J. Math. Computer Sci., 32 (2024), 229–240

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Solving fractional integro-differential equations by Aboodh transform

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Abstract

This study approaches some families of fractional integro-differential equations (FIDEs) using a simple fractional calculus method, which leads to several appealing consequences, including the classical Frobenius method, which is generalized. The method presented here is based mostly on certain general theorems on particular solutions of FIDEs using the Aboodh transform and binomial series extension coefficients. We additionally demonstrate techniques to solve FIDEs.

Keywords: Riemann-Liouville (RL), fractional integral, fractional-order differential equation, gamma function, Mittag-Leffler function, Wright function, Aboodh transform of the fractional derivative.

2020 MSC: 26A33, 31A10, 33C10, 34A05, 35K37.

1. Introduction

Fractional calculus is a branch of mathematics which explores the theory and applications of non-integer order of integrals and derivatives [4]. Fractional calculus has grown in importance and popularity as a result of its reliable applications in a wide range of scientific and engineering sectors. These contributions to science and engineering are established on mathematics. Because of the developing applications, there has been a lot of interest for creating transforms for solving FIDEs. Several widely established

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doi: 10.22436/jmcs.032.03.04

Received: 2023-06-26 *Revised:* 2023-06-30 *Accepted:* 2023-07-23

notions that are strongly associated to fractional calculus will also be encountered quite often. They include the gamma function (fn.), Beta fn., Error fn., Mittag-Leffler fn., and Mellin-Ross fn. [15]. Integral transformations are considered one of the most essential mathematical approaches for solving differential equations, partial differential equations, partial integro-differential equations, delay differential equations, and population development. We also refer the reader to [5–7, 9–12] and the references cited therein for the qualitative analysis of these equations. Aboodh transform [2] has been extracted from the classical Fourier integral. Khalid Aboodh introduced the Aboodh transform in order to demonstrate the method of solving several ordinary differential equations in the time domain because of its simplicity and mathematical attributes. Fourier, Laplace, Mohand, and Elzaki transforms [1, 8, 13, 14] are the most commonly used mathematical tools for solving differential equations. Aboodh transform and some of its basic features are also utilized to solve differential equations. The Aboodh transform is more closely related to the Laplace transform. In 2020, Aruldoss and Anusuya Devi [3] developed the concept of using the extension coefficients of binomial series and also the Aboodh transform of the fractional derivative to generate explicit solutions to homogeneous fractional differential equations (see, for example, the papers [16, 17]).

In this investigation, we capitalise on the Aboodh transform of the fractional derivative and binomial series extension coefficients to solve several FIDEs. Furthermore, we conceal several properties that have relevance to our main topic.

2. Preliminaries

1. For the fn. $h(t)$, the RL fractional integral of order $\omega > 0$ is defined as:

$$I_t^{\omega} h(t) = \frac{1}{\Gamma(\omega)} \int_a^t (t-\zeta)^{\omega-1} h(\zeta) d\zeta.$$

2. Caputo fractional derivative of the fn. $h(t)$ is defined by:

$$D_t^{\omega} h(t) = \begin{cases} h^{(q)}(t), & \text{if } \omega = q \in \mathbb{N}, \\ \frac{1}{\Gamma(q-\omega)} \int_0^{\zeta} \frac{h^{(q)}(t)}{(t-x)^{\omega-q+1}} dt, & \text{if } q-1 < \omega < q, \end{cases}$$

where the Euler gamma fn. $\Gamma(.)$ is defined by

$$\Gamma(\varphi) = \int_0^{\infty} t^{\varphi-1} e^{-t} dt, \quad (\mathbb{R}(\varphi) > 0).$$

3. The Aboodh transform of a fn. $h(t)$, $t \in (0, \infty)$ is defined by

$$A[h(t)](r) = \frac{1}{r} \int_0^{\infty} h(t) e^{-rt} dt \quad (r \in \mathbb{C}).$$

4. Mittag-Leffler fn. is defined by

$$E_{\omega,\phi}(\varphi) = \sum_{\aleph=0}^{\infty} \frac{\varphi^{\aleph}}{\Gamma(\omega\aleph + \phi)}, \quad (\varphi, \omega, \phi \in \mathbb{C}, \mathbb{R}(\omega) > 0).$$

5. Simplest Wright fn. is defined by

$$\rho(\omega, \phi; \varphi) = \sum_{\aleph=0}^{\infty} \frac{1}{\Gamma(\omega\aleph + \phi)} . \frac{\varphi^{\aleph}}{\aleph!}, \qquad (\nu, \omega, \phi \in \mathbb{C}).$$

6. The general Wright fn. ${}_i\lambda_j(\varphi)$ is classified as $\varphi \in \mathbb{C}$, $\mathfrak{v}_{1p}, \mathfrak{v}_{2q} \in \mathbb{C}$, and real $\omega_p, \phi_q \in \mathbb{R}$ ($p = 1, \dots, i; q = 1, \dots, j$) by the series

$$ {}_i\lambda_j(\nu) = {}_i\lambda_j\left(\begin{matrix}(\mathfrak{v}_{1p},\omega_p)_{1,i}\\ (\mathfrak{v}_{2q},\phi_q)_{1,j}\end{matrix}\,\middle|\,\varphi\right) = \sum_{\aleph=0}^{\infty} \frac{\prod_{p=1}^{i}\Gamma(\mathfrak{v}_{1p}+\omega_p\aleph)}{\prod_{q=1}^{j}\Gamma(\mathfrak{v}_{2q}+\phi_q\aleph)}.\frac{\varphi^{\aleph}}{\aleph!}, $$

where $\varphi, \mathfrak{v}_{1p}, \mathfrak{v}_{2q} \in \mathbb{C}, \omega_p, \phi_q \in \mathbb{R}, p = 1, 2, \dots, i$ and $q = 1, 2, \dots, j$.

7. Binomial formula is defined as

$$ \binom{\Delta}{\delta} = \frac{\Delta!}{\Delta!(\Delta-\delta)!} = \frac{\Delta(\Delta-1)(\Delta-\delta+1)}{\delta!}, $$

where Δ and δ are integers. Observe that $0! = 1$. Then

$$ \binom{\Delta}{0} = 1, \ \binom{\Delta}{\Delta} = 1, \text{ and } (1-\varphi)^{-\Delta} = \sum_{\aleph=0}^{\infty} \frac{(\Delta)_{\aleph}}{\aleph!}\varphi^{i} = \sum_{\aleph=0}^{\infty}\binom{\Delta+\aleph-1}{\aleph}\varphi^{\aleph}. $$

8. The n^{th} derivative of Aboodh transform of $h(t)$ is

$$ A[h^{(n)}(t)](r) = r^n A(r) - \sum_{k=0}^{n-1}\frac{h^{(k)}(0)}{r^{2-n+k}}. $$

9. Aboodh transform of $h(t) = t^{\alpha}$ is

$$ A[t^{\alpha}](r) = \frac{\Gamma(\alpha+1)}{r^{\alpha+2}}. $$

10. The convolution integral of Aboodh transform is

$$ A[(f*g)(t)] = rA[f(t)]A[g(t)]. $$

Remark 2.1. The Aboodh transform of RL fractional integral operator of order $\omega > 0$ of the fn. $h(t)$ is given by

$$ A[J^{\omega}h(t)] = \frac{f(r)}{r^{\omega}}. $$

Proof. RL fractional integral of the function $h(t)$ can be written as

$$ J^{\omega}h(x) = \frac{1}{\Gamma(\omega)}\int_0^x (x-a)^{\omega-1}h(t)dt. $$

Applying Aboodh transform on both sides, we get

$$ A[J^{\omega}h(x)] = A\left[\frac{1}{\Gamma(\omega)}\int_0^x (x-a)^{\omega-1}h(t)dt\right] = \frac{1}{\Gamma(\omega)}rf(r)g(r) = \frac{f(r)}{r^{\omega}}; \quad \text{where} \quad g(r) = A[x^{\omega-1}] = \frac{\Gamma(\omega)}{r^{\omega+1}}. $$

□

Remark 2.2. The Aboodh transform of Caputo fractional derivative of order $\omega > 0$ of the fn. $h(t)$ is given by

$$ A[D^{\omega}h(t)] = \frac{1}{r^{n-\omega}}\left[r^n A[h(t)] - r^{n-2}f(0) - r^{n-3}f^{(1)}(0) - \dots - \frac{f^{n-1}(0)}{r}\right] $$

for $m-1 < \omega \leqslant m, m \in \mathbb{N}$.

Proof. Applying Aboodh transform on the Caputo fractional derivative of $h(t)$, we have

$$\begin{aligned}
A[D^{\omega}h(t)] &= \frac{1}{r}\int_0^{\infty} e^{-rt}[D^{\omega}h(t)]d(t)\\
&= \frac{1}{r}\int_0^{\infty} e^{-rt}\frac{1}{\Gamma(n-\omega)}\int_0^{t}\frac{h^{(n)}(\tau)}{(t-\tau)^{\omega-n+1}}d\tau dt\\
&= \frac{1}{r\Gamma(n-\omega)}\int_0^{\infty}\int_{\tau}^{\infty} e^{-rt}\frac{h^{(n)}(\tau)}{(t-\tau)^{\omega-n+1}}dtd\tau\\
&= \frac{1}{r\Gamma(n-\omega)}\int_0^{\infty} h^{(n)}(\tau)\int_0^{\infty}\frac{e^{-r(u+\tau)}}{u^{\omega-n+1}}dud\tau\\
&= \frac{1}{r\Gamma(n-\omega)}\int_0^{\infty} h^{(n)}(\tau)\int_0^{\infty} e^{-ru}e^{-r\tau}u^{n-\omega-1}dud\tau\\
&= \frac{1}{r\Gamma(n-\omega)}\int_0^{\infty} h^{(n)}(\tau)e^{-r\tau}\int_0^{\infty} e^{-ru}u^{n-\omega-1}dud\tau\\
&= \frac{1}{\Gamma(n-\omega)}\int_0^{\infty} h^{(n)}(\tau)e^{-r\tau}\frac{\Gamma(n-\omega-1+1)}{r^{n-\omega-1+2}}d\tau\\
&= \frac{1}{\Gamma(n-\omega)}\int_0^{\infty} h^{(n)}(\tau)e^{-r\tau}\frac{\Gamma(n-\omega)}{r^{n-\omega+1}}d\tau\\
&= \frac{1}{r^{n-\omega+1}}\int_0^{\infty} h^{(n)}(\tau)e^{-r\tau}d\tau\\
&= \frac{1}{r^{n-\omega+1}}A[h^{(n)}(t)](r)\\
&= \frac{1}{r^{n-\omega}}\left[r^{n}A[h(t)] - r^{n-2}f(0) - r^{n-3}f^{(1)}(0) - \cdots - \frac{f^{n-1}(0)}{r}\right].
\end{aligned}$$

□

3. Solutions of the fractional integro-differential equations

In this part, we can firmly claim that $\mathfrak{q}(t)$ is enough to ensure that Aboodh transform $A[\mathfrak{g}]$ proceeds for some value of the parameter r.

Theorem 3.1. *If $1 < \tau \leqslant 2$ and $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathbb{R}$, then the FIDE is*

$$\mathfrak{q}^{(\tau)}(t) + \mathfrak{m}_1\mathfrak{q}'(t) + \mathfrak{m}_2\mathfrak{q}(t) = \int_0^{r}\frac{g(t)}{(r-t)^{\sigma}}dt, \quad 0 < \sigma < 1,$$

with the initial condition $\mathfrak{q}(0) = \mathfrak{l}_0$ and $\mathfrak{q}'(0) = \mathfrak{l}_1$ its proposal is provided by

$$\begin{aligned}
\mathfrak{q}(t) = \mathfrak{l}_0 &\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!}\mathfrak{m}_1{}^{\aleph}\mathfrak{m}_2{}^{\gamma}.\frac{t^{(\tau-1)\aleph+\tau\gamma}}{\Gamma[(\tau-1)\aleph+\tau\gamma+1]}\\
&+ \mathfrak{m}_1\mathfrak{l}_0\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!}\mathfrak{m}_1{}^{\aleph}\mathfrak{m}_2{}^{\gamma}.\frac{t^{(\tau-1)\aleph+\tau\gamma+\tau-1}}{\Gamma[(\tau-1)\aleph+\tau\gamma+\tau]}\\
&+ \mathfrak{l}_1\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!}\mathfrak{m}_1{}^{\aleph}\mathfrak{m}_2{}^{\gamma}.\frac{t^{(\tau-1)\aleph+\tau\gamma+1}}{\Gamma[(\tau-1)\aleph+\tau\gamma+2]}\\
&+ \frac{\sin\pi\sigma}{\pi}\frac{d}{dr}\int_0^{r}(r-t)^{\sigma-1}f(t)dt\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!}\mathfrak{m}_1{}^{\aleph}\mathfrak{m}_2{}^{\gamma}.\frac{t^{(\tau-1)\aleph+\tau\gamma+\tau-1}}{\Gamma[(\tau-1)\aleph+\tau\gamma+\tau]}.
\end{aligned}$$

Proof. Providing Aboodh transform on both sides, we get

$$A[q^{(\tau)}(t)] + m_1 A[q^{'}(t)] + m_2 A[q(t)] = A[f(t)],$$

where $f(t) = \int_0^r \frac{g(t)}{(r-t)^\sigma} dt$,

$$\begin{aligned}
\left[r^\tau A[q(t)] - \frac{q(o)}{r^{2-\tau}} - \frac{q'(o)}{r^{3-\tau}}\right] + m_1\left[rA[q(t)] - \frac{q(0)}{r}\right] + m_2 A[q(t)] &= A[f(t)],\\
r^\tau A[q(t)] - l_0 r^{\tau-2} - l_1 r^{\tau-3} + m_1 r A[q(t)] - m_1 l_0 r^{-1} + m_2 A[q(t)] &= A[f(t)],\\
A[q(t)](r^\tau + m_1 r + m_2) &= l_0 r^{\tau-2} + l_1 r^{\tau-3} + m_1 l_0 r^{-1} + A[f(t)],\\
A[q(t)] &= \frac{l_0 r^{\tau-2} + l_1 r^{\tau-3} + m_1 l_0 r^{-1} + A[f(t)]}{r^\tau + m_1 r + m_2},\\
A[q(t)] &= \frac{l_0(r^{\tau-2} + m_1 r^{-1})}{r^\tau + m_1 r + m_2} + \frac{l_1 r^{\tau-3}}{r^\tau + m_1 r + m_2} + \frac{A[f(t)]}{r^\tau + m_1 r + m_2}. \qquad (3.1)
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{r^\tau + m_1 r + m_2} &= \frac{r^{-1}}{r^{\tau-1} + m_1 + m_2 r^{-1}}\\
&= \frac{r^{-1}}{(r^{\tau-1} + m_1)\left(1 + \frac{m_2 r^{-1}}{r^{\tau-1}+m_1}\right)}\\
&= \frac{r^{-1}}{r^{\tau-1} + m_1}\sum_{\aleph=0}^{\infty}(-1)^\aleph\left(\frac{m_2 r^{-1}}{r^{\tau-1} + m_1}\right)^\aleph\\
&= \sum_{\aleph=0}^{\infty}\frac{(-m_2)^\aleph r^{-\aleph-1}}{(r^{\tau-1} + m_1)^{\aleph+1}} \qquad (3.2)\\
&= \sum_{\aleph=0}^{\infty}\frac{(-m_2)^\aleph r^{-\tau\aleph-\tau}}{(1 + m_1 r^{1-\tau})^{\aleph+1}}\\
&= \sum_{\aleph=0}^{\infty}(-m_2)^\aleph r^{-\tau\aleph-\tau}\sum_{\gamma=0}^{\infty}(-m_1 r^{1-\tau})^\gamma\binom{\aleph+\gamma}{\gamma}\\
&= \sum_{\aleph=0}^{\infty}(-m_2)^\aleph\sum_{\gamma=0}^{\infty}\binom{\aleph+\gamma}{\gamma}(-m_1)^\gamma r^{\gamma-(\tau)\gamma-\tau\aleph-\tau}
\end{aligned}$$

and

$$A[f(t)] = A\left[\int_0^r \frac{g(t)}{(r-t)^\sigma}dt\right].$$

This is convolution integral

$$F(p) = rK(p)L(p).$$

Here $K(p)$ is Aboodh transform of $K(r) = r^{-\sigma}$ and

$$\begin{aligned}
&A[K(r)] = r^{-\sigma}, && K(p) = \frac{\Gamma(-\sigma+1)}{r^{-\sigma+2}}, \quad L(p) = \frac{K(p)}{\Gamma(1-\sigma)r^{\sigma-2}r^1},\\
&L(p) = \frac{K(p)r^\sigma.r}{\Gamma(1-\sigma)}, && L(p) = \frac{K(p)r^\sigma.r\Gamma(\sigma)}{\Gamma(1-\sigma)\Gamma(\sigma)}, \quad L(p) = \frac{r^{-\sigma+1}\Gamma(\sigma)F(p)}{\pi \text{cosec}\pi\sigma}, \qquad (3.3)\\
&L(p) = \frac{\sin\pi\sigma}{\pi}rA\left[\int_0^r (r-t)^{\sigma-1}f'(t)dt\right].
\end{aligned}$$

Substituting equations (3.2) and (3.3) in (3.1), we get

$$\begin{aligned} A[q(t)] &= l_0 \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph} m_1{}^{\aleph} m_2{}^{\gamma} r^{(1-\tau)\aleph-\tau\gamma-2} \\ &+ m_1 l_0 \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph} m_1{}^{\aleph} m_2{}^{\gamma} r^{(1-\tau)\aleph-\tau\gamma-\tau-1} \\ &+ l_1 \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph} m_1{}^{\aleph} m_2{}^{\gamma} r^{(1-\tau)\aleph-\tau\gamma-3} \\ &+ \frac{\sin\pi\sigma}{\pi} A\left[\int_0^r (r-t)^{\sigma-1} f'(t)\,dt\right] \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph} m_1{}^{\aleph} m_2{}^{\gamma} r^{(1-\tau)\aleph-\tau\gamma-\tau+1}. \end{aligned} \tag{3.4}$$

Thus, providing inverse Aboodh transform on (3.4), we get

$$\begin{aligned} q(t) &= l_0 \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!} m_1{}^{\aleph} m_2{}^{\gamma}.\frac{t^{(\tau-1)\aleph+\tau\gamma}}{\Gamma[(\tau-1)\aleph+\tau\gamma+1]} \\ &+ m_1 l_0 \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!} m_1{}^{\aleph} m_2{}^{\gamma}.\frac{t^{(\tau-1)\aleph+\tau\gamma+\tau-1}}{\Gamma[(\tau-1)\aleph+\tau\gamma+\tau]} \\ &+ l_1 \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!} m_1{}^{\aleph} m_2{}^{\gamma}.\frac{t^{(\tau-1)\aleph+\tau\gamma+1}}{\Gamma[(\tau-1)\aleph+\tau\gamma+2]} \\ &+ \frac{\sin\pi\sigma}{\pi}\frac{d}{dr}\int_0^r (r-t)^{\sigma-1} f(t)\,dt \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!} m_1{}^{\aleph} m_2{}^{\gamma}.\frac{t^{(\tau-1)\aleph+\tau\gamma+\tau-1}}{\Gamma[(\tau-1)\aleph+\tau\gamma+\tau]}. \end{aligned}$$

This solution can be developed as Wright fn. as

$$\begin{aligned} q(t) &= l_0 \sum_{\aleph=0}^{\infty}\frac{(-m_2)^{\aleph} t^{\tau\aleph}}{\aleph!}\, {}_1\xi_1\left(\begin{matrix}(\aleph+1,1)\\(\tau\aleph+1,\tau-1)\end{matrix}\,\middle|\, -m_1 t^{\tau-1}\right) \\ &+ l_1 \sum_{\aleph=0}^{\infty}\frac{(-m_2)^{\aleph} t^{\tau\aleph+1}}{\aleph!}\, {}_1\xi_1\left(\begin{matrix}(\aleph+1,1)\\(\tau\aleph+1,\tau-1)\end{matrix}\,\middle|\, -m_1 t^{\tau-1}\right) \\ &+ m_1 l_0 \sum_{\aleph=0}^{\infty}\frac{(-m_2)^{\aleph} t^{\tau\aleph+\tau-1}}{\aleph!}\, {}_1\xi_1\left(\begin{matrix}(\aleph+1,1)\\(\tau\aleph+\tau,\tau-1)\end{matrix}\,\middle|\, -m_1 t^{\tau-1}\right) \\ &+ \frac{\sin\pi\sigma}{\pi}\frac{d}{dr}\int_0^r (r-t)^{\sigma-1} f(t)\,dt \sum_{\aleph=0}^{\infty}\frac{(-m_2)^{\aleph} t^{\tau\aleph+\tau-1}}{\aleph!}\, {}_1\xi_1\left(\begin{matrix}(\aleph+1,1)\\(\tau\aleph+\tau,\tau-1)\end{matrix}\,\middle|\, -m_1 t^{\tau-1}\right). \end{aligned}$$

□

Example 3.2. The FIDE is

$$q^{(\frac{3}{2})}(t) + 2q^{'}(t) + 3q(t) = \int_0^r \frac{g(t)}{(r-t)^{\frac{1}{2}}}\,dt,$$

with the initial condition $q(0) = 1$ and $q^{'}(0) = 0$ its proposal is provided by

$$q(t) = \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!} 2^{\aleph} 3^{\gamma}.\frac{t^{(\frac{1}{2})\aleph+(\frac{3}{2})\gamma}}{\Gamma[(\frac{1}{2})\aleph+(\frac{3}{2})\gamma+1]}$$

$$+2\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!}2^{\aleph}3^{\gamma}.\frac{t^{(\frac{1}{2})\aleph+(\frac{3}{2})\gamma+(\frac{1}{2})}}{\Gamma[(\frac{1}{2})\aleph+(\frac{3}{2})\gamma+\frac{1}{2}]}$$
$$+\frac{1}{\pi}\frac{d}{dr}\int_0^r(r-t)^{\frac{-1}{2}}f(t)dt\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}.\frac{\Gamma(\aleph+\gamma+1)}{\aleph!\gamma!}2^{\aleph}3^{\gamma}.\frac{t^{(\frac{1}{2})\aleph+\frac{3}{2}\gamma+\frac{1}{2}}}{\Gamma[(\frac{1}{2})\aleph+(\frac{3}{2})\gamma+\frac{3}{2}]}.$$

Theorem 3.3. *If $1<\tau\leqslant 2$ and $m_2\in\mathbb{R}$, then the FIDE is*

$$q^{(\tau)}(t)+m_2q(t)=\int_0^r\frac{g(t)}{(r-t)^{\sigma}}dt,\quad 0<\sigma<1, \tag{3.5}$$

with the initial condition $q(0)=l_0$ and $q'(0)=l_1$ its proposal is provided by

$$q(t)=l_0E_{\tau,1}(-m_2t^{\tau})+l_1tE_{\tau,2}(-m_2t^{\tau})+\frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r(r-t)^{\sigma-1}f(t)dt.t^{\tau-1}E_{\tau,\tau}(-m_2t^{\tau}). \tag{3.6}$$

Proof. Providing Aboodh transform on both sides in (3.5), we get

$$A[q^{(\tau)}(t)]+m_2A[q(t)]=A[f(t)],$$

where $f(t)=\int_0^r\frac{g(t)}{(r-t)^{\sigma}}dt$,

$$\left[r^{\tau}A[q(t)]-\frac{q(o)}{r^{2-\tau}}-\frac{q'(o)}{r^{3-\tau}}\right]+m_2A[q(t)]=A[f(t)],$$
$$r^{\tau}A[q(t)]-l_0r^{\tau-2}-l_1r^{\tau-3}+m_2A[q(t)]=A[f(t)],$$
$$A[q(t)](r^{\tau}+m_2)=l_0r^{\tau-2}+l_1r^{\tau-3}+A[f(t)],$$
$$A[q(t)]=\frac{l_0r^{\tau-2}+l_1r^{\tau-3}+A[f(t)]}{(r^{\tau}+m_2)}.$$

Now,

$$\frac{1}{r^{\tau}+m_2}=\frac{1}{r^{\tau}(1+\frac{m_2}{r^{\tau}})}$$
$$=\frac{r^{-\tau}}{1+m_2r^{-\tau}}=r^{-\tau}(1+m_2r^{-\tau})^{-1}=r^{-\tau}[1-m_2r^{-\tau}+(m_2r^{-\tau})^2-\cdots]=r^{-\tau}\sum_{\aleph=0}^{\infty}(-m_2r^{-\tau})^{\aleph},$$
$$A[q(t)]=l_0r^{\tau-2}r^{-\tau}\sum_{\aleph=0}^{\infty}(-m_2r^{-\tau})^{\aleph}+l_1r^{\tau-3}r^{-\tau}\sum_{\aleph=0}^{\infty}(-m_2r^{-\tau})^{\aleph}+A[f(t)]r^{-\tau}\sum_{\aleph=0}^{\infty}(-m_2r^{-\tau})^{\aleph},$$

$$\begin{aligned}A[q(t)]&=l_0\sum_{\aleph=0}^{\infty}(-m_2)^{\aleph}r^{-\tau\aleph-2}+l_1\sum_{\aleph=0}^{\infty}(-m_2)^{\aleph}r^{-\tau\aleph-3}\\&+\frac{\sin\pi\sigma}{\pi}A\left[\int_0^r(r-t)^{\sigma-1}f'(t)dt\right]\sum_{\aleph=0}^{\infty}(-m_2)^{\aleph}r^{-\tau\aleph-\tau+1}.\end{aligned} \tag{3.7}$$

Thus, providing inverse Aboodh transform on both sides in (3.7), we get

$$q(t)=l_0\sum_{\aleph=0}^{\infty}(-m_2)^{\aleph}\frac{t^{\tau\aleph}}{\Gamma(\tau\aleph+1)}+l_1\sum_{\aleph=0}^{\infty}(-m_2)^{\aleph}\frac{t^{\tau\aleph+1}}{\Gamma(\tau\aleph+2)}$$
$$+\frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r(r-t)^{\sigma-1}f(t)dt\sum_{\aleph=0}^{\infty}(-m_2)^{\aleph}\frac{t^{\tau\aleph+\tau-1}}{\Gamma(\tau\aleph+\tau)},$$
$$q(t)=l_0E_{\tau,1}(-m_2t^{\tau})+l_1tE_{\tau,2}(-m_2t^{\tau})+\frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r(r-t)^{\sigma-1}f(t)dt.t^{\tau-1}E_{\tau,\tau}(-m_2t^{\tau}).$$

□

Example 3.4. The FIDE

$$q^{(\tau)}(t)+q(t)=0, \qquad 1<\tau\leqslant 2; \quad 0<\sigma<1,$$

with the initial condition $q(0)=1$ and $q'(0)=1$, has the following solution

$$q(t)=\sum_{\aleph=0}^{\infty}(-1)^{\aleph}\frac{t^{\tau\aleph}}{\Gamma(\tau\aleph+1)}+\sum_{\aleph=0}^{\infty}(-1)^{\aleph}\frac{t^{\tau\aleph+1}}{\Gamma(\tau\aleph+2)}+\frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r(r-t)^{\sigma-1}f(t)dt\sum_{\aleph=0}^{\infty}(-1)^{\aleph}\frac{t^{\tau\aleph+\tau-1}}{\Gamma(\tau\aleph+\tau)},$$

$$q=E_{\tau,1}(-t^{\tau})+tE_{\tau,2}(-t^{\tau})+\frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r(r-t)^{\sigma-1}f(t)dt.t^{\tau-1}E_{\tau,\tau}(-t^{\tau}).$$

Proposition 3.5 ([4]). *The integro-differential equation of a nearly simple harmonic vibration is*

$$q^{(\tau)}(t)+\mathfrak{z}^2q(t)=\int_0^r\frac{g(t)}{(r-t)^{\sigma-1}}dt; \qquad 1<\tau\leqslant 2; \quad 0<\sigma<1,$$

with the initial condition $q(0)=l_0$ *and* $q'(0)=l_1$*, proposed by*

$$q(t)=l_0E_{\tau,1}(-\mathfrak{z}^2t^{\tau})+k_1tE_{\tau,2}(-\mathfrak{z}^2t^{\tau})+\frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r(r-t)^{\sigma-1}f(t)dt.t^{\tau-1}E_{\tau,\tau}(-\mathfrak{z}^2t^{\tau}).$$

Proof. We accomplish this proof by inputting $m_2=\mathfrak{z}^2$ into the equation (3.6). □

Theorem 3.6. *If* $1<\tau\leqslant 2$ *and* $m_1,m_2\in\mathbb{R}$*, then the FIDE is*

$$q''(t)+m_1q^{\tau}(t)+m_2q(t)=\int_0^r\frac{g(t)}{(r-t)^{\sigma}}dt, \qquad 0<\sigma<1, \tag{3.8}$$

with the initial condition $q(0)=l_0$ *and* $q^{'}(0)=l_1$*, its proposal is provided by*

$$\begin{aligned}q(t)=&\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}\frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1){m_1}^{\aleph}{m_2}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma+1]}\frac{t^{(2-\tau)\aleph+2\gamma}}{\aleph!\gamma!}\Big[l_0+\frac{l_1t}{(2-\tau)\aleph+2\gamma+1}\\&+\frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r(r-t)^{\sigma-1}f(t)dt\frac{t}{(2-\tau)\aleph+2\gamma+1}\Big]\\&+\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}\frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1){m_1}^{\aleph}{m_2}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma-\tau+3]}\frac{t^{(2-\tau)\aleph+2\gamma-\tau+2}}{\aleph!\gamma!}\Big[m_1l_0+\frac{m_1l_1t}{(2-\tau)\aleph+2\gamma-\tau+3}\Big].\end{aligned}$$

Proof. Providing Aboodh transfprm on both sides in (3.8), we get

$$A[q^{''}(t)]+m_1A[q^{(\tau)}(t)]+m_2A[q(t)]=A[f(t)],$$

where $f(t)=\int_0^r\frac{g(t)}{(r-t)^{\sigma}}dt$,

$$\begin{aligned}\left[r^2A[q(t)]-q(0)-\frac{q'(0)}{r}\right]+m_1\left[r^{\tau}A[q(t)]-\frac{q(o)}{r^{2-\tau}}-\frac{q'(o)}{r^{3-\tau}}\right]+m_2A[q(t)]&=A[f(t)],\\ r^2A[q(t)]-l_0-\frac{l_1}{r}+m_1r^{\tau}A[q(t)]-\frac{m_1l_0}{r^{2-\tau}}-\frac{m_1l_1}{r^{3-\tau}}+m_2A[q(t)]&=A[f(t)],\\ r^2A[q(t)]-l_0-l_1r^{-1}+m_1r^{\tau}A[q(t)]-m_1l_0r^{\tau-2}-m_1l_1r^{\tau-3}+m_2A[q(t)]&=A[f(t)],\\ r^2A[q(t)]+m_1r^{\tau}A[q(t)]+m_2A[q(t)]=l_0+l_1r^{-1}+m_1l_0r^{\tau-2}+m_1l_1r^{\tau-3}+A[f(t)],\end{aligned}$$

$$A[q(t)]=\frac{l_0+l_1r^{-1}+m_1l_0r^{\tau-2}+m_1l_1r^{\tau-3}+A[f(t)]}{(r^2+m_1r^{\tau}+m_2)}. \tag{3.9}$$

Now,

$$\begin{aligned}\frac{1}{r^2+m_1r^\tau+m_2} &= \frac{(r)^{-\tau}}{r^{2-\tau}+m_1+m_2r^{-\tau}}\\ &= \frac{r^{-\tau}}{(r^{2-\tau}+m_1)\left(1+\frac{m_2r^{-\tau}}{r^{2-\tau}+m_1}\right)}\\ &= \frac{r^{-\tau}}{r^{2-\tau}+m_1}\sum_{\aleph=0}^{\infty}(-1)^{\aleph}\left(\frac{m_2r^{-\tau}}{r^{2-\tau}+m_1}\right)^{\aleph}\\ &= \sum_{\aleph=0}^{\infty}\frac{(-m_2)^{\aleph}r^{-\aleph\tau-\tau}}{(r^{2-\tau}+m_1)^{\aleph+1}}\\ &= \sum_{\aleph=0}^{\infty}\frac{(-m_2)^{\aleph}r^{-2\aleph-2}}{(1+m_1r^{\tau-2})^{\aleph+1}}\\ &= \sum_{\aleph=0}^{\infty}(-m_2)^{\aleph}r^{-2\aleph-2}\sum_{\gamma=0}^{\infty}(-m_1r^{\tau-2})^{\gamma}\binom{\aleph+\gamma}{\gamma}\\ &= \sum_{\aleph=0}^{\infty}(-m_2)^{\aleph}\sum_{\gamma=0}^{\infty}\binom{\aleph+\gamma}{\gamma}(-m_1)^{\gamma}r^{(\tau-2)\aleph-2\gamma-2}\end{aligned} \tag{3.10}$$

and

$$A[f(t)] = A\left[\int_0^r \frac{g(t)}{(r-t)^{\sigma}}dt\right].$$

We know that,

$$L(p) = \frac{\sin\pi\sigma}{\pi}rA\Big[\int_0^r (r-t)^{\sigma-1}f'(t)dt\Big]. \tag{3.11}$$

Substituting equations (3.10) and (3.11) in (3.9), we get

$$\begin{aligned}A[q(t)] = l_0 &\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph}m_1{}^{\aleph}m_2{}^{\gamma}r^{(\tau-2)\aleph-2\gamma-2}\\ &+ m_1l_0\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph}m_1{}^{\aleph}m_2{}^{\gamma}r^{(\tau-2)\aleph-2\gamma+\tau-4}\\ &+ l_1\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph}m_1{}^{\aleph}m_2{}^{\gamma}r^{(\tau-2)\aleph-2\gamma-3}\\ &+ m_1l_1\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph}m_1{}^{\aleph}m_2{}^{\gamma}r^{(\tau-2)\aleph-2\gamma+\tau-5}\\ &+ \frac{\sin\pi\sigma}{\pi}A\Big[\int_0^r(r-t)^{\sigma-1}f'(t)dt\Big]\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}(-1)^{\aleph+\gamma}\binom{\aleph+\gamma}{\aleph}m_1{}^{\aleph}m_2{}^{\gamma}r^{(\tau-2)\aleph-2\gamma-1}.\end{aligned} \tag{3.12}$$

Thus, providing inverse Aboodh transform on both sides in equation (3.12), we get

$$\begin{aligned}q(t) = l_0 &\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}\frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1)m_1{}^{\aleph}m_2{}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma+1]}\frac{t^{(2-\tau)\aleph+2\gamma}}{\aleph!\gamma!}\\ &+ m_1l_0\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty}\frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1)m_1{}^{\aleph}m_2{}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma-\tau+3]}\frac{t^{(2-\tau)\aleph+2\gamma-\tau+2}}{\aleph!\gamma!}\end{aligned}$$

$$+ l_1 \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty} \frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1){m_1}^{\aleph}{m_2}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma+2]}\frac{t^{(2-\tau)\aleph+2\gamma+1}}{\aleph!\gamma!}$$
$$+ m_1 l_0 \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty} \frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1){m_1}^{\aleph}{m_2}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma-\tau+4]}\frac{t^{(2-\tau)\aleph+2\gamma-\tau+3}}{\aleph!\gamma!}$$
$$+ \frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r (r-t)^{\sigma-1} f(t)\,dt \sum_{\gamma=0}^{\infty} \frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1){m_1}^{\aleph}{m_2}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma+2]}\frac{t^{(2-\tau)\aleph+2\gamma+1}}{\aleph!\gamma!},$$
$$q(t) = \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty} \frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1){m_1}^{\aleph}{m_2}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma+1]}\frac{t^{(2-\tau)\aleph+2\gamma}}{\aleph!\gamma!}\Big[l_0 + \frac{l_1 t}{(2-\tau)\aleph+2\gamma+1}$$
$$+ \frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r (r-t)^{\sigma-1} f(t)\,dt \frac{t}{(2-\tau)\aleph+2\gamma+1}\Big]$$
$$+ \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty} \frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1){m_1}^{\aleph}{m_2}^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma-\tau+3]}\frac{t^{(2-\tau)\aleph+2\gamma-\tau+2}}{\aleph!\gamma!}\Big[m_1 l_0 + \frac{m_1 l_1 t}{(2-\tau)\aleph+2\gamma-\tau+3}\Big].$$

□

Example 3.7. The FIDE is

$$q''(t) + \sqrt{3}q^{\frac{3}{2}}(t) + 3q(t) = \int_0^r \frac{g(t)}{(r-t)^{\frac{1}{2}}}\,dt,$$

with the initial conditions $q(0) = 1$ and $q'(0) = 0$, then its proposal is provided by

$$q(t) = \sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty} \frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1)(\sqrt{3})^{\aleph}3^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma+1]}\frac{t^{(2-\tau)\aleph+2\gamma}}{\aleph!\gamma!}\Big[1 + \frac{1}{\pi}\frac{d}{dr}\int_0^r (r-t)^{\frac{-1}{2}} f(t)\,dt \frac{t}{(2-\tau)\aleph+2\gamma+1}\Big]$$
$$+ \sqrt{3}\sum_{\aleph=0}^{\infty}\sum_{\gamma=0}^{\infty} \frac{(-1)^{\aleph+\gamma}\Gamma(\aleph+\gamma+1)(\sqrt{3})^{\aleph}3^{\gamma}}{\Gamma[(2-\tau)\aleph+2\gamma-\tau+3]}\frac{t^{(2-\tau)\aleph+2\gamma-\tau+2}}{\aleph!\gamma!}.$$

Proposition 3.8. *If $1 < \tau \leqslant 2$ and $m_1 \in \mathbb{R}$, then the FIDE is*

$$q''(t) + m_1 q^{(\tau)}(t) = \int_0^r \frac{g(t)}{(r-t)^{\sigma}}\,dt, \qquad 0 < \sigma < 1,$$

with the initial condition $q(0) = l_0$ and $q'(0) = l_1$, its proposal is provided by

$$q(t) = \sum_{\aleph=0}^{\infty} \frac{(-1)^{\aleph}{m_1}^{\aleph} t^{(2-\tau)\aleph}}{\Gamma[(2-\tau)\aleph+1]}\Big[l_0 + \frac{l_1 t}{(2-\tau)\aleph+1} + \frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r (r-t)^{\sigma-1} f(t)\,dt.\frac{t}{(2-\tau)\aleph+1}\Big]$$
$$+ \sum_{\aleph=0}^{\infty} \frac{(-1)^{\aleph}{m_1}^{\aleph} t^{(2-\tau)\aleph-\tau+2}}{\Gamma[(2-\tau)\aleph-\tau+3]}\Big[m_1 l_0 + \frac{m_1 l_1 t}{(2-\tau)\aleph-\tau+3}\Big].$$

Proof. We accomplish this proof by inputting $m_2 = 0$ into the equation (3.8). □

Theorem 3.9. *If $0 < \tau$, $\sigma \leqslant 1$ and $m_2 \in \mathbb{R}$, then the FIDE is*

$$q^{(\tau)}(t) - m_2 q(t) = \int_0^r \frac{g(t)}{(r-t)^{\sigma}}\,dt, \tag{3.13}$$

with the initial condition $q(0) = l_0$ and $q^{'}(0) = l_1$, its proposal is provided by

$$q(t) = l_0 E_{\tau,1}(bt^{\tau}) + \frac{\sin\sigma\pi}{\pi}\frac{d}{dr}\int_0^r (r-t)^{\sigma-1} f(t)\,dt.t^{\tau-1}E_{\tau,\tau}(bt^{\tau}).$$

Proof. Providing Aboodh transfprm on both sides in (3.13), we get

$$A[q^{(\tau)}(t)] - m_2 A[q(t)] = A[f(t)],$$

where $f(t) = \int_0^r \frac{g(t)}{(r-t)^\sigma} dt$,

$$\begin{aligned}
&\left[r^\tau A[q(t)] - \frac{q(o)}{r^{2-\tau}} - \frac{q'(o)}{r^{3-\tau}}\right] - m_2 A[q(t)] = A[f(t)],\\
&r^\tau A[q(t)] - l_0 r^{\tau-2} - l_1 r^{\tau-3} - m_2 A[q(t)] = A[f(t)],\\
&A[q(t)](r^\tau - m_2) = l_0 r^{\tau-2} + l_1 r^{\tau-3} + A[f(t)],\\
&A[q(t)] = \frac{l_0 r^{\tau-2} + l_1 r^{\tau-3} + A[f(t)]}{(r^\tau - m_2)}.
\end{aligned}$$

We know that,

$$\begin{aligned}
&\frac{1}{(r^\tau - m_2)} = r^{-\tau} \sum_{\aleph=0}^{\infty} (m_2 r^{-\tau})^\aleph,\\
&A[q(t)] = l_0 r^{\tau-2} r^{-\tau} \sum_{\aleph=0}^{\infty} (m_2 r^{-\tau})^\aleph + l_1 r^{\tau-3} r^{-\tau} \sum_{\aleph=0}^{\infty} (m_2 r^{-\tau})^\aleph + A[f(t)] r^{-\tau} \sum_{\aleph=0}^{\infty} (m_2 r^{-\tau})^\aleph,\\
&A[q(t)] = l_0 \sum_{\aleph=0}^{\infty} (m_2)^\aleph r^{-\tau\aleph-2} + l_1 \sum_{\aleph=0}^{\infty} (m_2)^\aleph r^{-\tau\aleph-3}\\
&\qquad + \frac{\sin \pi\sigma}{\pi} A\left[\int_0^r (r-t)^{\sigma-1} f'(t) dt\right] \sum_{\aleph=0}^{\infty} (m_2)^\aleph r^{-\tau\aleph-\tau+1}.
\end{aligned} \tag{3.14}$$

Thus, providing inverse Aboodh transform on both sides in (3.14), we get

$$\begin{aligned}
&q(t) = l_0 \sum_{\aleph=0}^{\infty} (m_2)^\aleph \frac{t^{\tau\aleph}}{\Gamma(\tau\aleph+1)} + l_1 \sum_{\aleph=0}^{\infty} (m_2)^\aleph \frac{t^{\tau\aleph+1}}{\Gamma(\tau\aleph+2)} + \frac{\sin \sigma\pi}{\pi} \frac{d}{dr} \int_0^r (r-t)^{\sigma-1} f(t) dt \sum_{\aleph=0}^{\infty} (m_2)^\aleph \frac{t^{\tau\aleph+\tau-1}}{\Gamma(\tau\aleph+\tau)},\\
&q(t) = l_0 E_{\tau,1}(bt^\tau) + \frac{\sin \sigma\pi}{\pi} \frac{d}{dr} \int_0^r (r-t)^{\sigma-1} f(t) dt . t^{\tau-1} E_{\tau,\tau}(bt^\tau).
\end{aligned}$$

□

4. Conclusion

The Aboodh transform was implemented to solve some FIDEs in this article. The relationship between the Aboodh transform and the Laplace transform is much deeper, and we are able to identify even more Aboodh transform interactions through this connection. We presented a distinctive approach for solving the FIDE applying the Aboodh transform and binomial series extension coefficients. We also concentrated on some properties and examples.

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