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Optimal control for a variable-order diffusion-wave equation with a reaction term; A numerical study

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ABSTRACT

In this paper, optimal control for a variable-order diffusion-wave equation with a reaction term is numerically studied, where the variable-order operator is defined in the sense of Caputo proportional constant. Necessary optimality conditions for the control problem are derived. Existence and uniqueness for the solutions of fractional optimal control problem are derived. The nonstandard weighted average finite difference method and the nonstandard leap-frog method are developed to study numerically the proposed problem. Moreover, the stability analysis of the methods is proved. Finally, in order to characterise the memory property of the proposed model, three test examples are given. It is found that the nonstandard weighted average finite difference method can be applied to study such variable-order fractional optimal control problems simply and effectively.

1. Introduction

Partial differential equations (PDEs) are frequently used in many areas of the natural and social sciences to mathematically model phenomena and processes. The wave differential equation displays a sample partial differential model for describing the communication between reaction apparatuses, acoustic waves, chemical waves, convection effects, diffusion transports, and modelling of dynamics. The analysis of this equation is important to understanding different numerical and analytical techniques. This model also provides the basis for a classical wave theory. Several researchers have examined wave models for various mechanical wave problems. In recent years, the fractional order derivative has been used to improve the accuracy and suitability of mathematical systems. The fractional order derivatives of systems with the effects of historical memory, inherited properties of materials and processes cannot be described by the integer-order derivatives of those systems, .[1,2]. Therefore, it is no surprise that many researchers have dedicated their attention to the development of a new definition of the fractional order derivative, ranging from Riemann Liouville to Caputo.[3,4,5]. Generally, the difference between various definitions is chosen special kernels and the form of a differential operator. More recently, Baleanu et al., in[6] introduced a new type of derivative known as a hybrid fractional operator, which can be expressed as a linear combination of the Caputo fractional derivative and the Riemann-Liouville fractional integral. In recent years, the theory of fractional optimal control of PDEs has been widely applied in various areas such as science, engineering, and economics. In[7] and,[8] Agrawal suggested a generalized formulation and approach for solving the fractional scheme of optimal control problems using the Lagrange multiplier technique and the fractional variation principle. Several studies have analysed many optimal control of integer-order PDEs problems, but there have been few studies on the fractional order optimal control of PDEs, .[9–18] In[11] and,[12] Mophou has studied the fractional optimal control diffusion equation with and without state constraints. In,[13] Mophou and Joseph focused on the controlled fractional diffusion wave equation involving Riemann-Liouville fractional derivative with a final observation. In,[15] they investigated the fractional optimal control wave equation, which also has missing initial conditions and includes the fractional Riemann-Liouville derivative. The optimal control for fractional order wave equation has been discussed in a few publications. In general in this work, we study the optimal control of the source function (i.e., external forces) for the variable-order diffusion-wave equation

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(VOD-wave) with a reaction term. Specifically, optimal control is used to determine the minimum effects of the source of waves on the medium carrying them that must be delivered to minimize the wave function. In addition, we discuss the formulation and theoretical studies for a fractional optimal control of the VOD-wave with a reaction term. Therefore, we focus on the optimal control of variable-order partial differential equations (OCVPDEs), which are the fractional operators that consider the order as a function of time and space. The variable order calculus is a natural extension of the constant order calculus, i.e., integer or fractional. In this sense, the orders are functions of any variable, i.e., space variable, time variable, or any other variable, .[19–24] The variable order fractional derivatives can capture the fading memory and crossover behaviour found in many PDEs problems. Therefore, the memory effect appears in these problems that were using variable-order fractional derivatives instead of integer-order derivatives to provide better appropriate exact solutions. In,[19] Samko and Ross suggested the idea of a variable order operator and investigated the properties of integration and differentiation of the variable-order Riemann-Liouville type. There are various definitions of variable order differential operators in the literature, each with a specific meaning to suit desired goals; the majority of these definitions are extensions of fractional calculus definitions, as Riemann-Liouville, Caputo, and Riesz, .[19] This paper's major goal is to extend VOD-wave with a reaction term given in[15] by applying the new hybrid fractional operator derivative. This operator has two cases, there are the (PC) stands for Proportional-Caputo and the (CPC) stands for Constant-Proportional-Caputo. The new hybrid variable order fractional (CPC) operator is a general case of the variable order fractional Caputo operator. We will introduce a control variable to minimize the objective cost. To approximate the obtained fractional optimality system, two numerical approaches will be constructed. These methods are: the nonstandard leap-frog method (NLFM) and the nonstandard weighted average finite difference method (NWAFDM). The stability analysis of the NWAFDM will be proved. Non-standard finite-difference methods have been applied to numerically solve ordinary differential equations (ODEs) and partial differential equations (PDEs),. In [25]and[26], Mickens has illustrated the non-standard finite difference methods to solve PDE applications such as wave propagation, scattering and Maxwell's Equations.

Therefore, we use the NWAFDM to study numerically one and two-dimensional VOD-wave with a reaction term. Based on the weight factor value, we have three different cases of the NWAFDM, which are an explicit, an implicit and Crank-Nicholson method [27,28].

In this work, we developed a new numerical scheme NWAFDM for solving the optimal control for VOD-wave with reaction term (OCVOD-wave).

This paper is consisted of six sections. In Section 2, we reviewed fundamental fractional calculus definitions. In Section 3, optimal control for variable-order diffusion-wave with a reaction term equation is introduced. Additionally, necessary and sufficient optimality conditions are derived, as are the existence and uniqueness of the optimal solution. In Section 4, the nonstandard weighted average finite difference method of the OCVOD-wave equation is proposed. Numerical simulations of the proposed optimal control diffusion-wave with a reaction term problem are given in Section 5. In Section 6, the conclusions are given. An appendix is shown in the final section in order to give the details of the construction of NLFM using variable-order of CPC derivative.

2. Preliminaries and notations

In this section, we review some fundamental fractional calculus definitions employed in the remaining sections of this paper.

2.1. Fractional calculus definitions

The left and right variable-order of Caputo fractional derivative of $W(.,t)$ defined on $[a,b]$ at d^{th} dimensions is defined as follows,[29]:

$$ {}_{a}^{C}D_{t}^{\beta(.,t)}W(.,t)=\frac{1}{\Gamma(n-\beta(.,t))}\int_{a}^{t}(t-\nu)^{n-\beta(.,t)-1}\ \frac{d^{n}W(.,\nu)}{d\nu^{n}}\ d\nu,\ (Left\ Caputo), \tag{2.1}$$

$$ {}_{t}^{C}D_{b}^{\beta(.,t)}W(.,t)\ =\ \frac{(-1)^{n}}{\Gamma(n-\beta(.,t))}\int_{t}^{b}(\nu-t)^{n-\beta(.,t)-1}\ \frac{d^{n}W(.,\nu)}{d\nu^{n}}\ d\nu,\ (Right\ Caputo), \tag{2.2}$$

where, $n-1<\beta(.,t)\leq n,\ n\in N,$ and $\Gamma(.)$ is the gamma function.

For an integrable function $W(.,t)$ defined on an interval $[a,b]$ and $n-1<\beta(.,t)\leq n,\ n\in N,$ then the left and right variable-order of Riemann-Liouville fractional integrals of order $\beta(.,t)$ are defined as follows,[29]:

$$ {}_{a}^{RL}I_{t}^{\beta(.,t)}W(.,t)=\frac{1}{\Gamma(\beta(.,t))}\int_{a}^{t}(t-\nu)^{\beta(.,t)-1}W(.,\nu)d\nu,\quad (Left\ RL), \tag{2.3}$$

$$ {}_{t}^{RL}I_{b}^{\beta(.,t)}W(.,t)=\frac{1}{\Gamma(\beta(.,t))}\int_{t}^{b}(\nu-t)^{\beta(.,t)-1}W(.,\nu)d\nu,\quad (Right\ RL). \tag{2.4}$$

For the left and right variable-order of Riemann-Liouville fractional derivatives of order $\beta(.,t)$ are defined as follows,[29]:

$$ {}_{a}^{RL}D_{t}^{\beta(.,t)}W(.,t)=\frac{d^{n}}{dt^{n}}\ {}_{a}^{RL}I_{t}^{n-\beta(.,t)}W(.,t),\quad (Left\ RL), \tag{2.5}$$

$$ {}_{t}^{RL}D_{b}^{\beta(.,t)}W(.,t)=(-1)^{n}\ \frac{d^{n}}{dt^{n}}\ {}_{t}^{RL}I_{b}^{n-\beta(.,t)}W(.,t),\quad (Right\ RL). \tag{2.6}$$

The relation between the variable-order of Riemann-Liouville and Caputo fractional derivatives,[29,30]:

$$ {}_{a}^{RL}D_{t}^{\beta(.,t)}W(.,t)={}_{a}^{C}D_{t}^{\beta(.,t)}\ W(.,t)+\sum_{k=0}^{n-1}\frac{W^{(k)}(a)}{\Gamma(k-\beta(.,t)+1)}\ (t-a)^{k-\beta(.,t)}, \tag{2.7}$$

$$ {}_{t}^{RL}D_{b}^{\beta(.,t)}W(.,t)\ =\ {}_{t}^{C}D_{b}^{\beta(.,t)}\ W(.,t)\ +\sum_{k=0}^{n-1}\frac{W^{(k)}(a)}{\Gamma(k-\beta(.,t)+1)}\ \ (b-t)^{k-\beta(.,t)} \tag{2.8}$$

Therefore,

$$ if\ W(a)=W^{'}(a)=\cdots=W^{n-1}(a)=0,\ then,\ {}_{a}^{RL}D_{t}^{\beta(.,t)}W(.,t)=\ {}_{a}^{C}D_{t}^{\beta(.,t)}\ W(.,t), \tag{2.9}$$

and

$$ if\ W(b)=W^{'}(b)=\cdots=W^{n-1}(b)=0,\ then,\ {}_{t}^{RL}D_{b}^{\beta(.,t)}W(.,t)=\ {}_{t}^{C}D_{b}^{\beta(.,t)}\ W(.,t). \tag{2.10}$$

The new type of variable-order is defined as a variable-order of hybrid fractional derivative by combining the definitions of the proportional and Caputo derivatives,[6]:

In the left case, as follows:

$$ {}_{a}^{CP}D_{t}^{\beta(.,t)}W(.,t)=\frac{1}{\Gamma(n-\beta(.,t))}\int_{a}^{t}(t-\nu)^{n-\beta(.,t)-1}\Big(K_{1}(\beta(.,t),\nu)W(.,\nu) +K_{0}(\beta(.,t),\nu)\frac{d^{n}W(.,\nu)}{d\nu^{n}}\Big)d\nu. \tag{2.11}$$

In the right case, as follows:

$$ {}_{t}^{CP}D_{b}^{\beta(.,t)}W(.,t)=\frac{(-1)^{n}}{\Gamma(n-\beta(.,t))}\int_{t}^{b}(\nu-t)^{n-\beta(.,t)-1}\left(K_{1}(\beta(.,t),\nu)W(.,\nu)+K_{0}(\beta(.,t),\nu)\frac{d^{n}W(.,\nu)}{d\nu^{n}}\right)d\nu, \quad (2.12) $$

where $K_0(\beta(.,t),t) = (\beta(.,t)-1)t^{(2-\beta(.,t))}$ and $K_1(\beta(.,t),t) = (2-\beta(.,t))t^{(\beta(.,t)-1)}$ are functions of the variable t and the parameter $\beta(.,t) \in [1, 2]$, which satisfy the following conditions for all $t \in R$,[6] :

$$ \lim_{\beta\to 1^+} K_0(\beta(.,t),t) = 0,\ \lim_{\beta\to 2^-} K_0(\beta(.,t),t) = 1,\ K_0(\beta(.,t),t) \neq 0,\ \beta \in (1,2], $$

$$ \lim_{\beta\to 1^+} K_1(\beta(.,t),t) = 1,\ \lim_{\beta\to 2^-} K_1(\beta(.,t),t) = 0,\ K_1(\beta(.,t),t) \neq 0,\ \beta \in [1,2). $$

In a specific case where K_0 and K_1 depend only on $\beta(.,t)$ only, at $1 < \beta(.,t) \leq 2$.

The variable-order of constant-proportional-Caputo hybrid fractional operator is defined on an $[a,b]$ as,[6]:

In the left case, as follows:

$$ {}_{a}^{CPC}D_{t}^{\beta(.,t)}W(.,t) = K_1(\beta(.,t))\ {}_{a}^{RL}I_{t}^{2-\beta(.,t)}\ W(.,t) + K_0(\beta(.,t))\ {}_{a}^{C}D_{t}^{\beta(.,t)}\ W(.,t). \quad (2.13) $$

In the right case, as follows:

$$ {}_{t}^{CPC}D_{b}^{\beta(.,t)}\ W(.,t) = K_1(\beta(.,t))\ {}_{t}^{RL}I_{b}^{2-\beta(.,t)}\ W(.,t) + K_0(\beta(.,t))\ {}_{t}^{C}D_{b}^{\beta(.,t)}W(.,t). \quad (2.14) $$

Additionally, in this study, we consider the kernels in this study as follows:

$$ K_0(\beta(.,t)) = (\beta(.,t)-1)\ C^{2\beta(.,t)}\ w^{2-\beta(.,t)},\quad K_1(\beta(.,t)) = (2-\beta(.,t))\ w^{(\beta(.,t)-1)}, $$

where w and C are constants. The variable-order of CPC fractional integrals are defined on an $[a,b]$, as[6] :

In the left case, as follows:

$$ {}_{a}^{CPC}I_{t}^{\beta(.,t)}W(.,t) = \frac{1}{K_0(\beta(.,t))}\int_{a}^{t} exp\left[\frac{-K_1(\beta(.,t))}{K_0(\beta(.,t))}(t-\nu)\right]\ {}_{a}^{RL}D_{\nu}^{2-\beta(.,t)}W(.,\nu)\ d\nu. \quad (2.15) $$

In the right case, as follows:

$$ {}_{t}^{CPC}I_{b}^{\beta(.,t)}W(.,t) = \frac{1}{K_0(\beta(.,t))}\int_{t}^{b} exp\left[\frac{-K_1(\beta(.,t))}{K_0(\beta(.,t))}(\nu-t)\right]\ {}_{\nu}^{RL}D_{b}^{2-\beta(.,t)}W(.,\nu)\ d\nu. \quad (2.16) $$

Where ${}^{RL}D^{2-\beta(.,t)}$ refers to the variable-order of Riemann-Liouville at order $(2-\beta(.,t))$.

Lemma 2.1. [31] *For any functions W and H defined on* $[a,b]$ *and* $\beta(.,t) > 0$, *we have:* $\int_a^b W(t)\ {}_{a}^{RL}I_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b H(t)\ {}_{t}^{RL}I_{b}^{\beta(.,t)}\ W(t)\ dt.$

Lemma 2.2. [13] *For any functions W and H defined on* $[a,b]$ *and* $1 < \beta(.,t) \leq 2$, *we have:*

$$ \int_a^b W(t)\ {}_{a}^{RL}D_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b H(t)\ {}_{t}^{C}D_{b}^{\beta(.,t)}\ W(t)\ dt. $$

By, (2.10),

$$ \int_a^b W(t)\ {}_{a}^{RL}D_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b H(t)\ {}_{t}^{RL}D_{b}^{\beta(.,t)}\ W(t)\ dt. $$

And

$$ \int_a^b W(t)\ {}_{a}^{C}D_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b H(t)\ {}_{t}^{RL}D_{b}^{\beta(.,t)}\ W(t)\ dt. $$

By, (2.10),

$$ \int_a^b W(t)\ {}_{a}^{C}D_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b H(t)\ {}_{t}^{C}D_{b}^{\beta(.,t)}\ W(t)\ dt. $$

Lemma 2.3. *For any functions W and H defined on* $[a,b]$ *and* $1 < \beta(.,t) \leq 2$, *we have:*

$$ \int_a^b W(t)\ {}_{a}^{CPC}D_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b W(t)\ K_1(\beta(.,t))\ {}_{a}^{RL}I_{t}^{2-\beta(.,t)}\ H(t)\ dt + \int_a^b W(t)\ K_0(\beta(.,t))\ {}_{a}^{C}D_{t}^{\beta(.,t)}H(t)\ dt, $$

By lemme (2.1) and (2.2),

$$ \int_a^b W(t)\ {}_{a}^{CPC}D_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b H(t)\ K_1(\beta(.,t))\ {}_{t}^{RL}I_{b}^{2-\beta(.,t)}\ W(t)\ dt + \int_a^b H(t)\ K_0(\beta(.,t))\ {}_{t}^{C}D_{b}^{\beta(.,t)}\ W(t)\ dt = \int_a^b H(t)\ {}_{t}^{CPC}D_{b}^{\beta(.,t)}\ W(t)\ dt. $$

Lemma 2.4. *For any functions W and H defined on* $[a,b]$ *and* $1 < \beta(.,t) \leq 2$, *we have:*

$$ \int_a^b W(t)\ {}_{a}^{CPC}I_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b H(t)\ {}_{t}^{CPC}I_{b}^{\beta(.,t)}W(t)\ dt. $$

Proof. By using (2.15), we have:

$$ \int_a^b W(t)\ {}_{a}^{CPC}I_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b W(t)\ \frac{1}{K_0(\beta(.,t))}\int_a^t exp\left[\frac{-K_1(\beta(.,t))}{K_0(\beta(.,t))}(t-\nu)\right]\ {}_{a}^{RL}D_{\nu}^{2-\beta(.,t)}H(\nu)\ d\nu. $$

From lemme (2.2), and by changing the order of integrals, we have:

we obtain: $\int_a^b W(t)\ {}_{a}^{CPC}I_{t}^{\beta(.,t)}\ H(t)\ dt = \int_a^b H(\nu)\ {}_{\nu}^{CPC}I_{b}^{\beta(.,t)}W(\nu)\ d\nu$ □

$$\int_a^b W(t)\ {}^{CPC}_{a}I_t^{\beta(.,t)}\ H(t)\ dt = \int_a^b \frac{1}{K_0(\beta(.,t))}\int_\nu^b exp\left[\frac{-K_1(\beta(.,t))}{K_0(\beta(.,t))}(t-\nu)\right] W(t)\ {}^{RL}_{a}D_\nu^{2-\beta(.,t)}G(\nu)\ dt\ d\nu$$

$$= \int_a^b H(\nu)\ \frac{1}{K_0(\beta(.,t))}\int_\nu^b exp\left[\frac{-K_1(\beta(.,t))}{K_0(\beta(.,t))}(t-\nu)\right]\ {}^{RL}_{\nu}D_b^{2-\beta(.,t)}W(\nu)\ dt\ d\nu,$$

3. Optimal control of the VOD-wave equation

Assuming that, the spatial domain is $\Omega = (0,L)^d$, $d \geq 1$, with boundary $\Gamma := \partial\Omega$. Define $Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$over a finite period of time $T > 0$. Moreover, we consider the standard optimal control problem,[32] to minimize a quadratic tracking cost functional,

$$J(w,u) = \frac{1}{2}\| w - g \|^2_{L^2(Q)} + \frac{\gamma}{2}\| u \|^2_{L^2(Q)} \tag{3.1}$$

subject to the OCVOD-Wave with reaction term,[15] :

$$\begin{cases} {}^{CPC}D_t^{\beta(x,t)}w - \Delta w + \alpha w = f + u, \ in\ Q, \\ w = 0\ on\ \Sigma, \\ w(x,0) = w_0\ in\ \Omega, \\ {}^{CPC}D^{\beta(x,t)-1}w(x,0) = w_1\ in\ \Omega, \end{cases} \tag{3.2}$$

where $1 < \beta(x,t) < 2$, $\alpha > 0$ is the reaction coefficient, $g \in L^2(Q)$ is the desired tracking trajectory, $\gamma > 0$ represents the weight parameter, $f \in L^2(Q)$ denotes the source function, $u \in U := L^2(Q)$ is the optimal control function to minimize the effects of f on a wave at the medium, and the initial conditions $w_0 \in H^1_0(\Omega)$ and $w_1 \in L^2(Q)$ for more details see,[32] [11–15,33].

3.1. The formal variable-order Lagrange method

We reformulate the optimization problem (3.1)-(3.2). Using a kind of Lagrange multiplier function given in,[34] see also.[32,35,36]

Theorem 3.1. *Let (w,u) be the optimal solution to the optimal control problem (3.1)-(3.2). According to,[36] there is an P, satisfying (in the weak sense) the adjoint equation, it is given below. Moreover, the following system of partial differential equations and inequalities must be satisfied:*

State equation:

$$\begin{cases} {}^{CPC}_{0}D_t^{\beta(x,t)}\ w - \Delta\ w + \alpha\ w = f + u\ in\ Q,\ w = 0\ on\ \Sigma, \\ w(x,0) = w_0\ in\ \Omega,\ \ {}^{CPC}_{0}D_t^{\beta(x,t)-1}w(x,0) = w_1\ in\ \Omega. \end{cases} \tag{3.3}$$

Adjoint equation:

$$\begin{cases} {}^{CPC}_{t}D_T^{\beta(x,t)}\ P - \Delta\ P + \alpha\ P = w - g\ in\ Q,\ P = 0\ on\ \Sigma, \\ P(x,T) = 0\ in\ \Omega,\ \frac{\partial}{\partial t}\ {}^{CPC}_{t}I_T^{2-\beta(x,t)}P(x,T) = 0\ in\ \Omega. \end{cases} \tag{3.4}$$

Maximum conditions:

$$u = \frac{-P}{\gamma}. \tag{3.5}$$

Proof. According to[36] and.[34] We define the Lagrangian function as follows:

$$L(w,P,u) = J(w,u) - \iint_Q \left({}^{CPC}_{0}D_t^{\beta(x,t)}\ w - \Delta\ w + \alpha\ w - f - u\right)P_1\ dx\ dt - \iint_\Sigma w\ P_2\ ds\ dt \tag{3.6}$$

where P_1, P_2 are the Lagrange multipliers functions defined on Ω and Σ. The state variable w vanishes on Σ, since the boundary conditions on $w\ |_\Sigma = 0$ is already accounted for space $w \in H^1(\Omega)$. By using fractional Green's formula, [37]

Let $w(x,t)$ is the solution of the (3.1)-(3.2). Following that, for each $P_1(x,t) \in C^\infty(\Omega)$ such that $P_1(x,T) = 0$ in Ω and $P_1 = 0$ on Σ we have:

$$\int\int_Q \left({}^{CPC}_{0}D_t^{\beta(x,t)}w \quad -\Delta w\right)\ P_1\ dx\ dt = \int_\Omega P_1(x,T)\ \frac{\partial}{\partial t^0}{}^{CPC}I_t^{2-\beta}w(x,T) - {}^{CPC}_{0}I_t^{2-\beta}w(x,T)\ \frac{\partial P_1(x,T)}{\partial t}\ dx$$
$$- \int_\Omega P_1(x,0^+)\ \ \frac{\partial}{\partial t^0}{}^{CPC}I_t^{2-\beta}w(x,0^+)\ dx + \int_{\Omega 0}^{CPC} I_t^{2-\beta}w(x,0^+)\ \frac{\partial P_1(x,0^+)}{\partial t}\ dx$$
$$+ \int\int_Q \left({}^{CPC}_{t}D_T^{\beta(x,t)}P_1 - \Delta P_1\right)\ w\ dx\ dt.$$

We can obtain from (3.6) and lemma (2.4):

We can derive the following necessary optimality conditions:

$$
\begin{aligned}
L(w,P,u) = \; & J(w,u) - \int_\Omega P_1(x,T)\frac{\partial}{\partial t}\, {}_0^{CPC}I_t^{2-\beta} w(x,T)dx + \int_\Omega P_1(x,0^+)\frac{\partial}{\partial t}\, {}_0^{CPC}I_t^{2-\beta} w(x,0^+)dx \\
& + \int_\Omega {}_0^{CPC}I_t^{2-\beta} w(x,T)\frac{\partial P_1(x,T)}{\partial t}dx - \int_\Omega {}_a^{CPC}I_t^{2-\beta} w(x,0^+)\frac{\partial P_1(x,0^+)}{\partial t}dx \\
& - \iint_Q \left({}_t^{CPC}D_T^{\beta(x,t)} P_1 - \Delta P_1\right) w \; dx\; dt - \iint_Q (\alpha w - f - u) P_1 \; dx\; dt.
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
L(w,P,u) = \; & J(w,u) + \int_\Omega P_1(x,0^+)\frac{\partial}{\partial t}\, {}_a^{CPC}I_t^{2-\beta} w(x,0^+)dx + \int_\Omega w(x,T)\, {}_t^{CPC}I_T^{2-\beta}\frac{\partial P_1(x,T)}{\partial t}dx \\
& - \int_\Omega {}_a^{CPC}I_t^{2-\beta} w(x,0^+)\frac{\partial P_1(x,0^+)}{\partial t}dx - \iint_Q \left({}_t^{CPC}D_T^{\beta} P_1 - \Delta P_1\right) w \; dx\; dt \\
& - \iint_Q (\alpha w - f - u) P_1 \; dx\; dt.
\end{aligned} \tag{3.8}
$$

First condition:

$$D_w L(\overline{w},\overline{P},\overline{u})\,\overline{h} = 0, \; \forall\, \overline{h} \in H^1(\Omega), \; \overline{h}(0) = 0,$$

where $\overline{h} = w - \overline{w}$. The variable-order adjoint system resulted from $D_w L$, which it is the first derivative of the Lagrangian with respect to w, we note that for all $\overline{h} \in C^\infty(\Omega)$ so that $\overline{h} = \partial_V \overline{h} = 0$ vanish on Ω and Σ. We denote the element of surface area by ds and the outward unit normal to Γ at $x \in \Gamma$ by $V(x)$.

$$
\begin{aligned}
D_w L(\overline{w},\overline{P},\overline{u})\; \overline{h} = & \iint_Q \left(\overline{w} - g - {}_t^{CPC}D_T^{\beta(x,t)} P_1 + \Delta P_1 - \alpha\, P_1\right) \overline{h}\; dx\; dt \\
& + \int_\Omega {}_t^{CPC}I_T^{2-\beta}\frac{\partial}{\partial t} P_1(x,T)\; \overline{h}\, dx \; = 0
\end{aligned}
$$

First, we obtain that:

$$\iint_Q \left(\overline{w} - g - {}_t^{CPC}D_T^{\beta(x,t)} P_1 + \Delta P_1 - \alpha P_1\right)\overline{h}\; dx\; dt \; = 0, \; \forall \overline{h} \in C^\infty(\Omega),$$

implies that:

$${}_t^{CPC}D_T^{\beta(x,t)} P - \Delta P + \alpha P = w - g, \; in \; Q$$

Also,

$$\int_\Omega {}_t^{CPC}I_T^{2-\beta}\; \frac{\partial}{\partial t} P_1(x,T) \;\; \overline{h} \;\; dx \;\; = \;\; 0 \;, in \;\; \Omega$$

we deduce that:

$${}_t^{CPC}I_T^{2-\beta}\; \frac{\partial}{\partial t} P_1(x,T) \; = 0,$$

The adjoint system resulted as (3.4).

Second condition: The variational inequality:

$$D_u L(\overline{w},\overline{P},\overline{u})(u - \overline{u}) \geq 0, \;\; u \in u_{ab}.$$

where $D_u L$ is the first derivative of the Lagrangian with respect to u. And u_{ad} is admissible control, $u_{ad} = \{u \in L^2(\Sigma) : u_a(x,t) \leq u(x,t) \leq u_b(x,t)\}$.

$$\iint_Q (\gamma u + P)(u - \overline{u}) \geq 0,$$

It reduces to $u = \frac{-P}{\gamma}$, if there are no constraints on the control $u \in u_{ad}$.

3.2. Existence and uniqueness of the optimal solution

For study the existence and uniqueness of the optimal solution for the considered fractional optimal control problem, see,[913] and.[14]

Theorem 3.2. *Assume that the state $w = w(u, x, t)$ is solution of the system (3.3). Then there exists a unique optimal control u in u_{ad}. such that*

$$J(u) = \inf_{v \in u_{ad}} J(v).$$

Proof. According to.[32] Let $v \in u_{ad}$. be a minimizing sequence such that,

$$\lim_{n\to\infty} J(v_n) = \inf_{v \in u_{ad}} J(v). \tag{3.9}$$

Then there exists $C > 0$ such that

$$J(v_n) \leq C.$$

It then follows from the structure of J given by (3.1) that

$$\| u_n\|^2_{L^2(Q)} \leq C, \tag{3.10}$$

$$\| w_n - g \|^2_{L^2(Q)} \leq C. \tag{3.11}$$

Moreover $w_n = w(v_n, x, t)$ being solution of (3.3), w_n satisfies:

$$
\begin{cases}
{}^{CPC}D_t^{\beta(x,t)} w_n - \Delta\, w_n + \alpha\, w_n = f + u_n, \; in \; Q, \\
w_n = 0 \; on \; \Sigma, \\
w_n(x,0) = w_0 \; in \; \Omega, \\
{}_0^{CPC}D_t^{\beta(x,t)-1} w_n(x,0) = \; w_1 \; in \; \Omega,
\end{cases} \tag{3.12}
$$

Let

$$w_n \to w \; weakly \; in \; L^2(Q) \tag{3.13}$$

$${}_0^{CPC}D_t^{\beta(x,t)-1} \; w_n \to \kappa \; weakly \; in \; L^2(Q). \tag{3.14}$$

By using (3.10) and (3.12), we deduce that,

$$\| {}_0^{CPC}D_t^{\beta(x,t)} w_n - \Delta w_n + \alpha\, w_n \| \leq C. \tag{3.15}$$

Hence, from (3.10) and (3.15), we can extract subsequences of v_n and w_n such that

$$v_n \rightarrow u \text{ weakly in } L^2(Q), \tag{3.16}$$

$$ {}_0^{CPC}D_t^{\beta(x,t)} w_n - \Delta w_n + \alpha\, w_n \rightarrow \delta \text{ weakly in } L^2(Q). \tag{3.17}$$

Since u_{ad} is a convex closed subset of $L^2(Q)$ we have,

$u \in u_{ad}$.

Set $\mathbb{D}(Q) = \{\varphi \in C^\infty(\Omega) \text{ such that } \varphi|_\Gamma,\ \varphi(x, 0) = \varphi(x, T) \text{ in } \Omega\}$ and we denote by $\mathbb{D}^{'}(Q)$its dual. Then multiplying (3.12) by $\varphi \in \mathbb{D}(Q)$ and integrating by part over Q, we obtain

$$\int_0^T\int_\Omega \left(\left({}_0^{CPC}D_t^{\beta(x,t)} w_n - \Delta w_n + \alpha\, w_n\right)\varphi(x, t)\, dx\, dt\right.$$

$$= \int_0^T\int_\Omega (f + v_n)\, \varphi(x, t)\, dx\, dt,\ \forall \varphi \in \mathbb{D}(Q) \tag{3.18}$$

Therefore using fractional Green's formula,[37] and Passing this latter identity when $n \to \infty$ while using (3.13) and (3.16), we obtain that

$$\int_0^T\int_\Omega \left({}_0^{CPC}D_t^{\beta(x,t)} w - \Delta w + \alpha\, w\right)\varphi(x, t)\, dx\, dt$$

$$= \int_0^T\int_\Omega (f + u)\, \varphi(x, t)\, dx\, dt,\ \forall \varphi \in \mathbb{D}(Q) \tag{3.19}$$

This implies that

$${}_0^{CPC}D_t^{\beta(x,t)}\, w - \Delta\, w + \alpha\, w = f + u,\ (x, t) \in Q \tag{3.20}$$

On the other hand, we have

$$\int_\Omega\int_0^T \left({}_0^{CPC}D_t^{\beta(x,t)-1}\, w_n\right)\varphi(x, t)\, dt\, dx = -\int_\Omega\int_0^T w_n \left(\int_s^T {}_0^{CPC}D_s^{\beta(x,t)-1}\, \varphi(x, t)\, dt\right) ds\, dx,\ \forall \varphi \in \mathbb{D}(Q) \tag{3.21}$$

Passing this latter identity when $n \to \infty$ while using lemme (2.3), (3.13) and (3.14), we get

$${}^{CPC}D_t^{\beta(x,t)-1}\, w_n \rightarrow {}^{CPC}D_t^{\beta(x,t)-1}\, w \text{ weakly in } L^2(Q). \tag{3.22}$$

We have $f \in L^2(\Omega)$ and $w \in L^2((0, T); H_0^1(\Omega))$, be such that ${}^{CPC}D_t^{\beta(x,t)}\, w \in H^{-2}((0, T); H_0^1(\Omega)) \subset H^{-2}((0, T); L^2(\Omega))$. We get $\Delta w \in L^2(\Omega)$ because $\Delta w = {}_0^{CPC}D_t^{\beta(x,t)} w + \alpha\, w - f - u$. Then we have $w|_\Gamma$ and $\frac{\partial w}{\partial \nu}|_\Gamma$ exist and belong respectively to $H^{-\frac{1}{2}}(\Gamma)$ and $H^{-\frac{3}{2}}(\Gamma)$, for more details see[13]

In the following, we will verify that,[14]

$w(x, 0) = w_0,\ {}_0^{CPC}D_t^{\beta(x,t)-1}\, w(x, 0) = w_1 \text{ in } \Omega.$

For any function $\varphi \in C^\infty(\Omega)$ with $\varphi|_\Gamma$, $\varphi(x, 0) = \varphi(x, T) = 0$ in Ω, we obtain

$$\int (w_n(x,0) - w(x,0))\, \varphi(x)\, dx = \int (w_n(x,t) - w(x,t))\, \varphi(x)\, dx$$

$$-\int_0^t\int_\Omega \left({}_0^{CPC}D_s^{\beta(x,t)-1}\, w_n - {}_0^{CPC}D_s^{\beta(x,t)-1}\, w\right) \varphi(x)\, dx\, ds,$$

which implies

$$\int_0^T\int_\Omega (w_n(x,0) - w(x,0))\, \varphi(x)\, dx\, dt = \int_0^T\int_\Omega (w_n(x,t) - w(x,t))\, \varphi(x)\, dx\, dt$$

$$-\int_0^T\int_0^t\int_\Omega \left({}_0^{CPC}D_s^{\beta(x,t)-1}\, w_n - {}_0^{CPC}D_s^{\beta(x,t)-1}\, w\right) \varphi(x)\, dx\, ds\, dt,$$

From (3.13) and (3.22) and for any $t \in [0, T]$

$$\int_0^T\int_\Omega (w_n(x,0) - w(x,0))\, \varphi(x)\, dx\, dt \quad \rightarrow 0, \tag{3.23}$$

$$\int_0^T\int_0^t\int_\Omega \left({}_0^{CPC}D_s^{\beta(x,t)-1}\, w_n - {}_0^{CPC}D_s^{\beta(x,t)-1}\, w\right) \varphi(x)\, dx\, ds\, dt \rightarrow 0. \tag{3.24}$$

For any $\varphi \in L^2(\Omega)$, we get

$$\int_\Omega (w_n(x,0) - w(x,0))\, \varphi(x)\, dx \quad \rightarrow 0, \tag{3.25}$$

As $w_n(x, 0) = w_{n0} \rightarrow w_0$ in $L^2([0, T], H_0^1(\Omega))$, we get

$$w(x, 0) = w_0,\ \text{in } \Omega. \tag{3.26}$$

Similarly, we could verify that

$${}_0^{CPC}D_t^{\beta(x,t)-1}\, w(x, 0) = w_1 \text{ in } \Omega. \tag{3.27}$$

From (3.20), (3.26) and (3.27), we have $w = w(u, x, t)$ is solution of the system (3.3). It then follows from the lower semi-continuity of the functional J and $J(u) \le \lim_{n\to\infty}\inf J(v_n)$.

Hence in view of (3.9), we get

$$\lim_{n\to\infty} J(v_n) = \inf_{v \in u_{ad}} J(v).$$

From the strict convexity of J, we obtain the uniqueness of the optimal control u.

4. NWAFDM for OCVOD-wave equation

This section aims to study the OCVOD-wave with reaction term numerically. The numerical method that will be considered here is NWAFDM.[27,28,38,39]

4.1. NWAFDM

According to,[40] the numerical scheme is known as NSFDM, if at least one of the following conditions is satisfied:

- The nonlocal approximation is used,
- The discretization of a derivative is non-traditional and use positive function.

$$\psi(\tau) = \tau + O(\tau^2),\ 0 < \psi(\tau) < 1,\ \text{for all } \tau > 0,$$

where $\psi(\tau)$ is the continuous function of step size.

In the following, we construct NWAFDM with variable-order of CPC derivative to discretize of VFPDEs. Considering that:

$n, l, m \in \mathbb{N}$ and the points of the mesh

$$x_i = i\triangle x, \quad i = 0, 1, 2, \ldots, n,$$

$$y_j = j\triangle y, \quad j = 0, 1, 2, \ldots, l,$$

$$t_r = r\triangle t, \quad r = 0, 1, 2, \ldots, m,$$

where $\triangle x, \triangle y$ and $\triangle t$ space-step length and time-step length, respectively:

$$\triangle x = \frac{(x_n - x_o)}{n}, \quad \triangle y = \frac{(y_l - y_o)}{l}, \quad \triangle t = \frac{(t_r - t_o)}{r}.$$

The $W_{i,j}^r$ is the numerical value of u at the grid point $(x_i, y_j, t_r) = (i\triangle x, j\triangle y, r\triangle t)$. For the VFPDEs, the NWAFDM is given as follows:

$$W(x_i, y_j, t_r) = (1-\Theta)W_{i,j}^{r+1} + \Theta W_{i,j}^{r-1}.$$

Approximation of second-order derivatives:

$$W_{xx}(x_i, y_j, t_r) = (1-\Theta)\frac{W_{i+1,j}^{r+1} - 2W_{i,j}^{r+1} + W_{i-1,j}^{r+1}}{\psi(\triangle x)^2} + \Theta\frac{W_{i+1,j}^{r-1} - 2W_{i,j}^{r-1} + W_{i-1,j}^{r-1}}{\psi(\triangle x)^2} + O\big(\psi(\triangle x)^2\big),$$

$$W_{yy}(x_i, y_j, t_r) = (1-\Theta)\frac{W_{i,j+1}^{r+1} - 2W_{i,j}^{r+1} + W_{i,j-1}^{r+1}}{\psi(\triangle y)^2} + \Theta\frac{W_{i,j+1}^{r-1} - 2W_{i,j}^{r-1} + W_{i,j-1}^{r-1}}{\psi(\triangle y)^2} + O\big(\psi(\triangle y)^2\big).$$

and the discrete Laplacian in two dimensions:

$$\triangle W(x_i, y_j, t_r) = (1-\Theta)\big(W_{xx}(x_i, y_j, t_{r+1}) + W_{yy}(x_i, y_j, t_{r+1})\big) + \Theta\big(W_{xx}(x_i, y_j, t_{r-1}) + W_{yy}(x_i, y_j, t_{r-1})\big) + \mathrm{O}\big(\psi(\triangle x)^2, \psi(\triangle y)^2\big).$$

$$\sum_{r=0}^{k-1}\Big[K_1(\beta(.,t))w_{i,j}^{k+1-r} + K_0(\beta(.,t))\psi(\triangle t)^{-\beta(.,t)}\Big(w_{i,j}^{k+1-r} - 2w_{i,j}^{k-r} + w_{i,j}^{k-1-r}\Big)\Big]d_r$$
$$-(1-\Theta)\left[\triangle w_{i,j}^{k+1} - \alpha w_{i,j}^{k+1} - \frac{P_{i,j}^{k+1}}{\gamma} + f_{i,j}^{k+1}\right] - \Theta\left[\triangle w_{i,j}^{k-1} - \alpha w_{i,j}^{k-1} - \frac{P_{i,j}^{k-1}}{\gamma} + f_{i,j}^{k-1}\right] = R_{1_{i,j}^k},$$

$$\sum_{r=0}^{k-1}\Big[K_1(\beta(.,t))\ P_{i,j}^{k+1-r} + \ K_0(\beta(.,t))\ \psi(\triangle t)^{-\beta(.,t)}\ \Big(P_{i,j}^{k+1-r} - 2P_{i,j}^{k-r} + P_{i,j}^{k-1-r}\Big)\Big]\ b_r$$
$$-(1-\Theta)\ \Big[\triangle P_{i,j}^{k+1} - \alpha P_{i,j}^{k+1} + w_{i,j}^{k+1} - g_{i,j}^{k+1}\Big] - \Theta\ \Big[\triangle P_{i,j}^{k-1} - \alpha P_{i,j}^{k-1} + w_{i,j}^{k-1} - g_{i,j}^{k-1}\Big] = R_{2_{i,j}^k}.$$

4.2. Numerical approximate with variable-order CPC derivative

To discretize of the OCVOD-wave equation, we construct NWAFDM with a constant-proportional-Caputo variable-order derivative $\beta(.,t)$ stand for the function in space x, y and time t. By definition (2.13) and GL-approximation to approximate the CPC variable-order derivatives at one dimensional, we have the approximate for the OCVOD-wave equation given as follows:

The left case:

$$ {}_0^{CPC}D_t^{\beta(.,t)}w = K_1(\beta(.,t))\ \ {}_0^{RL}I_t^{2-\beta(.,t)}\ w(x,y,t)\ +\ K_0(\beta(.,t))\ \ {}_0^{C}D_t^{\beta(.,t)}\ w(x,y,t), \tag{4.1}$$

$$ {}_0^{CPC}D_t^{\beta(.,t)}w = \sum_{r=0}^{k-1}K_1(\beta(.,t))\,d_r\,w_{i,j}^{k+1-r} + K_0(\beta(.,t))\,d_r\,\psi(\triangle t)^{-\beta(.,t)}\Big(w_{i,j}^{k+1-r} - 2w_{i,j}^{k-r} + w_{i,j}^{k-1-r}\Big), \tag{4.2}$$

where

$$d_r = \frac{(r+1)^{2-\beta(x,t)} - (r)^{2-\beta(.,t)}}{\Gamma(3-\beta(.,t))}.$$

The right case:

$$ {}_t^{CPC}D_T^{\beta(.,t)}P = K_1(\beta(.,t))\ \ {}_t^{RL}I_T^{2-\beta(.,t)}\ P(x,y,t)\ +\ K_0(\beta(.,t))\ \ {}_t^{C}D_T^{\beta(.,t)}P(x,y,t), \tag{4.3}$$

$$ {}^{CPC}D_T^{\beta(.,t)}P_t = \sum_{r=0}^{k-1}K_1(\beta(.,t))\ b_r\ \ P_{i,j}^{k+1-r}\ +\ K_0(\beta(.,t))\ \psi(\triangle t)^{-\beta(.,t)} \Big(P_{i,j}^{k+1-r} - 2P_{i,j}^{k-r} + P_{i,j}^{k-1-r}\Big)\ b_r, \tag{4.4}$$

where

$$b_r = \frac{(r)^{2-\beta(.,t)} - (r+1)^{2-\beta(.,t)}}{\Gamma(3-\beta(.,t))}.$$

Where
$K_0(\beta(.,t)) = (1-\beta(.,t))\ C^{2\beta(.,t)}\ \ t^{2-\beta(.,t)},\ K_1(\beta(.,t)) = (2-\beta(.,t))\ \ t^{\beta(.,t)}$,
and C is a constant.

4.2.1. Construction of NWAFDM with variable-order CPC derivative

The discretization form of the variable-order optimal control of diffusion-wave with a reaction term system (3.3)-(3.5) using definition (4.3) is given as follows:

Where $R_{1_{i,j}^k}$ and $R_{2_{i,j}^k}$ are the truncation error.

If the truncation errors are neglected, the resulting difference scheme is as follows:

The state equation:

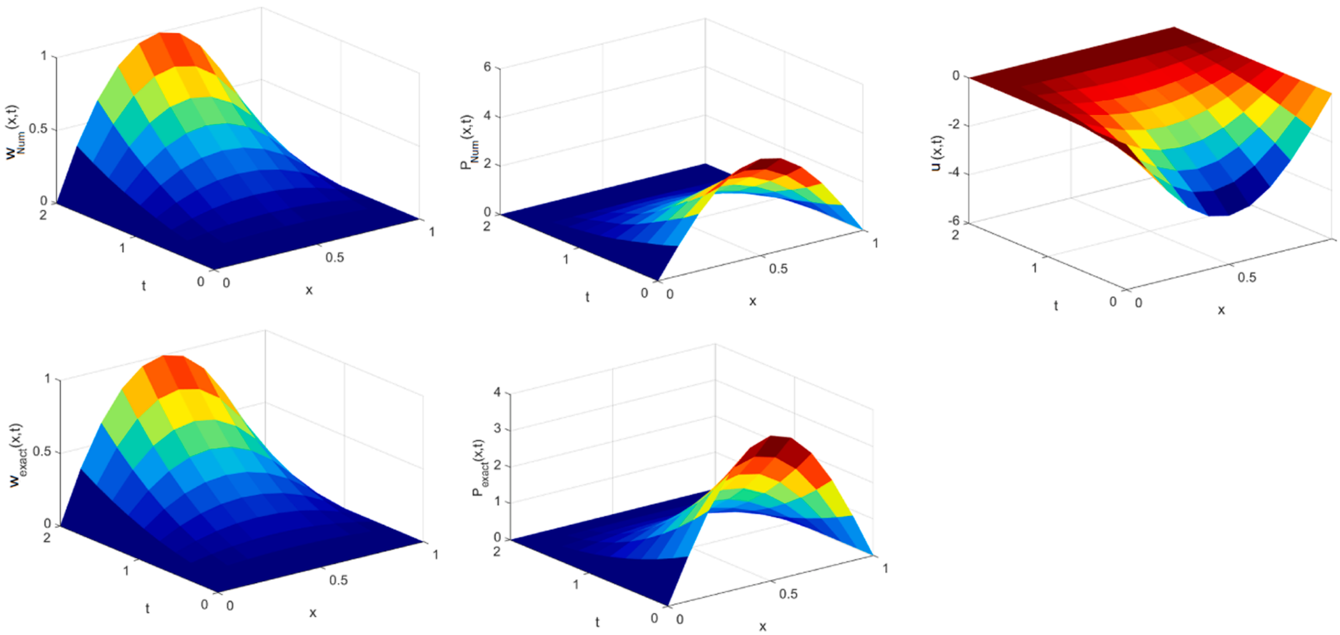

Fig. 1. For problem1, the behaviour of the numerical solutions when $\boldsymbol{\Theta = \frac{1}{2}}$, $\boldsymbol{\gamma = 1}$ **and** $\boldsymbol{\beta(x,t)} = 1.99 - \frac{t}{100}$.

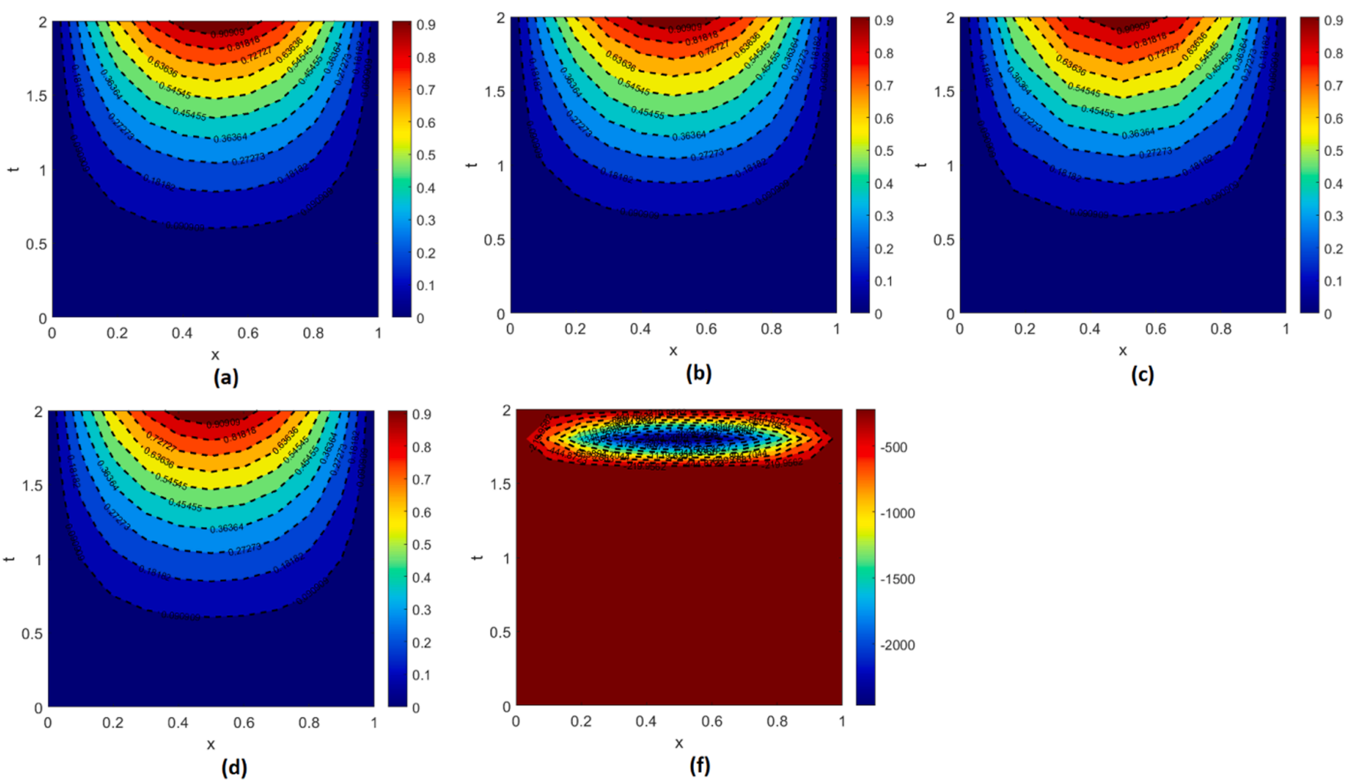

Fig. 2. For problem1, the behaviour of the numerical solutions using different methods at $\gamma = 1$ and $\beta(x,t) = 1.99 - \frac{t}{100}$ (a) Exact solution w (b) Numerical solution w using NLFM, (c) Numerical solution w using NWAFDM when $\Theta = \frac{1}{2}$, (d) Numerical solution w using NWAFDM when $\Theta = 0$, (f) Numerical solution w using NWAFDM when $\Theta = 1$.

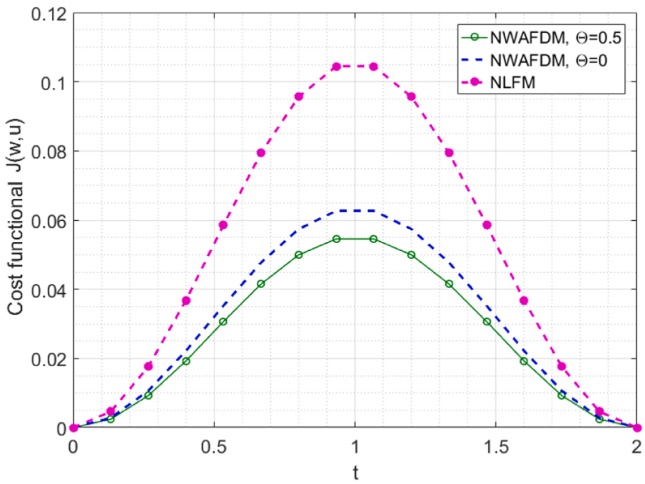

Fig. 3. For problem1, the behaviour of the cost functional $J(w,u)$of problem 1, using different methods at$\gamma = 1/10$ and $\beta(x,t) = 1.99 - \frac{t}{100}$.

$$-2e^{j\rho q_2\Delta y} + e^{(j-1)\rho q_2\Delta y}\big) - \alpha e^{\rho(iq_1\Delta x + jq_2\Delta y)}\big]$$
$$- \Theta\xi_1^{k-1}\big[e^{j\rho q_2\Delta y}\psi(\Delta x)^{-2}\big(e^{(i+1)\rho q_1\Delta x} - 2e^{i\rho q_1\Delta x}$$
$$+e^{(i-1)\rho q_1\Delta x}\big) + e^{i\rho q_1\Delta x}\psi(\Delta y)^{-2}\big(e^{(j+1)\rho q_2\Delta y} - 2e^{j\rho q_2\Delta y} + e^{(j-1)\rho q_2\Delta y}\big) - \alpha e^{\rho(iq_1\Delta x + jq_2\Delta y)}\big]$$
$$= 0,$$

The adjoint equation:

$$\sum_{r=0}^{k-1}\Big[K_1(\beta(.,t))\xi_2^{k+1-r} + K_0(\beta(.,t))\psi(\Delta t)^{-\beta(.,t)}\big(\xi_2^{k+1-r} - 2\xi_2^{k-r} + \xi_2^{k-1-r}\big)\Big]$$
$$e^{\rho(iq_1\Delta x + jq_2\Delta y)}b_r$$
$$-(1-\Theta)\xi_2^{k+1}\big[e^{j\rho q_2\Delta y}\psi(\Delta x)^{-2}\big(e^{(i+1)\rho q_1\Delta x} - 2e^{i\rho q_1\Delta x} + e^{(i-1)\rho q_1\Delta x}\big)$$
$$+ e^{i\rho q_1\Delta x}\psi(\Delta y)^{-2}\big(e^{(j+1)\rho q_2\Delta y}$$
$$-2e^{j\rho q_2\Delta y} + e^{(j-1)\rho q_2\Delta y}\big) - \alpha e^{\rho(iq_1\Delta x + jq_2\Delta y)}\big]$$
$$- \Theta\xi_2^{k-1}\big[e^{j\rho q_2\Delta y}\psi(\Delta x)^{-2}\big(e^{(i+1)\rho q_1\Delta x} - 2e^{i\rho q_1\Delta x}$$

$$\sum_{r=0}^{k-1}\Big[K_1(\beta(.,t))w_{i,j}^{k+1-r} + K_0(\beta(.,t))\psi(\Delta t)^{-\beta(.,t)}\Big(w_{i,j}^{k+1-r} - 2w_{i,j}^{k-r} + w_{i,j}^{k-1-r}\Big)\Big]d_r$$
$$-(1-\Theta)\left[\triangle w_{i,j}^{k+1} - \alpha w_{i,j}^{k+1} - \frac{P_{i,j}^{k+1}}{\gamma} + f_{i,j}^{k+1}\right] - \Theta\left[\triangle w_{i,j}^{k-1} - \alpha w_{i,j}^{k-1} - \frac{P_{i,j}^{k-1}}{\gamma} + f_{i,j}^{k-1}\right] = 0. \quad (4.5)$$

The adjoint equation:

$$\sum_{r=0}^{k-1}\Big[K_1(\beta(.,t))P_{i,j}^{k+1-r} + K_0(\beta(.,t))\psi(\Delta t)^{-\beta(.,t)}\Big(P_{i,j}^{k+1-r} - 2P_{i,j}^{k-r} + P_{i,j}^{k-1-r}\Big)\Big]b_r$$
$$-(1-\Theta)\Big[\triangle P_{i,j}^{k+1} - \alpha P_{i,j}^{k+1} + w_{i,j}^{k+1} - g_{i,j}^{k+1}\Big] - \Theta\Big[\triangle P_{i,j}^{k-1} - \alpha P_{i,j}^{k-1} + w_{i,j}^{k-1} - g_{i,j}^{k-1}\Big] = 0. \quad (4.6)$$

4.2.2. Stability analysis and convergence of NWAFDM with variable-order CPC derivative

In order to investigate the stability, a kind of von Neumann approach[41–43] will be applied to the proposed scheme (4.6)-(4.7), with considering the absence of the control source force. Assume that $w_{i,j}^k = \xi_1^k e^{i\rho q_1\Delta x}e^{\rho q_2 j\Delta y}$ and $P_i^k = \xi_2^k e^{i\rho p_1\Delta x}e^{j\rho P_2\Delta y}$ where

$\rho = \sqrt{-1}$ and q,p is a real spatial wave number, so that the requirement $|\xi_{1,2}| \le 1$. Therefore, the scheme (4.6)-(4.7) can be expressed in the following form:

The state equation:

$$\sum_{r=0}^{k-1}\Big[K_1(\beta(.,t))\xi_1^{k+1-r} + K_0(\beta(.,t))\psi(\Delta t)^{-\beta(.,t)}\big(\xi_1^{k+1-r} - 2\xi_1^{k-r} + \xi_1^{k-1-r}\big)\Big]$$
$$e^{\rho(iq_1\Delta x + jq_2\Delta y)}d_r$$
$$-(1-\Theta)\xi_1^{k+1}\big[e^{j\rho q_2\Delta y}\psi(\Delta x)^{-2}\big(e^{(i+1)\rho q_1\Delta x} - 2e^{i\rho q_1\Delta x} + e^{(i-1)\rho q_1\Delta x}\big)$$
$$+ e^{i\rho q_1\Delta x}\psi(\Delta y)^{-2}\big(e^{(j+1)\rho q_2\Delta y}$$

$$+e^{(i-1)\rho q_1\Delta x}\big) + e^{i\rho q_1\Delta x}\psi(\Delta y)^{-2}\big(e^{(j+1)\rho q_2\Delta y} - 2e^{j\rho q_2\Delta y} + e^{(j-1)\rho q_2\Delta y}\big) - \alpha e^{\rho(iq_1\Delta x + jq_2\Delta y)}\big]$$
$$= 0,$$

Diving by $\xi_{1,2}^k e^{i\rho q_1\Delta x}e^{j\rho q_2\Delta y}$, put $\eta = \frac{\xi_1^{k+1}}{\xi_1^k}$, $\lambda = \frac{\xi_2^{k+1}}{\xi_2^k}$, and using the Euler formula:

$$e^{\rho\theta} = \cos(\theta) + \rho\,\sin(\theta),$$

we have:

$$\sum_{r=0}^{k-1}\Big(K_1(\beta)\eta^{-r}\eta + K_0(\beta)\psi(\Delta t)^{-\beta}\big(\eta^{-r}\big(\eta - 2 + \eta^{-1}\big)\big)\Big)d_r$$
$$-2\left[\frac{\cos(q_1\Delta x) - 1}{\psi(\Delta x)^2} + \frac{\cos(q_2\Delta y) - 1}{\psi(\Delta y)^2}\right.$$
$$\left.-\beta\right]\big[(1-\Theta)\eta + \Theta\eta^{-1}\big] = 0.$$

Assume $A_1 = \sum_{r=0}^{k-1}K_1(\beta(.,t))\eta^{-r}d_r$, $A_2 = \sum_{r=0}^{k-1}K_0(\beta(.,$

$t))\psi(\Delta t)^{-\beta(.,t)}\eta^{-r}d_r$.

We have:

$$(A_1 + A_2 - B(1-\Theta))\eta^2 - 2A_2\eta + (A_2 - B\Theta) = 0,\ |\eta| \le 1,$$

where $B = 2\left[\frac{\cos(q_1\Delta x)-1}{\psi(\Delta x)^2} + \frac{\cos(q_2\Delta y)-1}{\psi(\Delta y)^2} - \beta\right]$.

$$|\eta_{1,2}| = \frac{\left|2A_2 \pm \sqrt{(2A_2)^2 - 4(A_1 + A_2 - B(1-\Theta))(A_2 - B\Theta)}\right|}{|2(A_1 + A_2 - B(1-\Theta))|} \le 1,$$

$$\left|2A_2 \pm \sqrt{(2A_2)^2 - 4(A_1 + A_2 - B(1-\Theta))(A_2 - B\Theta)}\right| \le |2(A_1 + A_2 - B(1-\Theta))|.$$

With the adjoint equation:

$$\sum_{r=0}^{k-1}\left(K_1(\beta)\lambda^{-r}\lambda + K_0(\beta)\psi(\Delta t)^{-\beta}\left(\lambda^{-r}\left(\lambda - 2 + \lambda^{-1}\right)\right)\right)d_r$$
$$-2\left[\frac{\cos(q_1\Delta x)-1}{\psi(\Delta x)^2} + \frac{\cos(q_2\Delta y)-1}{\psi(\Delta y)^2} - \beta\right]\left[(1-\Theta)\lambda + \Theta\lambda^{-1}\right] = 0.$$

Assume $A_3 = \sum_{r=0}^{k-1}K_1(\beta(.,t))\lambda^{-r}b_r$, $A_4 = \sum_{r=0}^{k-1}K_0(\beta(.,t))\psi(\Delta t)^{-\beta(.,t)}\lambda^{-r}b_r$.

We have:

$$(A_3 + A_4 - C(1-\Theta))\lambda^2 - 2A_4\lambda + (A_4 - C\Theta) = 0,\ |\lambda| \le 1,$$

where $C = 2\left[\frac{\cos(q_1\Delta x)-1}{\psi(\Delta x)^2} + \frac{\cos(q_2\Delta y)-1}{\psi(\Delta y)^2} - \beta\right]$.

$$|\lambda_{1,2}| = \frac{\left|2A_4 \pm \sqrt{(2A_4)^2 - 4(A_3 + A_4 - B(1-\Theta))(A_4 - C\Theta)}\right|}{|2(A_3 + A_4 - C(1-\Theta))|} \le 1,$$

$$\left|2A_4 \pm \sqrt{(2A_4)^2 - 4(A_3 + A_4 - C(1-\Theta))(A_4 - C\Theta)}\right| \le |2(A_3 + A_4 - C(1-\Theta))|.$$

5. Applications of the NWAFDM to OCVOD-wave

In this section, the three test problems of optimal control for VOD-Wave are examined to determine the viability and efficacy of the proposed method. To justify the efficiency and accuracy of the NWAFDM are checked by calculating the error norm L_∞, is given as follows:

$$e_w(x,y,t) = \| w_{exact} - w_{Num} \| \simeq \max\left|w_{exact}(x_i,y_j,t_r) - w_{Num}(x_i,y_j,t_r)\right|,\ \forall i,j,r, \quad (5.1)$$

where the analytical state solution of the considered problems is $w_{exact}(x_i,y_j,t_r)$, and the numerical state solution of is $w_{Num}(x_i,y_j,t_r)$.

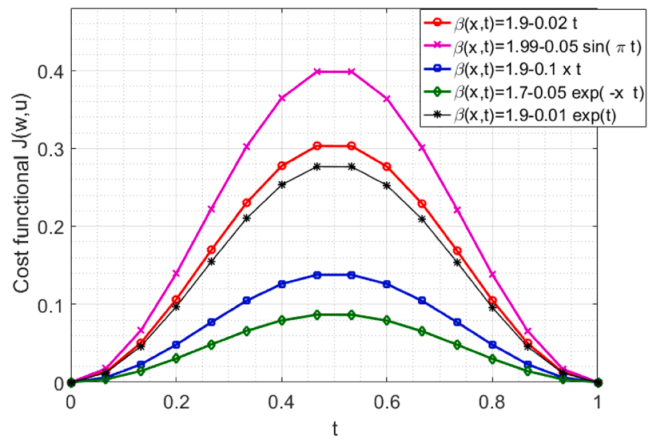

Fig. 4. For problem 1, the effect of variable order $\beta(x,t)$ on the cost functional $J(w,u)$ when $\gamma = \frac{1}{10}$.

$$e_P(x,y,t) = \| P_{exact} - P_{Num} \| \simeq \max\left|P_{exact}(x_i,y_j,t_r) - P_{Num}(x_i,y_j,t_r)\right|,\ \forall i,j,r, \quad (5.2)$$

where the analytical adjoint solution of the considered problems is $P_{exact}(x_i,y_j,t_r)$ and the numerical adjoint solution is $P_{Num}(x_i,y_j,t_r)$. For problem 1, Fig. 1, clarifies the behaviour of the numerical solutions of the state variable w and the adjoint variable P at $\beta(x,t) = 1.99 - \frac{t}{100}$ and the exact solutions, we observed that the proposed method NWAFDM $(\Theta = \frac{1}{2})$ provides excellent agreement between the numerical solutions and exact solutions. For problem 1, Fig. 2, the behaviour levels of the numerical solution w using WANFDM at various cases of Θ and NLFM. It observed that the explicit method $(\Theta = 1)$ is less effective than the implicit $(\Theta = 0)$, Crank-Nicholson $(\Theta = \frac{1}{2})$ and NLFM methods for approximating solutions. Fig. 3:For problem1, the behaviour of the numerical solutions when $\Theta = \frac{1}{2}$, $\gamma = 1$ and, depicts the cost functional $J(w,u)$ (3.1) solutions for problem 1, using NWANFDM $(\Theta = \frac{1}{2})$ at $\beta(x,t) = 1.99 - \frac{t}{100}$, we observe that the $J(w,u)$ solution obtained by the Crank-Niclson case (i.e., $\Theta = \frac{1}{2}$) is less valuable than the $J(w,u)$ solution obtained by implicit and NLFM.

$/\gamma = 10^{-1}$ $\alpha = 0$ $\psi(\Delta t) = \sinh(\Delta t)$ Table 1, illustrates that the maximum error at β(x, t) = 1.9 − t /100 using WANFDM at various values of Θ and NLFM for problem 1, illustrates that the maximum error at $\beta(x,t) = 1.9 - \frac{t}{100}$ using WANFDM at various values of Θ and NLFM for problem 1, with various values of a final space X_f and a final time T. For problem 1, the behaviour of the cost functional $J(w,u)$ solutions (3.1) at various values of linear and nonlinear $\beta(x,t)$ using NWANFDM $(\Theta = \frac{1}{2})$ is shown in Fig. 4.

For problem 2, the identical behaviour of the exact and numerical solutions of the state variable w, adjoint variable P, and optimal control variable u obtained using the suggested method NWAFDM $(\Theta = \frac{1}{2})$ at

Table 1
For problem 1, error norms (5.1–5.2) for when $n = 9, m = 10$, $\gamma = 10^{-1}$, $\alpha = 0$, $\psi(\Delta x) = sinh(\Delta x)$, and $\psi(\Delta t) = sinh(\Delta t)$ at different space and time levels.

(X_f,T)	NWAFDM				NLFM	
	$(\Theta = \frac{1}{2})$		$(\Theta = 0)$			
	$e_u(x,t)$	$e_P(x,t)$	$e_u(x,t)$	$e_P(x,t)$	$e_u(x,t)$	$e_P(x,t)$
(0.2,0.2)	2.037e-06	9.4458e-06	2.7817e-07	8.398e-06	6.336e-06	1.5223e-05
(0.2,0.5)	3.6865e-06	5.617e-05	1.758e-06	5.3056e-05	4.1820e-05	9.543e-05
(0.5,0.2)	2.904e-05	8.8512e-05	2.2013e-05	1.0668e-04	4.836e-05	2.1417e-04
(0.5,0.5)	1.7205e-04	6.7642e-04	1.4013e-04	5.3056e-05	7.71e-04	1.3 e-03
(0.5,1)	6.5369e-04	3.1e-03	5.625e-04	2.7e-03	2.9e-03	5e-03
(1,1)	1.86e-02	1.37e-02	1.13 e-02	1.36 e-02	3.47e-02	2.16e-02
(1,2)	6.54e-02	8.04e-02	4.51e-02	5.43e-02	1.417e-02	5.43e-02

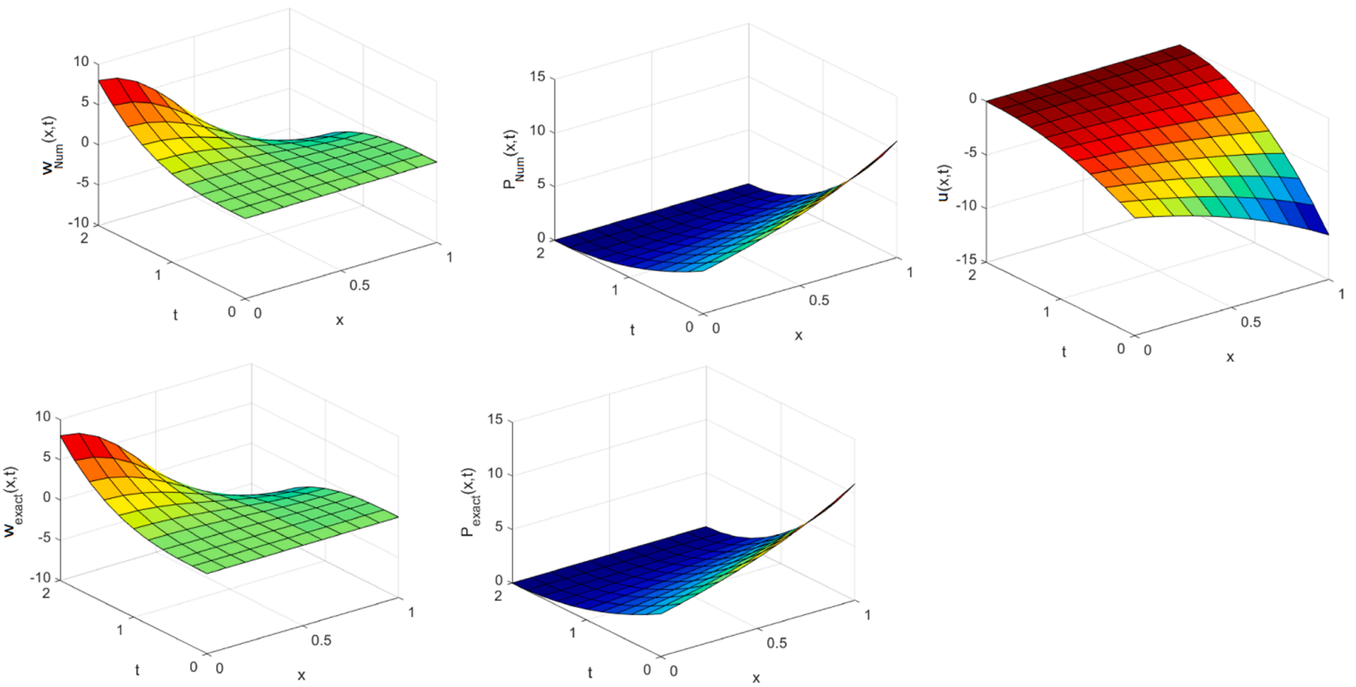

Fig. 5. For problem 2, the behaviour of the numerical solutions using NWAFDM at $\Theta = \frac{1}{2}$, and $\beta(x,t) = 1.9 - \frac{t}{100} \, sin(\pi x)$.

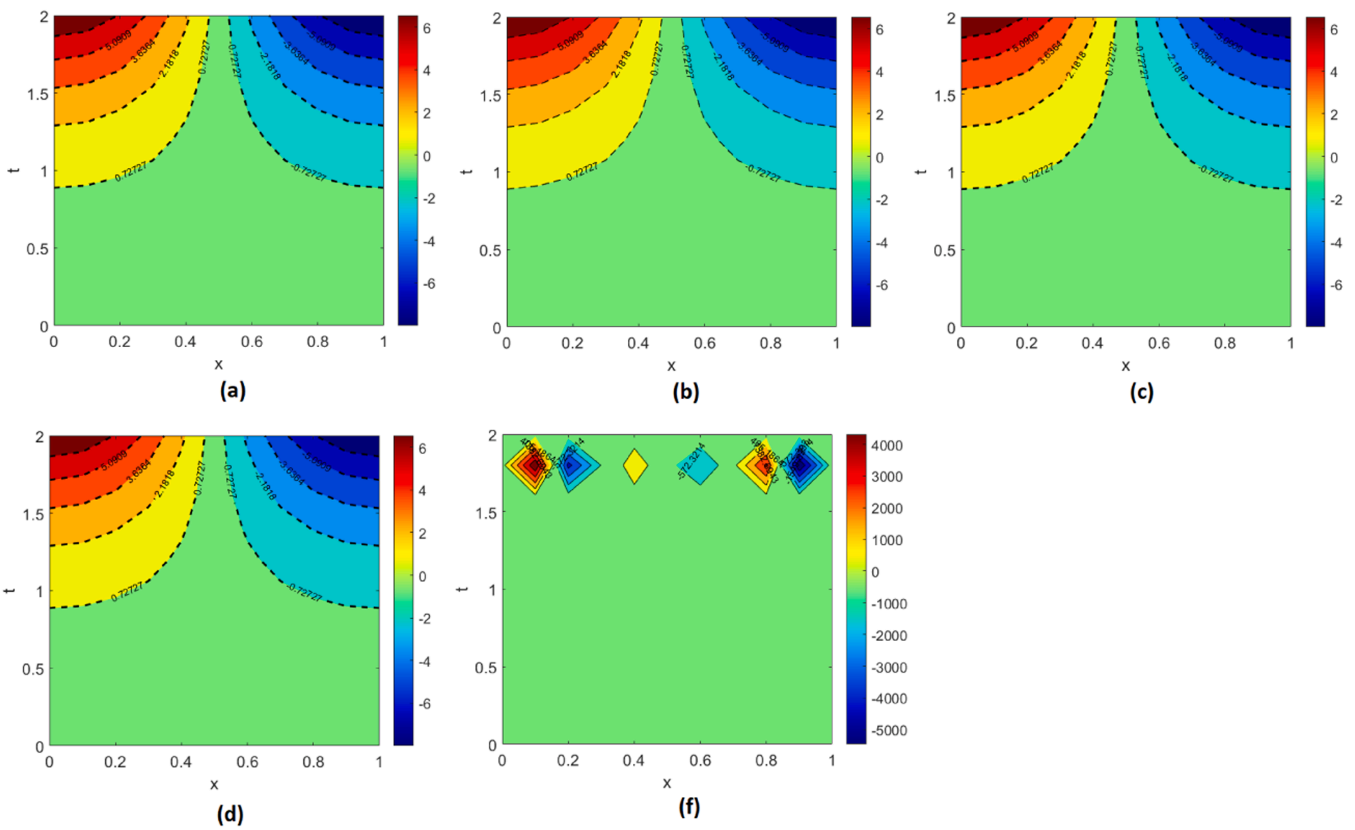

Fig. 6. For problem 2, the behaviour of the numerical solutions using different methods at $\gamma = \frac{1}{10}$ and $\beta(x,t) = 1.9 - \frac{t}{100} \, sin(\pi x)$ (a) Exact solution w (b) Numerical solution w using NLFM, (c) Numerical solution w using NWAFDM when $\Theta = \frac{1}{2}$, (d) Numerical solution w using NWAFDM when $\Theta = 0$, (f) Numerical solution w using NWAFDM when $\Theta = 1$.

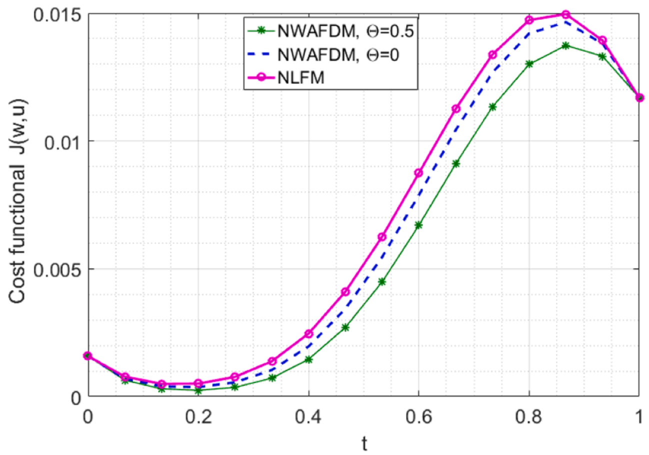

Fig. 7. The behaviour of the cost functional $J(w,u)$ of problem 2, using different methods at $\gamma = \frac{1}{10}$ and $\beta(x,t) = 1.9 - 0.02sin(\pi t)$.

Table 2
For problem 2, error norms (5.1–5.2) when $n = 9, m = 10$, $\gamma = 10^{-1}$, $\alpha = 1$, $\psi(\Delta x) = sinh(\Delta x)$ and $\psi(\Delta t) = sinh(\Delta t)$ at different space and time levels.

(X_f,T)	*NWAFDM*				*NLFM*	
	$(\Theta = \frac{1}{2})$		$(\Theta = 0)$			
	$e_u(x,t)$	$e_P(x,t)$	$e_u(x,t)$	$e_P(x,t)$	$e_u(x,t)$	$e_P(x,t)$
(0.2,0.2)	2.004e-05	3.364e-05	1.172e-05	2.591e-05	7.111e-05	3.2605e-05
(0.2,0.5)	1.544e-05	4.294e-04	2.325e-05	4.277e-04	1.196e-04	4.154e-04
(0.5,0.2)	9.173e-05	1.724e-04	6.010e-05	1.099e-04	2.916e-04	1.677e-04
(0.5,0.5)	4.177e-04	1.90e-03	5.246e-04	1.90e-03	8.862e-04	1.8 e-03
(0.5,1)	2.7e-03	1.52e-03	6.00e-03	1.51e-02	1.05e-02	1.46e-02
(1,1)	4.30e-03	1.58e-02	6.90 e-03	1.53 e-02	1.35e-02	1.60e-02
(1,2)	3.75e-02	1.219e-01	4.43e-02	1.216e-01	5.244e-02	1.228e-01

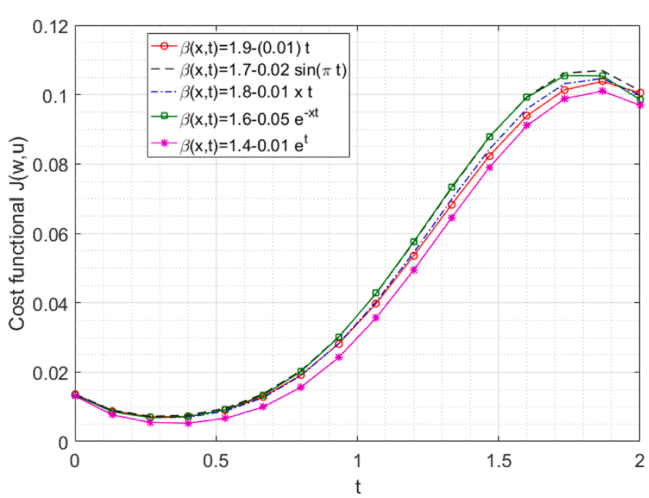

Fig. 8. For problem 2, the effect of variable order $\beta(x,t)$ on the cost functional $\gamma = \frac{1}{10}$ when $\gamma = \frac{1}{10}$

$\beta(x,t) = 1.9 - \frac{t}{100} sin(\pi x)$, is shown in Fig. 5. In Fig. 6, depicts the behaviour of the approximate solution levels of w using WANFDM at various cases of θ and NLFM for problem 2. Also it observed that the behaviour of the solutions in the implicit, Crank-Nicholson cases (i.e., $\Theta = 0$ and $\Theta = 0.5$) and NLFM are preferable to that of the solutions obtained by the explicit method at ($\Theta = 1$). For problem 2, Fig. 7, illustrates a lower value for the cost functional $J(w,u)$ (3.1) solution, using NWANFDM ($\Theta = \frac{1}{2}$) at $\beta(x,t) = 1.9 - 0.02\ sin(\pi t)$ than the $J(w,u)$ solutions obtained by implicit and NLFM. Table 2, illustrates that the maximum error at $\beta(x,t) = 1.9 - 0.02sin(\pi t)$ using WANFDM at various values of Θ and NLFM for problem 2, with various values of a final space X_f and a final time T. The behaviour of the numerical solutions of the cost functional $J(w,u)$ (3.1) for problem 2, at different values of linear and nonlinear $\beta(x,t)$ using NWANFDM ($\Theta = \frac{1}{2}$) is shown in Fig. 8. For problem 3, Fig. 9, illustrates the identical behaviour of the exact and numerical solutions of the state variable w, adjoint variable P, and optimal control variable u obtained using the suggested method NWAFDM ($\Theta = \frac{1}{2}$) at $\beta(x,t) = 1.99 - \frac{t}{100}$, for problem 3. In Fig. 10, depicts the behaviour of the numerical solution levels of w using WANFDM at various cases of Θ and NLFM for problem 3, at $\beta(x,t) = 1.99 - \frac{t}{100}$. In Fig. 11, illustrates a lower value for the cost functional $J(w,u)$ (3.1) solution, using NWANFDM ($\Theta = \frac{1}{2}$) at $\beta(x,y,t) = 1.8 - \frac{(xy)^t}{20}$ than the $J(w,u)$ solutions obtained by implicit and NLFM. Table 3, illustrates that the maximum error at $\beta(x,y,t) = 1.8 - \frac{(xy)^t}{20}$ using WANFDM at various values of Θ and NLFM for problem 3, with various values of a final space (X_f, Y_f) and a final time T. The behaviour of the numerical solutions of the cost functional $J(w,u)$ (3.1) for problem 3, at different values of linear and nonlinear $\beta(x,y,t)$ using NWANFDM ($\Theta = 1\ /2$) is shown in Fig. 12.

Problem 1

Let $\alpha = 0$, $\Omega = (0,1)$ and $T = 2$, $w_0 = 0, w_1(x) = 0$,

$$f = \frac{K_1 t^{4-\beta(x,t)} x(1-x)}{\Gamma(2-\beta(x,t))}\left[\frac{1}{4-\beta(x,t)} - \frac{2}{3-\beta(x,t)} + \frac{1}{2-\beta(x,t)}\right]$$

$$+\frac{2K_0}{\Gamma(3-\beta(x,t))} t^{2-\beta(x,t)} x(1-x) + 2t^2 + \left(\frac{1}{\gamma} + \alpha\right) sin(\pi x)(t-T)^2,$$

and

$$g(x,t) = \frac{-K_1 sin(\pi x)}{\Gamma(2-\beta(x,t))}\ \frac{(T-t)^{4-\beta(x,t)}}{4-\beta(x,t)} - \frac{2K_0 sin(\pi x)}{\Gamma(3-\beta(x,t))}(T-t)^{2-\beta(x,t)}$$
$$-\pi^2\ sin(\pi x)(t-T)^2 + t^2 x(1-x),$$

such that the exact solutions are: $w(x,t) = t^2 x(1-x)$ and $P(x,t) = sin(\pi x)(t-T)^2$.

Problem 2

Let $\alpha = 1$, $\Omega = (0,1)$ and $T = 1$, $w_0 = 0, w_1(x) = 0$,

$$f = \frac{6\ K_1\ t^{5-\beta(x,t)} cos(\pi\ x)}{\Gamma(6-\beta(x,t))} + \frac{6\ \ K_0\ cos(\pi x)}{\Gamma(4-\beta(x,t))} t^{3-\beta(x,t)} + \left(\alpha + \pi^2\right)\ t^3\ cos(\pi x)$$
$$+\frac{e^x (T-t)^2}{\gamma},$$

and

$$g(x,t) = t^3\ cos(\pi x) - \frac{K_1\ e^x}{\Gamma(2-\beta(x,t))}\frac{(T-t)^{4-\beta(x,t)}}{4-\beta(x,t)} - \frac{2\ K_0 e^x}{\Gamma(3-\beta(x,t))}(T-t)^{2-\beta(x,t)}$$
$$+(1-\alpha)\ e^x\ (T-t)^2,$$

such that the exact solutions are: $w(x,t) = t^3 cos(\pi x)$ and $P(x,t) = e^x (T-t)^2$.

Problem 3

Let $\alpha = 0$, $\Omega = (0,1)^2$ and $T = 2$, $w_0 = sin(\pi x)\ sin(\pi y), w_1(x,y) = sin(\pi x)\ sin(\pi y)$,

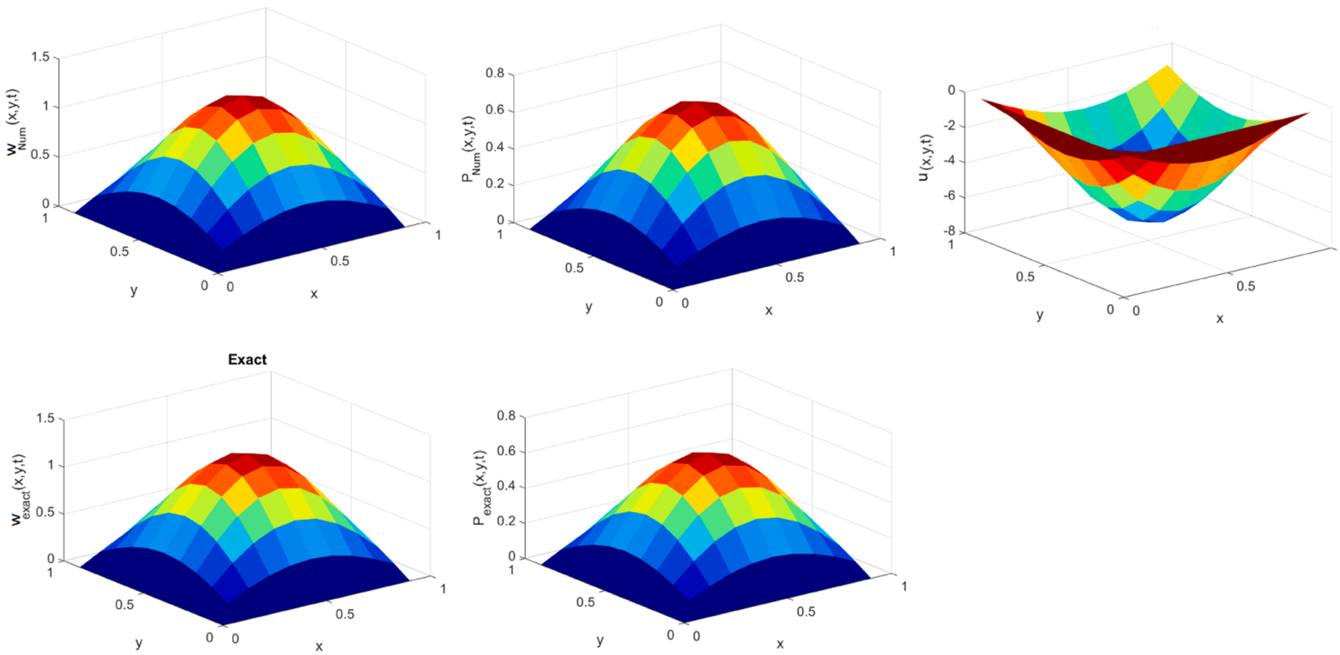

Fig. 9. For problem 3, the behaviour of the numerical solution when $\Theta = \frac{1}{2}$, $\gamma = \frac{1}{10}$ and $\beta(x,t) = 1.99 - \frac{t}{100}$.

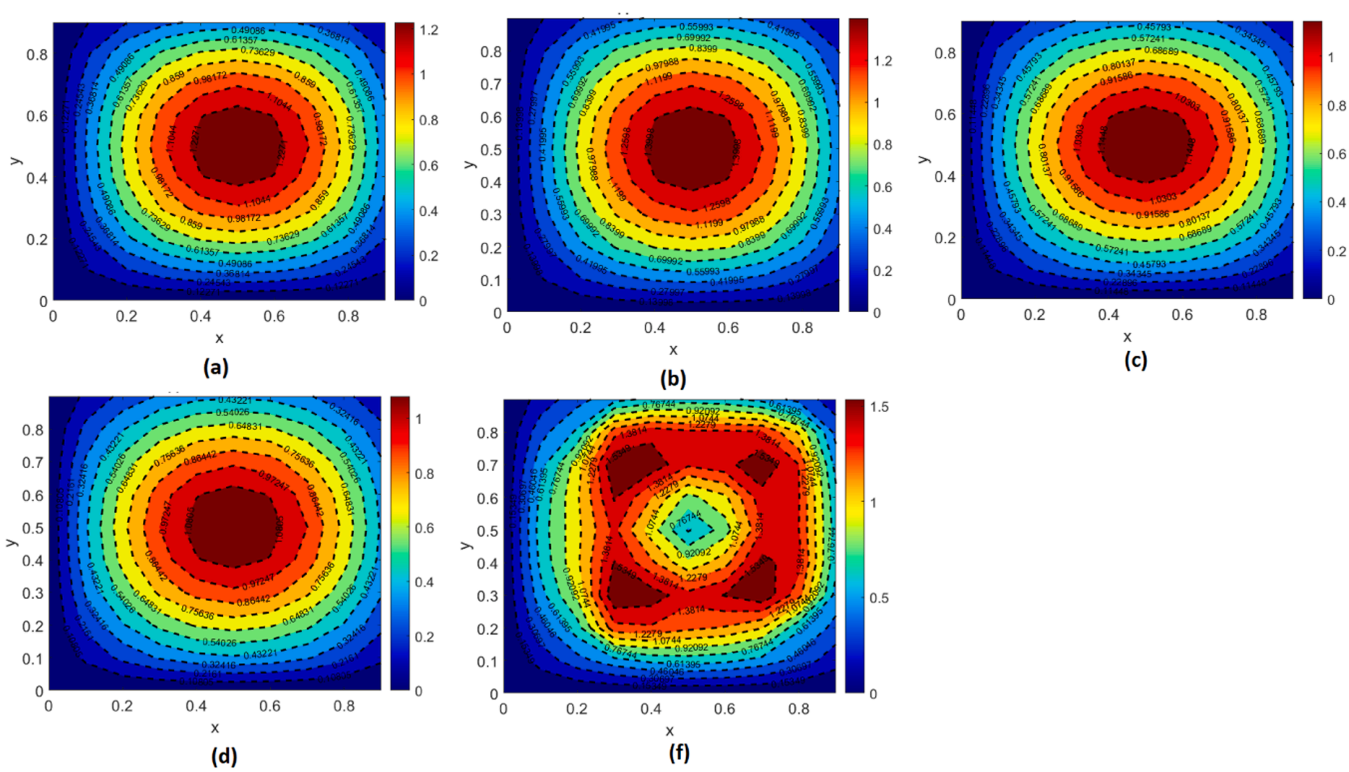

Fig. 10. For problem 3, the behaviour of the numerical solution using different methods at $\gamma = \frac{1}{10}$ and $\beta(x,t) = 1.99 - \frac{t}{100}$ (a) Exact solution w (b) Numerical solution w using NLFM, (c) Numerical solution w using NWAFDM when $\Theta = \frac{1}{2}$, (d) Numerical solution w using NWAFDM when $\Theta = 0$, (f) Numerical solution w using NWAFDM when $\Theta = 1$.

$$f(x,y,t) = \left(1 + 2\pi^2\right)e^t \ sin(\pi x) \ sin(\pi y) + \frac{1}{\gamma} \ sin(\pi x) \ sin(\pi y)(t - T)^2,$$

and

$$g(x,y,t) = \left(e^t - 2 - 2\pi^2(t-T)^2\right) \ sin(\pi x) \ sin(\pi y),$$

such that the exact solution is

$$w(x,y,t) = e^t sin(\pi x) \ sin(\pi y)$$

and

$$P(x,y,t) = sin(\pi x) \ sin(\pi y) \ (t-T)^2.$$

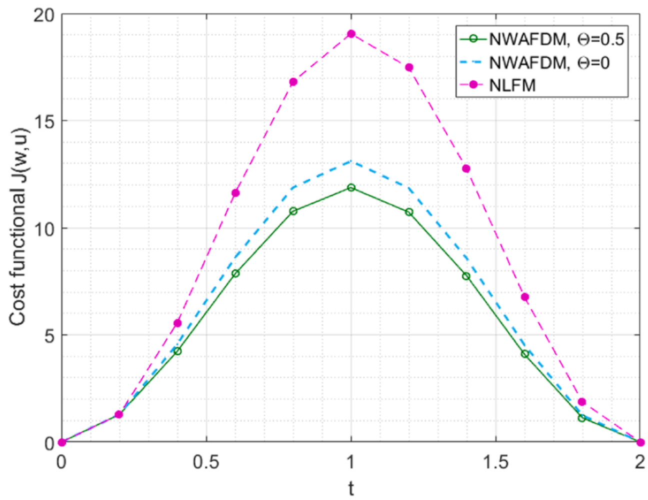

Fig. 11. For problem 3, the behaviour of the cost functional $J(w,u)$ using different methods at $\gamma=\frac{1}{10}$ and $\beta(x,y,t)=1.8-0.05(xy)^t$.

Table 3
For problem 3, error norms (5.1–5.2) when $n=l=6, m=7, \gamma=10^{-1}, \psi(\Delta x)=sinh(\Delta x)$, $\psi(\Delta y)=sinh(\Delta y)$, and $\psi(\Delta t)=sinh(\Delta t)$ at different space and time levels.

(X_f, Y_f, T)	NWAFDM				NLFM	
	$(\Theta=\frac{1}{2})$		$(\Theta=0)$			
	$e_u(x,y,t)$	$e_P(x,y,t)$	$e_u(x,y,t)$	$e_P(x,y,t)$	$e_u(x,y,t)$	$e_P(x,y,t)$
(0.2,0.2,0.2)	2.5e-03	1.3e-03	1.0e-03	6.555e-04	1.10e-03	4.91e-02
(0.2,0.2,0.5)	2.80e-03	1.7985e-04	3.10e-03	6.166e-05	1.10e-03	1.28e-03
(0.5,0.5,0.2)	4.51e-02	2.56e-02	8.54e-02	4.90e-03	7.11e-02	7.460e-01
(0.5,0.5,0.5)	4.69e-02	6.8e-03	8.45e-02	3.3e-03	4.99e-02	3.004 e-01
(0.5,0.5,1)	5.95e-02	5.9e-03	8.20e-02	3.6e-03	2.26e-02	1.246e-01
(1,1,1)	2.491e-01	8.52e-02	3.504 e-02	3.15 e-02	2.436e-01	3.852e-01
(1,1,2)	5.445e-01	1.077e-01	1.3e-01	8.01e-01	1.657e-01	3.321e-01

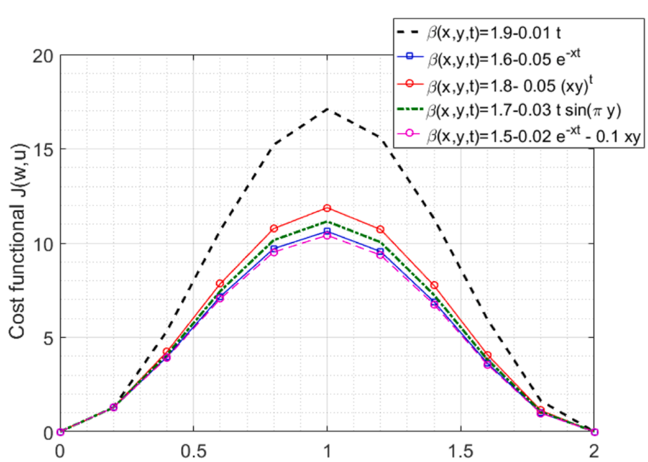

Fig. 12. For problem 3, the effect of variable order $\beta(x,y,t)$ on the cost functional $J(w,u)$ when $\gamma=\frac{1}{10}$.

6. Conclusions

In this paper, the necessary optimality conditions for the optimal solution of the fractional diffusion wave equation with a reaction term are derived, where the control function is the source function. The aim was to find the source function with the lowest cost functional. We demonstrated the existence and uniqueness of the optimal solution. The definitions of proportional-Caputo variable order derivatives are used. NWAFDM is constructed to study numerically the linear variable order control diffusion wave equation. Comparisons between NWAFDM and NLFM for the proposed problems are done. Furthermore, one of the advantages of NWAFDM is that we can get a lower value for the cost functional at $(\Theta=\frac{1}{2})$. Several figures have depicted the simulation of numerical solutions where the variable-order derivatives change with time and space. Furthermore, we claimed that NWAFDM can be applied to solve the variable-order optimal control problem. All the results in this paper were obtained using MATLAB (R2020a).

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Supplementary materials

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.padiff.2024.100658.

Appendix. Construction of NLFM using variable-order CPC derivative

The discretization form of the variable-order optimal control of diffusion-wave with a reaction term system (3.3)-(3.5) using definition (4.3) is given as follows:

Consider $n, l, m \in \mathbb{N}$ and the mesh points $x_i = i\triangle x,\ \ i=0,1,2,\ldots,n$, $t_r = r\triangle t,\ \ r=0,1,2,\ldots,m$. Where $\triangle x$ and $\triangle t$ space and time step lengths, respectively.

The state equation:

$$\sum_{r=0}^{k-1}\left[K_1(\beta(x,y,t))w_{i,j}^{k+1-r}+K_0(\beta(x,y,t))\psi(\Delta t)^{-\beta(x,y,t)}\left(w_{i,j}^{k+1-r}-2w_{i,j}^{k-r}+w_{i,j}^{k-1-r}\right)\right]d_r$$

$$-\Delta_h\frac{w_{i,j}^{k+1}+w_{i,j}^{k-1}}{2}+\alpha w_{i,j}^{k}+\frac{P_{i,j}^{k}}{\gamma}-f_{i,j}^{k}=0,$$

$$d_r=\frac{(r+1)^{2-\beta(x,y,t)}-(r)^{2-\beta(x,y,t)}}{\Gamma(3-\beta(x,y,t))}.$$

The adjoint equation:

$$\sum_{r=0}^{k-1}\left[K_1(\beta(x,y,t))P_{i,j}^{k+1-r}+K_0(\beta(x,y,t))\psi(\Delta t)^{-\beta(x,y,t)}\left(P_i^{k+1-j}-2P_i^{k-j}+P_i^{k-1-j}\right)\right]b_r$$

$$-\Delta_h\frac{P_{i,j}^{k+1}+P_{i,j}^{k-1}}{2}+\alpha P_{i,j}^k-w_{i,j}^k+g_{i,j}^k=0.$$

$$b_r=\frac{(r)^{2-\beta(x,y,t)}-(r+1)^{2-\beta(x,y,t)}}{\Gamma(3-\beta(x,y,t))},$$

where Δ_h be the second order approximation of the Laplacian operator Δ by using the finite difference method (e.g., central difference).

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