**Research Article**

Wen Sheng Zhu, Badreddine Meftah, Hongyan Xu, Fahd Jarad, and Abdelghani Lakhdari*

# On parameterized inequalities for fractional multiplicative integrals

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**Abstract:** In this article, we present a one-parameter fractional multiplicative integral identity and use it to derive a set of inequalities for multiplicatively $s$-convex mappings. These inequalities include new discoveries and improvements upon some well-known results. Finally, we provide an illustrative example with graphical representations, along with some applications to special means of real numbers within the domain of multiplicative calculus.

**Keywords:** parametrized identity, integral inequalities, fractional multiplicative integral, multiplicative $s$-convexity

**MSC 2020:** 26A33, 26A51, 26D10, 26D15

## 1 Introduction

Multiplicative calculus, first introduced by Grosman and Katz in 1967, emerged as a novel approach to classical calculus, addressing issues related to rates of change and multiplicative processes [1]. This calculus, primarily applied to positive functions, was formalized by Bashirov et al. in their comprehensive work in 2008, as outlined in [2]. Its significance lies in its enhanced ability to handle phenomena involving growth, decay, and proportional relationships more effectively. Over time, it has found relevance in various domains, including finance [3], biology [4], and physics [5], offering a fresh perspective on modeling and analysis for scenarios where traditional calculus may prove inadequate.

On the flip side, convexity stands as a fundamental mathematical concept with a crucial role in diverse scientific fields. Its significance comes from its ability to capture the essential characteristics of numerous real-world phenomena, making it a powerful tool for modeling and analysis. In particular, convex functions exhibit remarkable properties that simplify optimization, economics, and even the understanding of physical systems. A function $\mathcal{Z}$ is considered convex over the interval $[a, b]$ if, for all $y_1$ and $y_2$ within this interval, the following inequality holds for any $\eta$ in the range [0,1]:

$$\mathcal{Z}(\eta y_1 + (1 - \eta)y_2) \leq \eta\mathcal{Z}(y_1) + (1 - \eta)\mathcal{Z}(y_2).$$

*** Corresponding author: Abdelghani Lakhdari,** Department CPST, National Higher School of Technology and Engineering, Annaba 23005, Algeria, e-mail: a.lakhdari@ensti-annaba.dz

**Wen Sheng Zhu:** Center of Information and Educational Technology, Gannan Normal University, Ganzhou, Jiangxi, 341000, China, e-mail: 4300008@gnnu.edu.cn

**Badreddine Meftah:** Department of Mathematics, Laboratory of Analysis and Control of Differential Equations "ACED," Faculty MISM, University of 8 May 1945 Guelma, P.O. Box 401, 24000 Guelma, Algeria, e-mail: badrimeftah@yahoo.fr

**Hongyan Xu:** School of Arts and Sciences, Suqian University, Suqian, Jiangsu 223800, P. R. China; School of Mathematics and Computer Science, Shangrao Normal University, Shangrao Jiangxi, 334001, P. R. China, e-mail: xhysq321@126.com

**Fahd Jarad:** Department of Mathematics, Çankaya University, 06790 Ankara, Turkey, e-mail: fahd@cankaya.edu.tr

One of the key inequalities linked to convexity is the Hermite-Hadamard inequality, which asserts that for a convex function $\mathcal{Z}$ defined on an interval $[a, b]$, the following inequalities hold:

$$\mathcal{Z}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \mathcal{Z}(y)\mathrm{d}y \leq \frac{\mathcal{Z}(a)+\mathcal{Z}(b)}{2}.$$

In the literature, various extensions and variations of the concept of convexity have emerged and have been employed to estimate the error of certain quadrature formulas. However, the most suitable variant in conjunction with multiplicative calculus is logarithmic convexity, also known as multiplicative convexity, which can be formulated as follows:

**Definition 1.1.** [6] A function $\mathcal{Z} : I \to \mathbb{R}^{+}$ is considered multiplicatively convex if, for all $y_1, y_2 \in I$, the following inequality

$$\mathcal{Z}(\eta y_1 + (1-\eta)y_2) \leq [\mathcal{Z}(y_1)]^{\eta}[f(y_2)]^{1-\eta}$$

holds true for all $\eta \in [0, 1]$.

In [7], Ali et al. incorporated the Hermite-Hadamard inequality into the multiplicative calculus framework.

**Theorem 1.2.** *Let $\mathcal{Z}$ be a positive multiplicatively convex function on the interval $[a, b]$. Then, the following inequalities hold*:

$$\mathcal{Z}\left(\frac{a+b}{2}\right) \leq \left(\int_a^b \mathcal{Z}(y)^{\mathrm{d}y}\right)^{\frac{1}{b-a}} \leq \sqrt{\mathcal{Z}(a)\mathcal{Z}(b)}. \tag{1}$$

Significant research has been conducted in the field of multiplicative integrals. In [8], the authors established midpoint and trapezoid-type inequalities for multiplicatively convex functions. Ali et al. [9] conducted an examination of Ostrowski- and Simpson-type inequalities in the context of multiplicatively convex functions. Furthermore, another investigation detailed Maclaurin inequalities [10], while Meftah and Lakhdari [11] delved into dual Simpson-type inequalities. For additional resources on multiplicative integral inequalities, we encourage the reader to refer [10,12–16].

In [17], Abdeljawad and Grossman presented the multiplicative Riemann-Liouville fractional integrals in the following manner:

**Definition 1.3.** [17] The operators defining the multiplicative left- and right-sided Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, where $\mathrm{Re}(\alpha) > 0$, are as follows:

$$({}_{a}I_{*}^{\alpha}\mathcal{Z})(y) = e^{(J_{a^+}^{\alpha}(\ln \circ \mathcal{Z}))(y)}, \quad a < y \tag{2}$$

and

$$({}_{*}I_{b}^{\alpha}\mathcal{Z})(y) = e^{(J_{b^-}^{\alpha}(\ln \circ \mathcal{Z}))(y)}, \quad y < b, \tag{3}$$

where $J_{c^+}^{\alpha}$ and $J_{c^-}^{\alpha}$ denote the left- and right-sided Riemann-Liouville fractional integral operators, see [18].

The fractional Hermite-Hadamard inequalities in the context of multiplicative Riemann-Liouville fractional integrals was established by Budak and Özçelik in [19].

**Theorem 1.4.** *Let* $\mathcal{Z}$ *be a positive multiplicatively convex function on the interval* $[a, b]$. *Then, the following inequalities hold*:

$$\mathcal{Z}\left(\frac{a+b}{2}\right) \leq \left[{}_{*}I^{\alpha}_{\frac{a+b}{2}}\mathcal{Z}(a)\,{}_{\frac{a+b}{2}}I^{\alpha}_{*}\mathcal{Z}(b)\right]^{\frac{2^{\alpha-1}\eta(\alpha+1)}{(b-a)^{\alpha}}} \leq \sqrt{\mathcal{Z}(a)\mathcal{Z}(b)}. \tag{4}$$

In their work presented in [20], Fu et al. explored multiplicative tempered fractional integrals, extending the findings of Ali et al. [7] and Budak and Özčelik [19]. Furthermore, within the realm of fractional multiplicative integrals, Moumen et al. [21] established Simpson inequalities, while Boulares et al. [22] demonstrated Bullen-type inequalities. Additionally, Peng and Du contributed to the field with their work on fractional multiplicative Maclaurin-type inequalities in [23]. For further pertinent results, readers can refer to [24–29] and references mentioned therein.

Drawing upon the insights gleaned from the aforementioned works, this study begins by introducing a parameterized identity integral specifically designed for multiplicative differentiable functions. Building upon this foundational equality, we then proceed to derive a set of three-point Newton-Cotes-type inequalities, specifically tailored for multiplicative $s$-convex functions. To wrap up our investigation, we provide practical applications that vividly illustrate the usefulness and significance of the results we have established.

## 2 Preliminaries

In this section, we provide a review of fundamental concepts associated with multiplicative calculus essential for the subsequent development of our study.

**Definition 2.1.** [2] The multiplicative derivative of a positive function $\mathcal{Z}$, denoted as $\mathcal{Z}^{*}$ is defined as follows:

$$\frac{\mathrm{d}^{*}\mathcal{Z}}{\mathrm{d}y} = \mathcal{Z}^{*}(y) = \lim_{h\to 0}\left(\frac{\mathcal{Z}(y+h)}{\mathcal{Z}(y)}\right)^{\frac{1}{h}}.$$

**Remark 2.2.** Each positive and differentiable function $\mathcal{Z}$ inherently exhibits multiplicative differentiability, with the interconnection between $\mathcal{Z}'$ and $\mathcal{Z}^{*}$ governed by the following relationship:

$$\mathcal{Z}^{*}(y) = e^{(\ln \mathcal{Z}(y))'} = e^{\frac{\mathcal{Z}'(y)}{\mathcal{Z}(y)}}.$$

**Proposition 2.3.** [2] *Let* $\mathcal{Z}$ *and* $\mathcal{X}$ *be two multiplicatively differentiable functions, and* $Q$ *is differentiable. Let* $c$ *be an arbitrary positive constant, then functions* $c\mathcal{Z}$, $\mathcal{Z}\mathcal{X}$, $\mathcal{Z}+\mathcal{X}$, $\mathcal{Z}/\mathcal{X}$, $\mathcal{Z}^{Q}$, *and* $\mathcal{Z}\circ Q$ *are multiplicatively differentiable and we have*

- $(c\mathcal{Z})^{*}(y) = \mathcal{Z}^{*}(y)$,
- $(\mathcal{Z}\mathcal{X})^{*}(y) = \mathcal{Z}^{*}(y)\mathcal{X}^{*}(y)$,
- $(\mathcal{Z}+\mathcal{X})^{*}(y) = \mathcal{Z}^{*}(y)^{\frac{\mathcal{Z}(y)}{\mathcal{Z}(y)+\mathcal{X}(y)}}\mathcal{X}^{*}(y)^{\frac{\mathcal{X}(y)}{\mathcal{Z}(y)+\mathcal{X}(y)}}$,
- $\left(\frac{\mathcal{Z}}{\mathcal{X}}\right)^{*}(y) = \frac{\mathcal{Z}^{*}(y)}{\mathcal{X}^{*}(y)}$,
- $(\mathcal{Z}^{Q})^{*}(y) = \mathcal{Z}^{*}(y)^{Q(y)}\mathcal{Z}(y)^{Q'(y)}$,
- $(\mathcal{Z}\circ Q)^{*}(y) = \mathcal{Z}^{*}(Q(y))^{Q'(y)}$.

**Definition 2.4.** [2] The multiplicative integral of a positive function $\mathcal{Z}$ is defined as follows:

$$\int_{a}^{b}(\mathcal{Z}(y))^{\mathrm{d}y} = \exp\left\{\int_{a}^{b}\ln(\mathcal{Z}(y))\mathrm{d}y\right\}.$$

**Proposition 2.5.** [2] *Consider positive and Riemann integrable functions $\mathcal{Z}$ and $\mathcal{X}$, then $\mathcal{Z}$ and $\mathcal{X}$ are multiplicative integrable and the following properties hold true.*

- $\int_a^b ((\mathcal{Z}(y))^p)^{\mathrm{dy}} = \left(\int_a^b (\mathcal{Z}(y))^{\mathrm{dy}}\right)^p, \quad p \in \mathbb{R},$
- $\int_a^b (\mathcal{Z}(y)\mathcal{X}(y))^{\mathrm{dy}} = \int_a^b (\mathcal{Z}(y))^{\mathrm{dy}} \int_a^b (\mathcal{X}(y))^{\mathrm{dy}},$
- $\int_a^b \left(\frac{\mathcal{Z}(y)}{\mathcal{X}(y)}\right)^{\mathrm{dy}} = \frac{\int_a^b (\mathcal{Z}(y))^{\mathrm{dy}}}{\int_a^b (\mathcal{X}(y))^{\mathrm{dy}}},$
- $\int_a^b (\mathcal{Z}(y))^{\mathrm{dy}} = \int_a^c (\mathcal{Z}(y))^{\mathrm{dy}} \int_c^b (\mathcal{Z}(y))^{\mathrm{dy}}, a < c < b,$
- $\int_a^a (\mathcal{Z}(y))^{\mathrm{dy}} = 1$, *and* $\int_a^b (\mathcal{Z}(t))^{\mathrm{dy}} = \left(\int_b^a (\mathcal{Z}(y))^{\mathrm{dy}}\right)^{-1}.$

**Theorem 2.6.** [2] *Let $\mathcal{Z} : [a, b] \to \mathbb{R}$ be multiplicative differentiable, and let $Q : [a, b] \to \mathbb{R}$ be differentiable, so the function $\mathcal{Z}^Q$ is multiplicative integrable. Then*

$$\int_a^b (\mathcal{Z}^*(y)^{\mathcal{X}(y)})^{\mathrm{dy}} = \frac{\mathcal{Z}(b)^{\mathcal{X}(b)}}{\mathcal{Z}(a)^{\mathcal{X}(a)}} \cdot \frac{1}{\int_a^b (\mathcal{Z}(y)^{\mathcal{X}'(y)})^{\mathrm{dy}}}.$$

**Lemma 2.7.** [9] *Let $\mathcal{Z} : [a, b] \to \mathbb{R}^+$ be multiplicative differentiable, let $\mathcal{X} : [a, b] \to \mathbb{R}$, and let $Q : J \subset \mathbb{R} \to \mathbb{R}$ be two differentiable functions. Then we have*

$$\int_a^b (\mathcal{Z}^*(Q(y))^{Q'(y)\mathcal{X}(y)})^{\mathrm{dy}} = \frac{\mathcal{Z}(Q(b))^{\mathcal{X}(b)}}{\mathcal{Z}(Q(a))^{\mathcal{X}(a)}} \cdot \frac{1}{\int_a^b (\mathcal{Z}(Q(y))^{\mathcal{X}'(y)})^{\mathrm{dy}}}.$$

## 3 Main results

First, let us revisit the definition of multiplicatively $s$-convex functions.

**Definition 3.1.** [30] A function $\mathcal{Z} : I \subset \mathbb{R} \to \mathbb{R}^+$ is considered multiplicatively $s$-convex for some fixed $s \in (0, 1]$ if for all $y_1, y_2 \in I$ the following inequality

$$\mathcal{Z}(\eta y_1 + (1 - \eta)y_2) \le [\mathcal{Z}(y_1)]^{\eta^s}[\mathcal{Z}(y_2)]^{(1-\eta)^s}$$

holds true for all $\eta \in [0, 1]$.

Now, we introduce a lemma that will serve as the main tool for establishing the key results.

**Lemma 3.2.** *Let $\mathcal{Z}$:$[a, b] \to \mathbb{R}^+$ be a multiplicative differentiable mapping on $[a, b]$ with $a < b$. If $\mathcal{Z}^*$ is multiplicative integrable on $[a, b]$, then we have the following identity for multiplicative integrals*:

$$\left((\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-\upsilon}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}\right)(\mathcal{J}(a, b;\ \mathcal{Z};\ \alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}}$$
$$= \left(\int_0^1\left(\left(\mathcal{Z}^*\left((1-\eta)a + \eta\frac{a+b}{2}\right)\right)^{1-\upsilon-(1-\eta)^\alpha}\right)^{\mathrm{d}\eta}\right)^{\frac{b-a}{4}}\left(\int_0^1\left(\left(\mathcal{Z}^*\left((1-\eta)\frac{a+b}{2} + \eta b\right)\right)^{\eta^\alpha-(1-\upsilon)}\right)^{\mathrm{d}\eta}\right)^{\frac{b-a}{4}},$$

*where $\upsilon$ belongs to the interval* $[0, 1]$, *and $\mathcal{J}$ is defined as*

$$\mathcal{J}(a,b;\,\mathcal{Z};\,\alpha)=\left({}_{x}I_{*}^{\alpha}\mathcal{Z}\right)\left(\frac{a+b}{2}\right)^{2^{\alpha-1}}\left({}_{*}I_{b}^{\alpha}\mathcal{Z}\right)\left(\frac{a+b}{2}\right)^{2^{\alpha-1}}. \tag{5}$$

**Proof.** Let

$$\mathcal{I}_1=\left(\int_0^1\left(\mathcal{Z}^*\left((1-\eta)a+\eta\frac{a+b}{2}\right)\right)^{(1-\upsilon-(1-\eta)^{\alpha})\mathrm{d}\eta}\right)^{\frac{b-a}{4}}$$

and

$$\mathcal{I}_2=\left(\int_0^1\left(\mathcal{Z}^*\left((1-\eta)\frac{a+b}{2}+\eta b\right)\right)^{(\eta^{\alpha}-(1-\upsilon))\mathrm{d}\eta}\right)^{\frac{b-a}{4}}.$$

Using Theorem 2.6, then from $\mathcal{I}_1$ we have

$$\begin{aligned}
\mathcal{I}_1&=\left(\int_0^1\left(\mathcal{Z}^*\left((1-\eta)a+\eta\frac{a+b}{2}\right)\right)^{(1-\upsilon-(1-\eta)^{\alpha})\mathrm{d}\eta}\right)^{\frac{b-a}{4}}\\
&=\left(\int_0^1\left(\mathcal{Z}^*\left((1-\eta)a+\eta\frac{a+b}{2}\right)\right)^{\frac{b-a}{4}(1-\upsilon-(1-\eta)^{\alpha})\mathrm{d}\eta}\right)\\
&=\frac{\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\upsilon}{2}}}{(\mathcal{Z}(a))^{-\frac{\upsilon}{2}}}\,\frac{1}{\int_0^1\left(\mathcal{Z}\left((1-\eta)a+\eta\frac{a+b}{2}\right)\right)^{\frac{\alpha}{2}(1-\eta)^{\alpha-1}\mathrm{d}\eta}}\\
&=\frac{(\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\upsilon}{2}}}{\exp\left\{\frac{\alpha}{2}\int_0^1(1-\eta)^{\alpha-1}\ln\left(\mathcal{Z}\left((1-\eta)a+\eta\frac{a+b}{2}\right)\right)\mathrm{d}\eta\right\}}\\
&=\frac{(\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\upsilon}{2}}}{\exp\left\{\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left(\frac{1}{\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-y\right)^{\alpha-1}\ln(\mathcal{Z}(y))\mathrm{d}y\right)\right\}}\\
&=(\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\upsilon}{2}}\left({}_{a}I_{*}^{\alpha}\mathcal{Z}\right)\left(\frac{a+b}{2}\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}.
\end{aligned} \tag{6}$$

Similarly,

$$
\begin{aligned}
\mathcal{I}_2 &= \left(\int_0^1\left(\left(\mathcal{Z}^*\left((1-\eta)\frac{a+b}{2}+\eta b\right)\right)^{\eta^{\alpha}-(1-\upsilon)}\right)^{\mathrm{d}\eta}\right)^{\frac{b-a}{4}}\\
&= \int_0^1\left(\left(\mathcal{Z}^*\left((1-\eta)\frac{a+b}{2}+\eta b\right)\right)^{\frac{b-a}{4}(\eta^{\alpha}-(1-\upsilon))}\right)^{\mathrm{d}\eta}\\
&= \frac{(\mathcal{Z}(b))^{\frac{\upsilon}{2}}}{\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{-\frac{1-\upsilon}{2}}}\,\frac{1}{\int_0^1\left(\left(\mathcal{Z}\left((1-\eta)\frac{a+b}{2}+\eta b\right)\right)^{\frac{\alpha}{2}\eta^{\alpha-1}}\right)^{\mathrm{d}\eta}}\\
&= \frac{\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\upsilon}{2}}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}}{\exp\left\{\frac{\alpha}{2}\int_0^1\eta^{\alpha-1}\ln\left(\mathcal{Z}\left((1-\eta)\frac{a+b}{2}+\eta b\right)\right)\mathrm{d}\eta\right\}}\\
&= \frac{\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\upsilon}{2}}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}}{\left(\exp\left\{\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}\left(u-\frac{a+b}{2}\right)^{\alpha-1}\ln(\mathcal{Z}(y))\mathrm{d}y\right\}\right)^{\frac{2^{\alpha-1}\eta(\alpha-1)}{(b-a)^{\alpha}}}}\\
&= \left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\upsilon}{2}}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}\left(\left({}_*I_b^{\alpha}\mathcal{Z}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}}.
\end{aligned}
\tag{7}
$$

Multiplying equalities (6) and (7) yields the desired result, thus concluding the proof. □

**Theorem 3.3.** *Let* $\mathcal{Z}:[a,b]\to\mathbb{R}^+$ *be an increasing and multiplicative differentiable mapping on* $[a,b]$. *If* $\mathcal{Z}^*$ *is multiplicative s-convex on* $[a,b]$, *then for all* $\upsilon\in[0,1]$ *we have*

$$
\left|\left((\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-\upsilon}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}\right)(\mathcal{J}(a,b;\,\mathcal{Z};\,\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right| \le (\mathcal{Z}^*(a)\mathcal{Z}^*(b))^{\frac{b-a}{4}\mathcal{V}(\upsilon,\alpha,s)}\mathcal{Z}^*\left(\frac{a+b}{2}\right)^{\frac{b-a}{2}\mathcal{W}(\upsilon,\alpha,s)},
$$

*where* $\mathcal{J}$ *is defined by* (5), *while* $\mathcal{V}$ *and* $\mathcal{W}$ *are expressed as* (8) *and* (9), *respectively*.

**Proof.** From Lemma 3.2 and the properties of multiplicative integral, we have

$$
\begin{aligned}
&\left|\left((\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-\upsilon}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}\right)(\mathcal{J}(a,b;\,\mathcal{Z};\,\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|\\
&\le \exp\left(\frac{b-a}{4}\int_0^1|1-\upsilon-(1-\eta)^{\alpha}|\ln\mathcal{Z}^*\left((1-\eta)a+\eta\frac{a+b}{2}\right)\mathrm{d}\eta\right)\\
&\quad\times\exp\left(\frac{b-a}{4}\int_0^1|\eta^{\alpha}-(1-\upsilon)|\ln\mathcal{Z}^*\left((1-\eta)\frac{a+b}{2}+\eta b\right)\mathrm{d}\eta\right).
\end{aligned}
$$

Using the multiplicative $s$-convexity of $\mathcal{Z}^*$, we obtain

$$\begin{aligned}
&\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)(\mathcal{J}(a,b;\,\mathcal{Z};\,\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}}\right|\\
&\leq \exp\left(\frac{b-a}{4}\int_0^1|1-v-(1-\eta)^\alpha|\left((1-\eta)^s\ln\mathcal{Z}^*(a)+\eta^s\ln\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)\mathrm{d}\eta\right)\\
&\quad\times\exp\left(\frac{b-a}{4}\int_0^1|\eta^\alpha-(1-v)|\left((1-\eta)^s\ln\mathcal{Z}^*\left(\frac{a+b}{2}\right)+\eta^s\ln\mathcal{Z}^*(b)\right)\mathrm{d}\eta\right)\\
&=(\mathcal{Z}^*(a)\mathcal{Z}^*(b))^{\frac{b-a}{4}\mathcal{V}(v,\alpha,s)}\mathcal{Z}^*\left(\frac{a+b}{2}\right)^{\frac{b-a}{2}\mathcal{W}(v,\alpha,s)},
\end{aligned}$$

where we have used

$$\begin{aligned}
\mathcal{V}(v,\alpha,s)&=\int_0^1|1-v-(1-\eta)^\alpha|(1-\eta)^s\mathrm{d}\eta\\
&=\int_0^1|1-v-\eta^\alpha|\eta^s\mathrm{d}\eta\\
&=\frac{(\alpha+s+1)v-\alpha+2\alpha(1-v)^{\frac{\alpha+s+1}{\alpha}}}{(s+1)(\alpha+s+1)}
\end{aligned}\tag{8}$$

and

$$\begin{aligned}
\mathcal{W}(v,\alpha,s)&=\int_0^1|\eta^\alpha-(1-v)|(1-\eta)^s\mathrm{d}\eta\\
&=\int_0^1|1-v-(1-\eta)^\alpha|\eta^s\mathrm{d}\eta\\
&=\frac{1-v}{s+1}\left(1-2\left(1-(1-v)^{\frac{1}{\alpha}}\right)^{s+1}\right)-\Phi_{(1-v)^{\frac{1}{\alpha}}}(\alpha+1,s+1),
\end{aligned}\tag{9}$$

with

$$\Phi_x(u,v)=B_x(u,v)-B_{1-x}(v,y),$$

where $B_p(.,.)$ is the incomplete beta function.

This completes the proof. □

**Corollary 3.4.** *From* Theorem 3.3, *we deduce that for any positive function $\mathcal{Z}$ that is increasing and multiplicative differentiable on $[a,b]$, if $\mathcal{Z}^*$ is multiplicatively convex on $[a,b]$, then for all $v\in[0,1]$, the following inequalities hold for fractional multiplicative integrals.*

$$\begin{aligned}
&\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)(\mathcal{J}(a,b;\,\mathcal{Z};\,\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}}\right|\\
&\leq(\mathcal{Z}^*(a)\mathcal{Z}^*(b))^{\frac{b-a}{4}\mathcal{V}(v,\alpha,1)}\mathcal{Z}^*\left(\frac{a+b}{2}\right)^{\frac{b-a}{2}\mathcal{W}(\alpha,v,1)},
\end{aligned}$$

*where*

$$\mathcal{V}(v,\alpha,1)=\frac{(\alpha+2)v-\alpha+2\alpha(1-v)^{\frac{\alpha+2}{\alpha}}}{2(\alpha+2)}\tag{10}$$

*and*

$$\mathcal{W}(\alpha, v, 1) = \frac{v(\alpha^2 + 3\alpha + 2) + 4\alpha(\alpha + 2)(1 - v)^{\frac{\alpha+1}{\alpha}} - 2\alpha(\alpha + 1)(1 - v)^{\frac{\alpha+2}{\alpha}} - (\alpha^2 + 3\alpha)}{2(\alpha + 1)(\alpha + 2)}. \tag{11}$$

**Remark 3.5.** In Corollary 3.4, if we take

1/$v = \frac{1}{3}$, then we obtain Theorem 10 from [21].

2/$v = \frac{1}{2}$, then we obtain Theorem 5 from [22].

**Corollary 3.6.** *From* Theorem 3.3, *we deduce that for any positive function $\mathcal{Z}$ that is increasing and multiplicative differentiable on* $[a, b]$, *if $\mathcal{Z}^*$ is multiplicatively s-convex on* $[a, b]$, *then for all* $v \in [0, 1]$, *the following inequalities hold for multiplicative integrals.*

$$\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)\left(\int_a^b (\mathcal{Z}(y))^{\mathrm{d}y}\right)^{\frac{1}{a-b}}\right|$$
$$\leq \left(\mathcal{Z}^*(a)\mathcal{Z}^*(b)\right)^{\frac{(s+2)v-1+2(1-v)^{s+2}}{4(s+1)(s+2)}(b-a)}\mathcal{Z}^*\left(\frac{a+b}{2}\right)^{\frac{s+1-v(s+2)+2v^{s+2}}{2(s+1)(s+2)}(b-a)}.$$

**Corollary 3.7.** *From* Theorem 3.3, *we deduce that for any positive function $\mathcal{Z}$ that is increasing and multiplicative differentiable on* $[a, b]$, *if $\mathcal{Z}^*$ is multiplicatively convex on* $[a, b]$, *then for all* $v \in [0, 1]$, *the following inequalities hold for multiplicative integrals.*

$$\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)\left(\int_a^b (\mathcal{Z}(y))^{\mathrm{d}y}\right)^{\frac{1}{a-b}}\right|$$
$$\leq \left(\mathcal{Z}^*(a)\mathcal{Z}^*(b)\right)^{\frac{(1-3v+6v^2-2v^3)(b-a)}{24}}\mathcal{Z}^*\left(\frac{a+b}{2}\right)^{\frac{(2-3v+2v^3)(b-a)}{12}}.$$

**Remark 3.8.** In Corollary 3.7, if we take

1/$v = 1$, then using the multiplicative convexity of $\mathcal{Z}^*$ we obtain Theorem 3.6 from [8].

2/$v = \frac{1}{3}$, then we obtain Corollary 3 from [21].

3/$v = \frac{1}{2}$, then we obtain Corollary 3 from [22].

**Theorem 3.9.** *Let* $\mathcal{Z} : [a, b] \to \mathbb{R}^+$ *be an increasing multiplicative differentiable function on* $[a, b]$. *If* $(\ln \mathcal{Z}^*)^q$ *is s-convex on* $[a, b]$, *where* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then for all* $v \in [0, 1]$ *we have*

$$\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)\left(\mathcal{J}(a, b; \mathcal{Z}; \alpha)\right)^{-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}}\right|$$
$$\leq \left(\mathcal{Z}^*(a)\left(\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{Z}^*(b)\right)^{\frac{b-a}{4}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}(\Psi(\alpha,p,v))^{\frac{1}{p}}},$$

*where* $\mathcal{J}$ *is defined as* (5), *and*

$$\Psi(\alpha, p, v) = \frac{(1 - v)^{p+\frac{1}{\alpha}}}{\alpha}B\left(\frac{1}{\alpha}, p + 1\right) + \frac{v^{p+1}}{\alpha(p + 1)}\,{}_2F_1\left(1 - \frac{1}{\alpha}, 1, p + 2; v\right), \tag{12}$$

*with* $B$ *and* ${}_2F_1$ *are beta and hypergeometric functions, respectively.*

**Proof.** From Lemma 3.2, properties of multiplicative integral, and Hölder's inequality, we have

$$\begin{aligned}&\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)(\mathcal{J}(a,b;\mathcal{Z};\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|\\ &\le \exp\left(\frac{b-a}{4}\left(\int_0^1|1-v-(1-\eta)^{\alpha}|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_0^1\left|\ln\mathcal{Z}^*\left((1-\eta)a+\eta\frac{a+b}{2}\right)\right|^q\mathrm{d}\eta\right)^{\frac{1}{q}}\right)\\ &\times\exp\left(\frac{b-a}{4}\left(\int_0^1|\eta^{\alpha}-(1-v)|^p\mathrm{d}\eta\right)^{\frac{1}{p}}\left(\int_0^1\left|\ln\mathcal{Z}^*\left((1-\eta)\frac{a+b}{2}+\eta b\right)\right|^q\mathrm{d}\eta\right)^{\frac{1}{q}}\right).\end{aligned}$$

Using the $s$-convexity of $(\ln\mathcal{Z}^*)^q$, we obtain

$$\begin{aligned}&\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)(\mathcal{J}(a,b;\mathcal{Z};\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|\\ &\le \exp\left(\frac{b-a}{4}(\Psi(\alpha,p,v))^{\frac{1}{p}}\left(\int_0^1\left((1-\eta)^s(\ln\mathcal{Z}^*(a))^q+\eta^s\left(\ln\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^q\right)\mathrm{d}\eta\right)^{\frac{1}{q}}\right)\\ &\times\exp\left(\frac{b-a}{4}(\Psi(\alpha,p,v))^{\frac{1}{p}}\left(\int_0^1\left((1-\eta)^s\left(\ln\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^q+\eta^s(\ln\mathcal{Z}^*(b))^q\right)\mathrm{d}\eta\right)^{\frac{1}{q}}\right)\\ &=\exp\left(\frac{b-a}{4}(\Psi(\alpha,p,v))^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left((\ln\mathcal{Z}^*(a))^q+\left(\ln\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^q\right)^{\frac{1}{q}}\right)\\ &\times\exp\left(\frac{b-a}{4}(\Psi(\alpha,p,v))^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\left(\ln\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^q+(\ln\mathcal{Z}^*(b))^q\right)^{\frac{1}{q}}\right),\end{aligned} \tag{13}$$

where we have used

$$\int_0^1|\eta^{\alpha}-(1-v)|^p\mathrm{d}\eta=\int_0^{(1-v)^{\frac{1}{\alpha}}}((1-v)-\eta^{\alpha})^p\mathrm{d}\eta+\int_{(1-v)^{\frac{1}{\alpha}}}^1(\eta^{\alpha}-(1-v))^p\mathrm{d}\eta=\Psi(\alpha,p,v).$$

Using the fact that $\mathcal{A}^q+\mathcal{B}^q\le(\mathcal{A}+\mathcal{B})^q$ for $\mathcal{A}\ge 0$, $\mathcal{B}\ge 0$ with $q\ge 1$, (13) gives

$$\begin{aligned}&\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)(\mathcal{J}(a,b;\mathcal{Z};\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|\\ &\le \exp\left(\frac{b-a}{4}(\Psi(\alpha,p,v))^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\ln\mathcal{Z}^*(a)+\ln\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)\right)\\ &\times\exp\left(\frac{b-a}{4}(\Psi(\alpha,p,v))^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\ln\mathcal{Z}^*\left(\frac{a+b}{2}\right)+\ln\mathcal{Z}^*(b)\right)\right)\\ &=\left(\mathcal{Z}^*(a)\left(\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{Z}^*(b)\right)^{\frac{b-a}{4}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}(\Psi(\alpha,p,v))^{\frac{1}{p}}}.\end{aligned}$$

The proof is completed. □

**Corollary 3.10.** *From* Theorem 3.9, *it follows that for any positive function $\mathcal{Z}$ that is increasing and multiplicative differentiable on* $[a, b]$, *if* $(\ln \mathcal{Z}^*)^q$ *is convex on* $[a, b]$, *where* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the subsequent inequalities apply for fractional multiplicative integrals across all* $\upsilon \in [0, 1]$.

$$\left|\left((\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-\upsilon}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}\right)(\mathcal{J}(a, b;\ \mathcal{Z};\ \alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|$$
$$\leq \left(\mathcal{Z}^*(a)\left(\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{Z}^*(b)\right)^{\frac{b-a}{4}\left(\frac{1}{2}\right)^{\frac{1}{q}}(\Psi(\alpha,p,\upsilon))^{\frac{1}{p}}},$$

*where $\mathcal{J}$ and $\Psi$ are defined as* (5) *and* (12), *respectively.*

**Corollary 3.11.** *From* Theorem 3.9, *if* $(\ln \mathcal{Z}^*)^q$ *is s-convex on* $[a, b]$, *where* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequalities hold for multiplicative integrals across all* $\upsilon \in [0, 1]$.

$$\left|\left((\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-\upsilon}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}\right)\left(\int_a^b(\mathcal{Z}(y))^{\mathrm{d}y}\right)^{\frac{1}{a-b}}\right|$$
$$\leq \left(\mathcal{Z}^*(a)\left(\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{Z}^*(b)\right)^{\frac{b-a}{4}((1-\upsilon)^{p+1}+\upsilon^{p+1})^{\frac{1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}}.$$

**Corollary 3.12.** *From* Theorem 3.9, *if* $(\ln \mathcal{Z}^*)^q$ *is convex on* $[a, b]$, *where* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequalities hold for multiplicative integrals across all* $\upsilon \in [0, 1]$.

$$\left|\left((\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-\upsilon}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}\right)\left(\int_a^b(\mathcal{Z}(y))^{\mathrm{d}y}\right)^{\frac{1}{a-b}}\right|$$
$$\leq \left(\mathcal{Z}^*(a)\left(\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{Z}^*(b)\right)^{\frac{b-a}{4}((1-\upsilon)^{p+1}+\upsilon^{p+1})^{\frac{1}{p}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}}.$$

**Theorem 3.13.** *Let* $\mathcal{Z} : [a, b] \to \mathbb{R}^+$ *be an increasing multiplicative differentiable function on* $[a, b]$. *If* $(\ln \mathcal{Z}^*)^q$ *is s-convex on* $[a, b]$ *for* $q > 1$, *then for all* $\upsilon \in [0, 1]$ *we have*

$$\left|\left((\mathcal{Z}(a))^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-\upsilon}(\mathcal{Z}(b))^{\frac{\upsilon}{2}}\right)(\mathcal{J}(a, b;\ \mathcal{Z};\ \alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|$$
$$\leq (\mathcal{Z}^*(a)\mathcal{Z}^*(b))^{\frac{b-a}{4}\left(\frac{\upsilon-\alpha(1-\upsilon)+2\alpha(1-\upsilon)^{\frac{\alpha+1}{\alpha}}}{\alpha+1}\right)^{1-\frac{1}{q}}(\mathcal{V}(\upsilon,\alpha,s))^{\frac{1}{q}}}$$
$$\times\left(\mathcal{Z}^*\left(\frac{a+b}{2}\right)\right)^{\frac{b-a}{2}\left(\frac{\upsilon-\alpha(1-\upsilon)+2\alpha(1-\upsilon)^{\frac{\alpha+1}{\alpha}}}{\alpha+1}\right)^{1-\frac{1}{q}}(\mathcal{W}(\upsilon,\alpha,s))^{\frac{1}{q}}},$$

*where $\mathcal{J}$ is defined by* (5), *while $\mathcal{V}$ and $\mathcal{W}$ are expressed as* (8) *and* (9), *respectively.*

**Proof.** From Lemma 3.2, modulus, and power mean inequality, we have

$$\begin{aligned}
&\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)(\mathcal{J}(a,b;\mathcal{Z};\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|\\
&\le \exp\left(\frac{b-a}{4}\left(\int_0^1|1-v-(1-\eta)^{\alpha}|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\left(\int_0^1|1-v-(1-\eta)^{\alpha}|\left|\ln\mathcal{Z}^{*}\left((1-\eta)a+\eta\frac{a+b}{2}\right)\right|^{q}\mathrm{d}\eta\right)^{\frac{1}{q}}\right)\\
&\quad\times\exp\left(\frac{b-a}{4}\left(\int_0^1|\eta^{\alpha}-(1-v)|\mathrm{d}\eta\right)^{1-\frac{1}{q}}\left(\int_0^1|\eta^{\alpha}-(1-v)|\left|\ln\mathcal{Z}^{*}\left((1-\eta)\frac{a+b}{2}+\eta b\right)\right|^{q}\mathrm{d}\eta\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Utilizing $s$-convexity of $(\ln\mathcal{Z}^{*})^{q}$, we obtain

$$\begin{aligned}
&\left|\left((\mathcal{Z}(a))^{\frac{v}{2}}\left(\mathcal{Z}\left(\frac{a+b}{2}\right)\right)^{1-v}(\mathcal{Z}(b))^{\frac{v}{2}}\right)(\mathcal{J}(a,b;\mathcal{Z};\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}}\right|\\
&\le \exp\left(\frac{b-a}{4}\left(\frac{v-\alpha(1-v)}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-v)^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}\right.\\
&\quad\left.\times\left(\int_0^1|1-v-(1-\eta)^{\alpha}|\left((1-\eta)^{s}(\ln\mathcal{Z}^{*}(a))^{q}+\eta^{s}\left(\ln\mathcal{Z}^{*}\left(\frac{a+b}{2}\right)\right)^{q}\right)\mathrm{d}\eta\right)^{\frac{1}{q}}\right)\\
&\quad\times\exp\left(\frac{b-a}{4}\left(\frac{v-\alpha(1-v)}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-v)^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}\right.\\
&\quad\left.\times\left(\int_0^1|\eta^{\alpha}-(1-v)|\left((1-\eta)^{s}\left(\ln\mathcal{Z}^{*}\left(\frac{a+b}{2}\right)\right)^{q}+\eta^{s}(\ln\mathcal{Z}^{*}(b))^{q}\right)\mathrm{d}\eta\right)^{\frac{1}{q}}\right)\\
&= \exp\left(\frac{b-a}{4}\left(\frac{v-\alpha(1-v)}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-v)^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}\right.\\
&\quad\left.\times\left(\left((\mathcal{V}(v,\alpha,s))^{\frac{1}{q}}\ln\mathcal{Z}^{*}(a)\right)^{q}+\left((\mathcal{W}(v,\alpha,s))^{\frac{1}{q}}\ln\mathcal{Z}^{*}\left(\frac{a+b}{2}\right)\right)^{q}\right)^{\frac{1}{q}}\right)\\
&\quad\times\exp\left(\frac{b-a}{4}\left(\frac{v-\alpha(1-v)}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-v)^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}\right.\\
&\quad\left.\times\left(\left((\mathcal{W}(v,\alpha,s))^{\frac{1}{q}}\ln\mathcal{Z}^{*}\left(\frac{a+b}{2}\right)\right)^{q}+\left((\mathcal{V}(v,\alpha,s))^{\frac{1}{q}}\ln\mathcal{Z}^{*}(b)\right)^{q}\right)^{\frac{1}{q}}\right),
\end{aligned} \tag{14}$$

where we have used (8), (9), and

$$\int_0^1|1-v-(1-\eta)^{\alpha}|\mathrm{d}\eta=\int_0^1|1-v-\eta^{\alpha}|\mathrm{d}\eta=\frac{v-\alpha(1-v)}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-v)^{\frac{\alpha+1}{\alpha}}.$$

Using the fact that $\mathcal{A}^q + \mathcal{B}^q \leq (\mathcal{A} + \mathcal{B})^q$ for $\mathcal{A} \geq 0$, $\mathcal{B} \geq 0$ with $q \geq 1$, (14) gives

$$\begin{aligned}
&\left| \left( (\mathcal{Z}(a))^{\frac{v}{2}} \left( \mathcal{Z}\left( \frac{a+b}{2} \right) \right)^{1-v} (\mathcal{Z}(b))^{\frac{v}{2}} \right) (\mathcal{J}(a,b;\mathcal{Z};\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}} \right| \\
&\leq \exp\left( \frac{b-a}{4} \left( \frac{v-\alpha(1-v)}{\alpha+1} + \frac{2\alpha}{\alpha+1}(1-v)^{\frac{\alpha+1}{\alpha}} \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left. \left( (\mathcal{V}(v,\alpha,s))^{\frac{1}{q}} \ln \mathcal{Z}^*(a) + (\mathcal{W}(v,\alpha,s))^{\frac{1}{q}} \ln \mathcal{Z}^*\left( \frac{a+b}{2} \right) \right) \right) \\
&\quad \times \exp\left( \frac{b-a}{4} \left( \frac{v-\alpha(1-v)}{\alpha+1} + \frac{2\alpha}{\alpha+1}(1-v)^{\frac{\alpha+1}{\alpha}} \right)^{1-\frac{1}{q}} \right. \\
&\quad \times \left. \left( (\mathcal{W}(v,\alpha,s))^{\frac{1}{q}} \ln \mathcal{Z}^*\left( \frac{a+b}{2} \right) + (\mathcal{V}(v,\alpha,s))^{\frac{1}{q}} \ln \mathcal{Z}^*(b) \right) \right) \\
&= (\mathcal{Z}^*(a)\mathcal{Z}^*(b))^{\frac{b-a}{4}\left( \frac{v-\alpha(1-v)+2\alpha(1-v)^{\frac{\alpha+1}{\alpha}}}{\alpha+1} \right)^{1-\frac{1}{q}} (\mathcal{V}(v,\alpha,s))^{\frac{1}{q}}} \\
&\quad \times \left( \mathcal{Z}^*\left( \frac{a+b}{2} \right) \right)^{\frac{b-a}{2}\left( \frac{v-\alpha(1-v)+2\alpha(1-v)^{\frac{\alpha+1}{\alpha}}}{\alpha+1} \right)^{1-\frac{1}{q}} (\mathcal{W}(v,\alpha,s))^{\frac{1}{q}}}.
\end{aligned}$$

The proof is completed. □

**Corollary 3.14.** *From* Theorem 3.13, *it follows that for any positive function* $\mathcal{Z}$ *that is increasing and multiplicative differentiable on* $[a,b]$, *if* $(\ln \mathcal{Z}^*)^q$ *is convex on* $[a,b]$, *where* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the subsequent inequalities apply for fractional multiplicative integrals across all* $v \in [0,1]$.

$$\begin{aligned}
&\left| \left( (\mathcal{Z}(a))^{\frac{v}{2}} \left( \mathcal{Z}\left( \frac{a+b}{2} \right) \right)^{1-v} (\mathcal{Z}(b))^{\frac{v}{2}} \right) (\mathcal{J}(a,b;\mathcal{Z};\alpha))^{-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}} \right| \\
&\leq (\mathcal{Z}^*(a)\mathcal{Z}^*(b))^{\frac{b-a}{4}\left( \frac{v-\alpha(1-v)+2\alpha(1-v)^{\frac{\alpha+1}{\alpha}}}{\alpha+1} \right)^{1-\frac{1}{q}} (\mathcal{V}(v,\alpha,1))^{\frac{1}{q}}} \\
&\quad \times \left( \mathcal{Z}^*\left( \frac{a+b}{2} \right) \right)^{\frac{b-a}{2}\left( \frac{v-\alpha(1-v)+2\alpha(1-v)^{\frac{\alpha+1}{\alpha}}}{\alpha+1} \right)^{1-\frac{1}{q}} (\mathcal{W}(\alpha,v,1))^{\frac{1}{q}}},
\end{aligned}$$

*where* $\mathcal{J}$, $\mathcal{V}$, *and* $\mathcal{W}$ *are defined as* (5), (10), *and* (11), *respectively*.

**Corollary 3.15.** *From* Theorem 3.13, *if* $(\ln \mathcal{Z}^*)^q$ *is s-convex on* $[a,b]$, *where* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequalities hold for multiplicative integrals across all* $v \in [0,1]$.

$$\begin{aligned}
&\left| \left( (\mathcal{Z}(a))^{\frac{v}{2}} \left( \mathcal{Z}\left( \frac{a+b}{2} \right) \right)^{1-v} (\mathcal{Z}(b))^{\frac{v}{2}} \right) \left( \int_a^b (\mathcal{Z}(y))^{\mathrm{d}y} \right)^{\frac{1}{a-b}} \right| \\
&\leq (\mathcal{Z}^*(a)\mathcal{Z}^*(b))^{\frac{b-a}{4}\left( \frac{1-2v+2v^2}{2} \right)^{1-\frac{1}{q}} \left( \frac{(s+2)v-1+2(1-v)^{s+2}}{(s+1)(s+2)} \right)^{\frac{1}{q}}} \\
&\quad \times \left( \mathcal{Z}^*\left( \frac{a+b}{2} \right) \right)^{\frac{b-a}{2}\left( \frac{1-2v+2v^2}{2} \right)^{1-\frac{1}{q}} \left( \frac{s+1-(s+2)v+2v^{s+2}}{(s+1)(s+2)} \right)^{\frac{1}{q}}}.
\end{aligned}$$

**Corollary 3.16.** *From* Theorem 3.9, *if* $(\ln \mathcal{Z}^*)^q$ *is convex on* $[a, b]$, *where* $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequalities hold for multiplicative integrals across all* $\upsilon \in [0, 1]$.

$$\left| \left( (\mathcal{Z}(a))^{\frac{\upsilon}{2}} \left( \mathcal{Z}\left( \frac{a+b}{2} \right) \right)^{1-\upsilon} (\mathcal{Z}(b))^{\frac{\upsilon}{2}} \right) \left( \int_a^b (\mathcal{Z}(y))^{\mathrm{d}y} \right)^{\frac{1}{a-b}} \right|$$

$$\leq (\mathcal{Z}^*(a)\mathcal{Z}^*(b))^{\frac{b-a}{4}\left(\frac{1-2\upsilon+2\upsilon^2}{2}\right)^{1-\frac{1}{q}}\left(\frac{1-3\upsilon+6\upsilon^2-2\upsilon^3}{6}\right)^{\frac{1}{q}}}$$

$$\times \left( \mathcal{Z}^*\left( \frac{a+b}{2} \right) \right)^{\frac{b-a}{2}\left(\frac{1-2\upsilon+2\upsilon^2}{2}\right)^{1-\frac{1}{q}}\left(\frac{2-3\upsilon+2\upsilon^3}{6}\right)^{\frac{1}{q}}}.$$

# 4 Illustrative examples and applications

In this section, we provide a practical example and visual representations to validate our study's results, along with some applications.

## 4.1 Graphical illustration

**Example 4.1.** Let us consider the function $\mathcal{Z}(y) = e^{(y+1)^{s+1}}$ for $s \in (0, 1]$ with $a = 0$ and $b = 1$, the multiplicative derivative of this function is $\mathcal{Z}^*(y) = e^{(s+1)(y+1)^s}$, which is multiplicatively $s$-convex on $[0, 1]$. Then, from Theorem 3.3, we have for $0 < \alpha \leq 1$:

$$e^{\left[(1+2^{s+1})\frac{\upsilon}{2}+\left(\frac{3}{2}\right)^{s+1}(1-\upsilon)-\alpha 2^{\alpha-1}\left(\int_0^{\frac{1}{2}}(y+1)^{s+1}\left(\frac{1}{2}-y\right)^{\alpha-1}\mathrm{d}y+\int_{\frac{1}{2}}^{1}(y+1)^{s+1}\left(y-\frac{1}{2}\right)^{\alpha-1}\mathrm{d}y\right)\right]} \leq e^{(s+1)\left(\frac{(1+2^s)\mathcal{V}(\upsilon,\alpha,s)}{4}+\frac{3^s\mathcal{W}(\upsilon,\alpha,s)}{2^{s+1}}\right)},$$

where $\mathcal{V}$ and $\mathcal{W}$ are defined as (8) and (9), respectively.

Given that the aforementioned outcome is contingent on three parameters, we will explore two scenarios where one parameter is held constant. We will then illustrate the outcome in relation to the remaining two parameters.

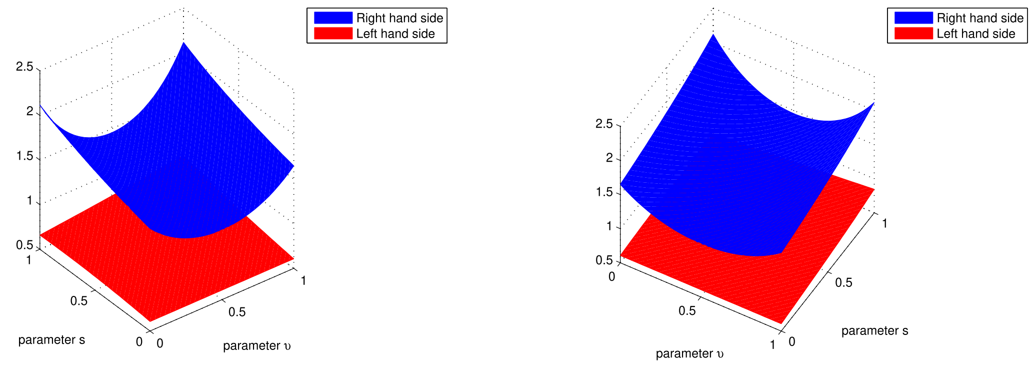
**Case 1:** By setting $\alpha = 1$, we achieve from Corollary 3.6 the following result, represented graphically in Figure 1:

$$e^{\left[(1+2^{s+1})\frac{\upsilon}{2}+\left(\frac{3}{2}\right)^{s+1}(1-\upsilon)-\frac{2^{s+2}}{s+2}\right]} \leq e^{\frac{1}{s+2}\left(\frac{(1+2^s)[(s+2)\upsilon-1+2(1-\upsilon)^{s+2}]}{4}+\frac{3^s[s+1-\upsilon(s+2)+2\upsilon^{s+2}]}{2^{s+1}}\right)}.$$

**Case 2:** When setting $s = 1$, we obtain the following result from Corollary 3.4, which is depicted graphically in Figure 2.

$$e^{\left[\frac{10\upsilon+9(1-\upsilon)}{4}-\alpha 2^{\alpha-1}\left(\frac{(\alpha+2)(5\alpha+4)+1}{2^\alpha\alpha(\alpha+1)(\alpha+2)}\right)\right]} \leq e^{\left(\frac{5(\alpha+2)\upsilon-\alpha+2\alpha(1-\upsilon)^{\frac{\alpha+2}{\alpha}}}{4(\alpha+2)}+\frac{3\upsilon(\alpha^2+3\alpha+2)+4\alpha(\alpha+2)(1-\upsilon)^{\frac{\alpha+1}{\alpha}}-2\alpha(\alpha+1)(1-\upsilon)^{\frac{\alpha+2}{\alpha}}-(\alpha^2+3\alpha)}{4(\alpha+1)(\alpha+2)}\right)}.$$

Leveraging the insights gleaned from Figures 1 and 2, it becomes evident that the right-hand term consistently surpasses the left-hand term. This observation holds consistent across the two cases under consideration, offering substantial evidence to validate our findings.




(a) (b)

**Figure 1:** $\alpha = 1$, $\upsilon \in [0,1]$, and $s \in (0,1]$. (a) View no. 1 and (b) view no. 2.

## 4.2 Some applications

Let us consider the following special means for arbitrary real numbers $a, b$.

(1) The arithmetic mean: $A(a,b) = \frac{a+b}{2}$.

(2) The harmonic mean: $H(a,b) = \frac{2ab}{a+b}$, $a, b > 0$.

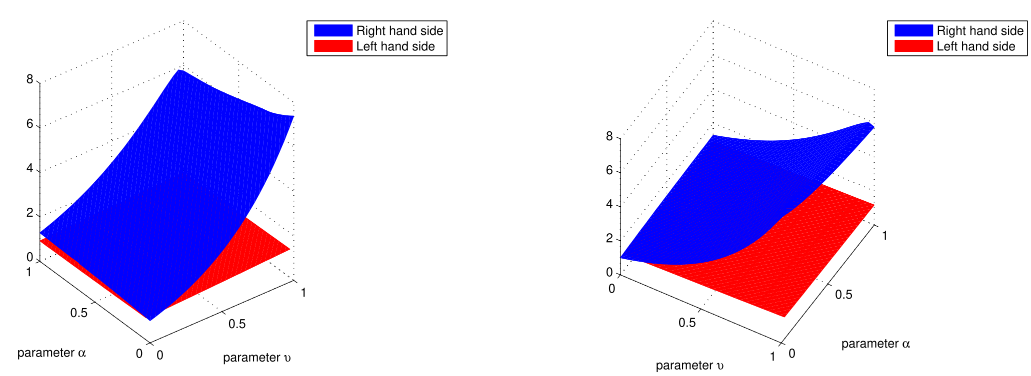
(3) The geometric mean: $G(a,b) = \sqrt{ab}$, $a, b > 0$.

(4) The logarithmic mean: $L(a,b) = \frac{b-a}{\ln b - \ln a}$, $a, b > 0$ and $a \neq b$.

(5) The $p$-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0$, $a \neq b$, and $p \in \mathbb{R} \setminus \{-1, 0\}$.

**Proposition 4.2.** *Let $a$ and $b$ be two positive real numbers with $a < b$ and let $\upsilon \in [0,1]$, then we have*

$$e^{\upsilon H^{-1}(a^p,b^p)+(1-\upsilon)H^{-p}(a,b)-G^{-2p}(a,b)L_p^p(a,b)} \le \left(e^{p(1-3\upsilon+6\upsilon^2-2\upsilon^3)\left(\left(\frac{1}{b}\right)^{p-1}+\left(\frac{1}{a}\right)^{p-1}\right)} e^{2(2-3\upsilon+2\upsilon^3)\left(\frac{a+b}{2ab}\right)^{p-1}}\right)^{p\frac{b-a}{24ab}}.$$




(a) (b)

**Figure 2:** $s = 1$, $\upsilon \in [0,1]$, and $\alpha \in (0,1]$. (a) View no. 1 and (b) view no. 2.

**Proof.** Using Corollary 3.7 on $\left[\frac{1}{b}, \frac{1}{a}\right]$ we obtain

$$\begin{aligned} &\left|\left(\mathcal{Z}\left(\frac{1}{b}\right)\right)^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)\right)^{1-\upsilon}\left(\mathcal{Z}\left(\frac{1}{a}\right)\right)^{\frac{\upsilon}{2}}\left(\int_{\frac{1}{b}}^{\frac{1}{a}}(\mathcal{Z}(y))^{\mathrm{d}y}\right)^{-\frac{1}{\frac{1}{a}-\frac{1}{b}}}\right| \\ &\leq \left(\mathcal{Z}^{*}\left(\frac{1}{b}\right)\mathcal{Z}^{*}\left(\frac{1}{a}\right)\right)^{\frac{(1-3\upsilon+6\upsilon^2-2\upsilon^3)\left(\frac{1}{a}-\frac{1}{b}\right)}{24}}\mathcal{Z}^{*}\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^{\frac{(2-3\upsilon+2\upsilon^3)\left(\frac{1}{a}-\frac{1}{b}\right)}{12}}. \end{aligned} \tag{15}$$

Clearly we have $\frac{\frac{1}{b}+\frac{1}{a}}{2} = \frac{a+b}{2ab} = H^{-1}(a,b)$, $\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab}$ and for $\mathcal{Z}(y) = e^{y^p}$ we have

$$\begin{aligned} \left(\int_{\frac{1}{b}}^{\frac{1}{a}}(\mathcal{Z}(y))^{\mathrm{d}y}\right)^{-\frac{1}{\frac{1}{a}-\frac{1}{b}}} &= \left(\exp\left\{\int_{\frac{1}{b}}^{\frac{1}{a}}\ln(e^{y^p})dt\right\}\right)^{-\frac{ab}{b-a}} \\ &= \exp\left(-\frac{ab}{b-a}\left\{\int_{\frac{1}{b}}^{\frac{1}{a}} y^p\,dt\right\}\right) \\ &= \exp\left(-\frac{ab}{b-a}\left\{\frac{1}{p+1}\left(\left(\frac{1}{a}\right)^{p+1}-\left(\frac{1}{b}\right)^{p+1}\right)\right\}\right) \\ &= \exp\left(-\frac{ab}{b-a}\left\{\frac{1}{p+1}\left(\frac{1}{a^{p+1}}-\frac{1}{b^{p+1}}\right)\right\}\right) \\ &= \exp\left(-\frac{ab}{b-a}\left\{\frac{1}{p+1}\left(\frac{b^{p+1}-a^{p+1}}{a^{p+1}b^{p+1}}\right)\right\}\right) \\ &= \exp\left(-\frac{ab}{a^{p+1}b^{p+1}}\left\{\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right\}\right) \\ &= \exp\left(-\frac{1}{a^p b^p}\left\{\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right\}\right) \\ &= \exp\left(-\left\{\frac{1}{ab}\right\}^p L_p^p(a,b)\right) \\ &= \exp\left(-\left\{\frac{1}{(\sqrt{ab})^2}\right\}^p L_p^p(a,b)\right) \\ &= \exp(-\{\sqrt{ab}\}^{-2p} L_p^p(a,b)) \\ &= \exp(-G^{-2p}(a,b) L_p^p(a,b)). \end{aligned} \tag{16}$$

On the other hand, we have

$$\mathcal{Z}^{*}(y) = e^{(\ln\mathcal{Z}(y))'} = e^{\frac{\mathcal{Z}'(y)}{\mathcal{Z}(y)}} = e^{\frac{(e^{y^p})'}{e^{y^p}}}e^{\frac{(y^p)'e^{y^p}}{e^{y^p}}} = e^{(y^p)'} = e^{py^{p-1}}. \tag{17}$$

Let us substitute (16) and (17) into (15), and using the fact that

$$\left(\mathcal{Z}\left(\frac{1}{b}\right)\right)^{\frac{\upsilon}{2}}\left(\mathcal{Z}\left(\frac{1}{a}\right)\right)^{\frac{\upsilon}{2}} = \left(e^{\frac{1}{b^p}+\frac{1}{a^p}}\right)^{\frac{\upsilon}{2}} = \left(e^{\frac{a^p+b^p}{a^pb^p}}\right)^{\frac{\upsilon}{2}} = e^{\upsilon\frac{a^p+b^p}{2a^pb^p}} = e^{\upsilon\left(\frac{2a^pb^p}{a^p+b^p}\right)^{-1}} = e^{\upsilon H^{-1}(a^p,b^p)}$$

and

$$\left(\mathcal{Z}\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)\right)^{1-\upsilon} = \left(\mathcal{Z}\left(\frac{a+b}{2ab}\right)\right)^{1-\upsilon} = \left(e^{\left(\frac{a+b}{2ab}\right)^{p}}\right)^{1-\upsilon} = e^{(1-\upsilon)H^{-P}(a,b)},$$

we obtain

$$\begin{aligned}
&\left|e^{\upsilon H^{-1}(a^p,b^p)+(1-\upsilon)H^{-P}(a,b)-G^{-2p}(a,b)L_p^p(a,b)}\right| \\
&\leq \left(e^{p\frac{(1-3\upsilon+6\upsilon^2-2\upsilon^3)}{24}\left(\frac{b-a}{ab}\right)\left(\left(\frac{1}{b}\right)^{p-1}+\left(\frac{1}{b}\right)^{p-1}\right)+p\frac{(2-3\upsilon+2\upsilon^3)}{12}\left(\frac{b-a}{ab}\right)\left(\frac{a+b}{2ab}\right)^{p-1}}\right) \\
&= \left(e^{(1-3\upsilon+6\upsilon^2-2\upsilon^3)\left(\left(\frac{1}{b}\right)^{p-1}+\left(\frac{1}{b}\right)^{p-1}\right)+2(2-3\upsilon+2\upsilon^3)\left(\frac{a+b}{2ab}\right)^{p-1}}\right)^{p\frac{b-a}{24ab}},
\end{aligned}$$

which is the required result. □

**Proposition 4.3.** *Let $a$ and $b$ be two positive real numbers with $a < b$, then we have*

$$e^{\frac{H^{-1}(a,b)+A^{-1}(a,b)-2L^{-1}(a,b)}{2}} \leq e^{-\left(\frac{1}{a^2}+\frac{8}{(a+b)^2}+\frac{1}{b^2}\right)\frac{b-a}{8\sqrt{6}}}.$$

**Proof.** The assertion follows from Corollary 3.12 with $p = q = 2$ and $\upsilon = \frac{1}{2}$, applied to the function $\mathcal{Z}(y) = e^{\frac{1}{y}}$ where $\mathcal{Z}^{*}(y) = e^{-\frac{1}{y^2}}$ and $\left(\int_a^b \mathcal{Z}(y)^{\mathrm{d}y}\right)^{\frac{1}{a-b}} = \exp\{-L^{-1}(a,b)\}$. □

## 5 Conclusion

In conclusion, the introduced one-parameter fractional multiplicative integral identity has proven to be a versatile tool, allowing us to establish a diverse range of inequalities tailored for multiplicative $s$-convex mappings. Our exploration not only unveiled novel findings but also refined existing results, highlighting the significance and potential of this mathematical framework. The illustrative example accompanied by graphical representations further enriches our understanding and serves as a visual testament to the validity of our results. Moreover, the demonstrated applications to special means of real numbers within the realm of multiplicative calculus underscore the practical utility and broader applicability of the derived outcomes. This work contributes significantly to the ongoing discourse on multiplicative calculus, offering new insights and paving the way for further research and advancements in this domain.

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