)
13. Ramkumar, K., Ravikumar, K., Varshini, S.: Fractional neutral stochastic differential equations with Caputo-fractional derivative: fractional Brownian motion, Poisson jumps and optimal control. Stoch. Anal. Appl. **39**, 157–176 (2021)
14. Naghshtabrizi, P., Hespanha, J.P., Teel, A.R.: Exponential stability of impulsive systems with application to uncertain sampled-data systems. Syst. Control Lett. **57**(5), 378–385 (2008)
15. Wang, T., Wu, S.: Random impulsive model for stock prices and its application for insurers. Ph.D. thesis, Master thesis (in Chinese), Shanghai, East China Normal University (2008)
16. Hu, W., Zhu, Q.: Exponential stability of stochastic differential equations with impulse effects at random times. Asian J. Control **22**(2), 779–787 (2020)
17. Hu, W., Zhu, Q.: Moment exponential stability of stochastic nonlinear delay systems with impulse effects at random times. Int. J. Robust Nonlinear Control **29**(12), 3809–4284 (2019)
18. Hu, W., Zhu, Q.: Moment exponential stability of stochastic delay systems with delayed impulse effects at random times and applications in the stabilisation of stochastic neural networks. Int. J. Control **93**(10), 2505–2515 (2021)
19. Sakthivel, R., Luo, J.: Asymptotic stability of nonlinear impulsive stochastic differential equations. Stat. Probab. Lett. **79**(9), 1219–1223 (2009)
20. Li, Z., Shu, X.B., Miao, T.: The existence of solutions for Sturm–Liouville differential equation with random impulses and boundary value problems. Bound. Value Probl. **2021**, 1 (2021)

21. Guo, Y., Shu, X.B., Xu, F., Yang, C.: HJB equation for optimal control system with random impulses. Optimization, 1–25 (2022)
22. Anguraj, A., Ravikumar, K., Nieto, J.J.: On stability of stochastic differential equations with random impulses driven by Poisson jumps. Stoch. Int. J. Probab. Stoch. Process. **93**(5), 682–696 (2021)
23. Chalishajar, D.N., Ravikumar, K., Ramkumar, K.: Existence and stability results for second-order neutral stochastic differential equations with random impulses and Poisson jumps. Eur. J. Math. Anal. **1**, 1–18 (2021)
24. Yin, Q.B., Guo, Y., Wu, D., Shu, X.B.: Existence and multiplicity of mild solutions for first-order Hamilton random impulsive differential equations with Dirichlet boundary conditions. Qual. Theory Dyn. Syst. **22**(2), 47 (2023)
25. Grimmer, R.C.: Resolvent operators for integral equations in a Banach space. Trans. Am. Math. Soc. **273**(1), 333–349 (1982)
26. Yang, Q., Wu, D., Shu, X.: Existence and stability results of mild soltuions for random impulsive stochastic partial differential equations with noncompact semigroups. Stochastics (2022)

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