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SOME SYMMETRIC PROPERTIES AND APPLICATIONS OF WEIGHTED FRACTIONAL INTEGRAL OPERATOR

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Abstract

In this paper, a weighted generalized fractional integral operator based on the Mittag-Leffler function is established, and it exhibits symmetric characteristics concerning classical operators. We demonstrate the semigroup property as well as the boundedness of the operator in absolute continuous like spaces. In this work, some applications with graphical representation are also considered. Finally, we modify the weighted generalized Laplace transform and then applied it to the newly defined weighted fractional integral operator. The defined operator is an extension and generalization of classical Riemann–Liouville and Prabhakar integral operators.

Keywords: Mittag-Leffler Function; Symmetric Properties; Weighted Fractional Integral; Weighted Laplace Transform; Modified (k, s)-fractional Integral.

1. INTRODUCTION

For many researchers, such as mathematicians, biologists, chemists, economists, engineers, and physicists, fractional calculus is a valuable tool for solving real-world problems. Abel solved the Tautocrone issue,[1] he was the first to use fractional calculus theory, according to the history. The comprehensive study of fractional calculus is available in the books.[2–5] The hypothesis and developments of fractional calculus are briefly explored in the articles.[6–8]

There are many different kinds of fractional operators, the ones that have been the most carefully investigated, together with their applications were discussed in Refs. 9–11. Moreover, there are several physical problems in which fractional operators play key role. For example, Samraiz *et al.* investigated the physical models and their solutions by involving different fractional operators[7,8] and Adjabi *et al.* discussed Cauchy problems in Ref. 12, Zhao *et al.* studied the memory effects via general fractional derivative in Ref. 13, Chen *et al.* introduced a new approach of fractional calculus and related it with probability density function in Ref. 14.

Recently, some new fractional operators[15–18] have also been proposed with a corresponding nonsingular kernel such as Mittag-Leffler function.[19,20] The researchers investigated plenty of applications of fractional operators in different eras. Many physical models[21,22] were currently designed by the scientists. Some of these models use the Mittag-Leffler nonsingular kernel to solve specific fractional order differential equations.[23,24] For further details and applications of fractional operators, we refer the readers to Refs. 25–28.

The following fundamental definitions are necessary in order to describe the key findings in subsequent sections.

Definition 1 (Ref. 29). The extended gamma function is characterized by the following relation:

$$\Gamma_k(\tau) = \lim_{m\to\infty} \frac{m!k^m(mk)^{\frac{\tau}{k}-1}}{(\tau)_{m,k}}, \quad k > 0, \ \Re(\tau) > 0,$$

where $(\tau)_{m,k}$, $m \geq 1$, represents the k-Pochhammer symbol. The alternative representation is given by the equation

$$\Gamma_k(\tau) = \int_0^\infty \zeta^{\tau-1} e^{\frac{-\zeta^k}{k}} d\zeta, \quad \Re(\tau) > 0.$$

It is notable that $\Gamma(\tau) = \lim_{k\to 1} \Gamma_k(\tau)$ and $\Gamma_k(\tau) = k^{\frac{\tau}{k}-1}\Gamma(\frac{\tau}{k})$.

Definition 2. Let $\Re(\alpha), \Re(\beta) > 0$ and $k > 0$, then we have the following k-beta function

$$B_k(\alpha,\beta) = \frac{1}{k}\int_0^1 \tau^{\frac{\alpha}{k}-1}(1-\tau)^{\frac{\beta}{k}-1}d\tau.$$

The Γ_k and $\mathbf{B}_k$ functions are related by the identity $\mathbf{B}_k(\alpha,\beta) = \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}$.

Definition 3 (Ref. 30). Let $n \in \mathbb{N}$ and k be the positive real number, then $\eta, \varrho, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\eta) > 0$, the extended Mittag-Leffler function is defined by

$$\mathcal{E}^{\gamma}_{k,\varrho,\eta}(\vartheta) = \sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}\vartheta^n}{\Gamma_k(\varrho n+\eta)n!}.$$

Definition 4 (Ref. 31). Let $\alpha > 0$ and $k > 0$, then k-Riemann–Liouville fractional integral $I^{\alpha}_{a,k}f$ of order α is given by

$$I^{\alpha}_{a,k}f(\vartheta) = \frac{1}{k\Gamma_k(\alpha)}\int_a^{\vartheta}(\vartheta-\kappa)^{\frac{\alpha}{k}-1}f(\kappa)d\kappa, \quad \kappa \in (a,b). \tag{1}$$

Definition 5 (Ref. 32). Let $\Phi \in L^1[r,q]$, $s \in \mathbb{R}\backslash\{-1\}$, $k \in \mathbb{R}^+$, $\eta \in \mathbb{C}$ with $\Re(\eta) > 0$, then the (k,s)-fractional integral of Riemann-type is defined by

$$\left({}^s_kJ^{\eta}_r\Phi\right)(\vartheta) = \frac{(s+1)^{1-\frac{\eta}{k}}}{k\Gamma_k(\eta)}\int_r^{\vartheta}(\vartheta^{s+1}-\kappa^{s+1})^{\frac{\eta}{k}-1} \times \kappa^s\Phi(\kappa)d\kappa. \tag{2}$$

Definition 6 (Ref. 33). The Prabhakar integral operator for the choice of $\eta, \varrho, \omega, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\eta) > 0$ and $\Phi \in L^1[r,q]$ is defined by

$$(\mathbf{P}^{\gamma}_{\varrho,\eta,\omega}\Phi)(\vartheta) = \int_0^{\vartheta}(\vartheta-\kappa)^{\eta-1}\mathcal{E}^{\gamma}_{k,\varrho,\eta}(\lambda(\vartheta-\kappa)^{\varrho}) \times \Phi(\kappa)d\kappa. \tag{3}$$

Definition 7 (Ref. 34). Let $\eta, \varrho, \lambda, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\eta) > 0$, $n = \lceil\eta\rceil$ and $\Phi \in L^1[0,q]$, then Prabhakar derivative operator is given by

$$(\mathfrak{D}^{\gamma}_{\varrho,\eta,\lambda}\Phi)(\vartheta) = (\mathbf{P}^{-\gamma}_{\varrho,n-\eta,\lambda}\Phi)(\vartheta). \tag{4}$$

Dorrego *et al.*[35] presented the extended Prabhakar fractional integral operator which is defined as follows.

Definition 8. For all positive real numbers k and $\eta, \varrho, \lambda, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\eta) > 0$ and $\Phi \in L^1[r,q]$, the k-Prabhakar integral operator is given as

$$({}_k\mathbf{P}^{\gamma}_{\varrho,\eta,\lambda}\Phi)(\vartheta) = \frac{1}{k}\int_0^{\vartheta}(\vartheta-\kappa)^{\frac{\eta}{k}-1}\mathcal{E}^{\gamma}_{k,\varrho,\eta} \times (\lambda(\vartheta-\kappa)^{\frac{\varrho}{k}})\Phi(\kappa)d\kappa. \tag{5}$$

In mathematical analysis, Hölder's inequality, named after Otto Hölder, is a fundamental inequality among integrals and indispensable tool for the study of L^p spaces. The Hölder's inequality is given in the following definition.

Definition 9. Let S is a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure, and f and g are measurable real or complex valued functions on S, $\frac{1}{p}+\frac{1}{q} = 1$, $p > 0$, $q > 0$, then we have the inequality

$$\int_S |f(x)g(x)|dx \leq \left(\int_S |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_S |g(x)|^q dx\right)^{\frac{1}{q}}. \tag{6}$$

The generalized Minkowski inequality given in Ref. 5 is as follows.

Proposition 10. *Let $\Omega_i = [a_i, b_i], i = 1,2$ be such that $-\infty \leq a_i \leq b_i \leq \infty$ and Φ be Lebesgue integrable function. Then*

$$\left\{\int_{\Omega_1}\left|\int_{\Omega_2}\Phi(u,v)dv\right|^p du\right\}^{\frac{1}{p}} \leq \int_{\Omega_2}\left\{\int_{\Omega_1}|\Phi(u,v)|^p du\right\}^{\frac{1}{p}} dv, \quad 1 \leq p < \infty.$$

Theorem 11 (Ref. 26). *Let $s \in \mathbb{R}\backslash\{-1\}$, $k \in \mathbb{R}^+$, $\eta, \nu, \varrho, \lambda, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\eta) > 0$ and $\Re(\nu) > 0$, then*

$$\begin{aligned}{}^s_kJ^{\eta}_{r^+}&\left[(\kappa^{s+1}-r^{s+1})^{\frac{\nu}{k}-1}\mathcal{E}^{\gamma}_{k,\varrho,\nu}(\lambda(\kappa^{s+1}-r^{s+1})^{\frac{\varrho}{k}})\right] \\ &= \frac{(\vartheta^{s+1}-r^{s+1})^{\frac{\eta+\nu}{k}-1}}{(s+1)^{\frac{\eta}{k}}} \times \mathcal{E}^{\gamma}_{k,\varrho,\eta+\nu}(\lambda(\vartheta^{s+1}-r^{s+1})^{\frac{\varrho}{k}}).\end{aligned}$$

From the point of view of research, currently there are several differing perspectives and directions of exploration. Our aim in this paper is to enrich the theory of fractional calculus. In the following section, we propose weighted fractional integral involving nonsingular kernel. We are hopeful that this can serve more adequately and more generally than those fractional operators that are already known in the literature.

2. WEIGHTED (k, s)-FRACTIONAL INTEGRAL OPERATORS INVOLVING EXTENDED MITTAG-LEFFLER FUNCTION

In this section, we first modify the weighted (k, s)-fractional integral operator by incorporating the Mittag-Leffler function into its kernel. Secondly, we establish a new integral operator and discuss its properties.

The modification of (k, s)-fractional integral operator is given by the following definition.

Definition 12. The modified (k, s)-fractional integral operator of order η for $s \in \mathbb{R}\backslash\{-1\}$, $k \in \mathbb{R}^+$, $\eta, \varrho, \lambda, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\gamma) > 0$, $\Re(\eta) > 0$, $w(\vartheta) \neq 0$ and $\Phi \in L^1[0, q]$ is defined by

$$({}^s_k\mathfrak{J}^{w,\lambda,\gamma}_{0^+;\varrho,\eta}\Phi)(\vartheta)\frac{(s+1)^{1-\frac{\eta}{k}}}{k}w^{-1}(\vartheta) \times \int_0^{\vartheta}(\vartheta^{s+1}-t^{s+1})^{\frac{\eta}{k}-1}t^s \times \mathcal{E}^{\gamma}_{k,\varrho,\eta}(\lambda(\vartheta^{s+1}-t^{s+1})^{\frac{\varrho}{k}})w(t)\Phi(t)dt \quad \vartheta > 0. \tag{7}$$

Definition 13. Let s be a real number excluding 1, k a positive real number, $\eta, \varrho, \lambda, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\gamma) > 0$, $\Re(\eta) > 0$, $w(\vartheta) \neq 0$. Let $\Psi > 0$ and increasing function on $(0, q]$, having continuous derivative Ψ' on $(0, q)$, and $\Phi \in L^1[0, q]$, then the modified form of (k, s)-fractional integral with order η is as follows:

$$({}_{\Psi;}{}^s_k\mathfrak{J}^{w,\lambda,\gamma}_{0^+;\varrho,\eta}\Phi)(\vartheta) = \frac{(s+1)^{1-\frac{\eta}{k}}}{k}w^{-1}(\vartheta)\int_0^{\vartheta}(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\eta}{k}-1} \times \mathcal{E}^{\gamma}_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\varrho}{k}}) \times \Omega^s(t)w(t)\Omega'(t)\Phi(t)dt, \tag{8}$$

where $\Omega^{s+1}(t) = (\Omega(t))^{s+1}$.

(i) Substituting $\Omega(x) = x$ in (8), we get weighted modified (k, s)-fractional operator given in (7).

(ii) Substituting $\Omega(x) = x$, $w(x) = 1$ in (8), we get modified (k, s)-fractional operator introduced by Samraiz *et al.* in Ref. 7 given by Definition 2.1.

(iii) Substituting $\Omega(x) = x$, $w(x) = 1$ and $\gamma = 0$ in (8) it represents (k, s)-Riemann–Liouville fractional integral (2) introduced by Sarikaya *et al.*[32]

(iv) Substituting $\Omega(x) = x$, $s = 0$, $w(x) = 1$ and $k = 1$, in (8), we get fractional operator (3) introduced by Prabhakar.[33]

(v) Substituting $\Omega(x) = x$, $s = 0$, $w(x) = 1$ in (8), we obtain (5) given by Dorrego.[35]

(vi) Substituting $\Omega(x) = x$, $\gamma = 0$ and $s = 0$ in (8) it reduces to k-Riemann–Liouville fractional integral (2) defined by Mubeen *et al.* in Ref. 31.

(vii) Substituting $\Omega(x) = x$, $\gamma = 0$, $s = 0$ and $k = 1$ in (8) it reduces to classical Reimann–Liouville fractional integral.

The above relations of the new fractional operator with classical fractional integrals proved the importance of the fractional operator. Moreover, the defined operator may use to model the integro-differential equations.

Lemma 14. *Let* $k \in \mathbb{R}^+$, $\eta, \eta_1, \varrho, \lambda, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\eta) > 0$, $\Re(\nu) > 0$, $a > 0$, *then*

$$_{\Omega;}{}^s_k\mathfrak{J}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\left(w^{-1}(t)(\Omega^{s+1}(t)-\Omega^{s+1}(a))^{\frac{\eta_1}{k}-1}\right)(\vartheta) = w^{-1}(\vartheta)\Gamma_k(\eta_1)\frac{(s+1)^{-\frac{\eta}{k}}}{k} \times (\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1}{k}-1} \times \mathcal{E}^{\gamma}_{k,\varrho,\eta+\eta_1}(\lambda(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\varrho}{k}}). \tag{9}$$

Proof. By using Definition 13, we get

$$_{\Omega;}{}^s_k\mathfrak{J}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\left(w^{-1}(t)(\Omega^{s+1}(t)-\Omega^{s+1}(a))^{\frac{\eta_1}{k}-1}\right)(\vartheta) = \frac{(s+1)^{1-\frac{\eta}{k}}}{k}w^{-1}(\vartheta) \times \int_a^{\vartheta}(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\eta}{k}-1} \times \Omega^s(t)(\Omega^{s+1}(t)-\Omega^{s+1}(a))^{\frac{\eta_1}{k}-1} \times \mathcal{E}^{\gamma}_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\varrho}{k}})\Omega'(t)dt$$

$$= w^{-1}(\vartheta)\frac{(s+1)^{1-\frac{\eta}{k}}}{k} \times \int_a^{\vartheta}(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\eta}{k}-1} \times \Omega^s(t)(\Omega^{s+1}(t)-\Omega^{s+1}(a))^{\frac{\eta_1}{k}-1}$$

$$\times \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta)n!}$$

$$\times (\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\varrho n}{k}}\Omega'(t)dt$$

$$= w^{-1}(\vartheta)\frac{(s+1)^{1-\frac{\eta}{k}}}{k}\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta)n!}$$

$$\times (\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\varrho n}{k}-1}$$

$$\times \int_a^{\vartheta}\left(\frac{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t)}{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a)}\right)^{\frac{\eta+\varrho n}{k}-1}$$

$$\times (\Omega^{s+1}(t)-\Omega^{s+1}(a))^{\frac{\eta_1}{k}-1}\Omega^s(t)\Omega'(t)dt.$$

Substituting $u=\frac{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t)}{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a)}$ and using Definition 2, we have

$$= w^{-1}(\vartheta)\frac{(s+1)^{-\frac{\eta}{k}}}{k}\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta)n!}$$

$$\times (\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1+\varrho n}{k}-1}$$

$$\times \int_a^1 u^{\frac{\eta+\varrho n}{k}-1}(1-u)^{\frac{\eta_1}{k}-1}du$$

$$= w^{-1}(\vartheta)\frac{(s+1)^{-\frac{\eta}{k}}}{k}\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta)n!}$$

$$\times (\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1+\varrho n}{k}-1}B_k(\eta+\varrho n,\eta_1)$$

$$= w^{-1}(\vartheta)\frac{(s+1)^{-\frac{\eta}{k}}}{k}\sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta)n!}$$

$$\times (\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1+\varrho n}{k}-1}$$

$$\times \frac{\Gamma_k(\varrho n+\eta)\Gamma_k(\eta_1)}{\Gamma_k(\varrho n+\eta+\eta_1)}$$

$$= w^{-1}(\vartheta)\frac{(s+1)^{-\frac{\eta}{k}}}{k}(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1}{k}-1}$$

$$\times \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta+\eta_1)n!}$$

$$\times \left((\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\varrho}{k}}\right)^n$$

$$= w^{-1}(\vartheta)\Gamma_k(\eta_1)\frac{(s+1)^{-\frac{\eta}{k}}}{k}$$

$$\times (\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1}{k}-1}$$

$$\times \mathcal{E}^{\gamma}_{k,\varrho,\eta+\eta_1}(\lambda(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\varrho}{k}}).$$

This completes the proof of the result. □

Example 15. Corresponding to the choice of the function $f(t)=w^{-1}(t)(\Omega^{s+1}(t)-\Omega^{s+1}(a))^{\frac{\eta_1}{k}-1}$, we obtained the value of ${}_{\Omega,k}^{\;\;s}\mathfrak{J}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}f(t)$ presented on right side of (9). Now for the choice of the parameters $s=0,\gamma=0,k=1,\eta_1=3,a=0$ and $w(t)=1$, we get the graphs in Figs. 1–3 with different choices of the function $\Omega(t)$.

Next, we present the space where the weighted (k,s)-Riemann–Liouville fractional integrals are bounded.

Definition 16. Let f be a function defined on $[a,b]$. The space $X^p_\lambda(a,b)$, $1\le p\le\infty$ of all Lebesgue

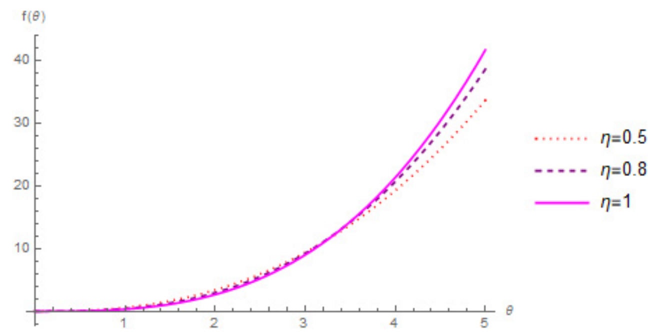

Fig. 1 For $\Omega(\vartheta)=\vartheta$, we get this figure, where $0^+\le\vartheta\le 5$.

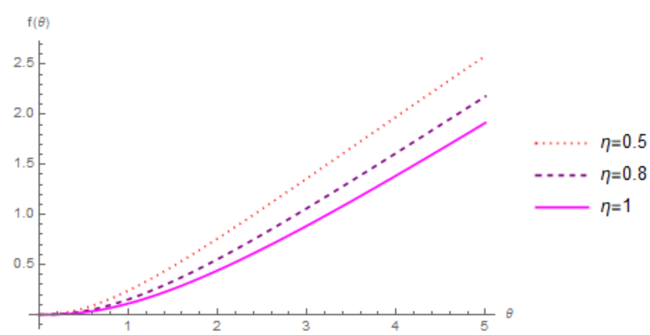

Fig. 2 For $\Omega(\vartheta)=\ln(\vartheta+1)$, we get this figure, where $0^+\le\vartheta\le 5$.

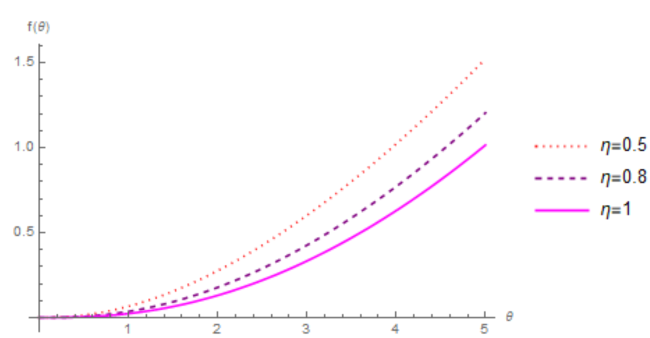

Fig. 3 For $\Omega(\vartheta)=\sqrt{\vartheta+1}$, we get this figure, where $0^+\le\vartheta\le 5$.

measurable functions for which $\|\varphi\|_{X^p_\lambda} < \infty$, i.e.

$$\|\varphi\|_{X^p_\lambda} = \left[(s+1)\int_a^b |\lambda(t)\varphi(t)|^p \Omega^s(t)\Omega'(t)dt\right]^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

$\lambda(t) \neq 0$, $s \in \mathbb{R}$, and

$$\|\varphi\|_{X^\infty_\lambda} = \operatorname*{ess\,sup}_{a\le\xi\le b} |\lambda(t)\varphi(t)| < \infty.$$

Theorem 17. *Let $k \in \mathbb{R}^+$, $\eta, \varrho, \lambda, \gamma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\eta) > 0$, $a > 0$, $1 \le p \le \infty$ and $\Phi \in X^p_w(r,q)$. Then ${}_{\Omega,k}^{\ s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}$ is bounded on $X^p_w(r,q)$, i.e.*

$$\begin{aligned}\|{}_{\Omega,k}^{\ s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi\|_{X^p_w} &\le (s+1)^{-\frac{\eta}{k}} w^{-1}(\vartheta)\big(\Omega^{s+1}(q) - \Omega^{s+1}(a)\big)^{\frac{\eta}{k}} \\ &\quad \times \mathcal{E}^\gamma_{k,\varrho,\eta+k}\big(\lambda(\Omega^{s+1}(q) - \Omega^{s+1}(a))^{\frac{\varrho}{k}}\big)\|\Phi\|_{X^p_w}.\end{aligned}$$

Proof. For $1 \le p < \infty$, we have

$$\begin{aligned}\|{}_{\Omega,k}^{\ s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi\|_{X^p_w} &= \frac{(s+1)^{1-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\Bigg(\int_a^q (s+1) \\ &\quad \times \Bigg|\int_a^\vartheta \big(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t)\big)^{\frac{\eta}{k}-1} \\ &\quad \times \mathcal{E}^\gamma_{k,\varrho,\eta}\big(\lambda(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t))^{\frac{\varrho}{k}}\big) \\ &\quad \times \Omega^s(t)\Omega'(t)w(t)\Phi(t)dt\Bigg|^p \Omega^s(\vartheta)\Omega'(\vartheta)d\vartheta\Bigg)^{\frac{1}{p}}.\end{aligned}$$

Substituting $u = \Omega^{s+1}(t)$ and $x = \Omega^{s+1}(\vartheta)$, we get

$$\begin{aligned}\|{}_{\Omega,k}^{\ s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi\|X^p_w &= \frac{(s+1)^{-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\sum_{n=0}^\infty \frac{|(\gamma)_{n,k}\lambda^n|}{|\Gamma_k(\varrho n+\eta)|n!} \\ &\quad \times \Bigg(\int_{\Omega^{(s+1)}(a)}^{\Omega^{(s+1)}(q)}\Bigg|\int_{\Omega^{(s+1)}(a)}^{\Omega^{(s+1)}(\vartheta)} (x-u)^{\frac{\eta+\varrho n}{k}-1} \\ &\quad \times w(\Omega^{-1}(u^{\frac{1}{s+1}}))\Phi(\Omega^{-1}(u^{\frac{1}{s+1}}))du\Bigg|^p dx\Bigg)^{\frac{1}{p}},\end{aligned}$$

where Ω^{-1} is the inverse function of Ω. Now using the generalized Minkowski inequality, we can write

$$\begin{aligned}\|{}_{\Omega,k}^{\ s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi\|_{X^p_w} &\le \frac{(s+1)^{1-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\sum_{n=0}^\infty \frac{|(\gamma)_{n,k}\lambda^n|}{|\Gamma_k(\varrho n+\eta)|n!} \\ &\quad \times \int_{\Omega^{(s+1)}(a)}^{\Omega^{(s+1)}(q)}\Bigg(|w(\Omega^{-1}(u^{\frac{1}{s+1}}))\Phi(\Omega^{-1}(u^{\frac{1}{s+1}}))|^p \\ &\quad \times \int_u^{\Omega^{(s+1)}(q)} (x-u)^{(\frac{\eta+\varrho n}{k}-1)p}dx\Bigg)^{\frac{1}{p}} du \\ &= \frac{(s+1)^{-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\sum_{n=0}^\infty \frac{|(\gamma)_{n,k}\lambda^n|}{|\Gamma_k(\varrho n+\eta)|n!} \\ &\quad \times \int_{\Omega^{(s+1)}(a)}^{\Omega^{(s+1)}(q)} \big|w(\Omega^{-1}(u^{\frac{1}{s+1}}))\Phi(\Omega^{-1}(u^{\frac{1}{s+1}}))\big| \\ &\quad \times \left(\frac{(\Omega^{s+1}(q)-u)^{(\frac{\eta+\varrho n}{k}-1)p+1}}{(\frac{\eta+\varrho n}{k}-1)p+1}\right)^{\frac{1}{p}} du.\end{aligned}$$

Using the Hölder's inequality (6) and by simple calculation, we get

$$\begin{aligned}\|{}_{\Omega,k}^{\ s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi\|_{X^p_w} &\le \frac{(s+1)^{-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\sum_{n=0}^\infty \frac{|(\gamma)_{n,k}\lambda^n|}{|\Gamma_k(\varrho n+\eta)|n!} \\ &\quad \times \left(\int_{\Omega^{(s+1)}(a)}^{\Omega^{(s+1)}(q)} \big|w(\Omega^{-1}(u^{\frac{1}{s+1}}))\Phi(\Omega^{-1}(u^{\frac{1}{s+1}}))\big|^p du\right)^{\frac{1}{p}} \\ &\quad \times \left[\int_{\Omega^{(s+1)}(a)}^{\Omega^{(s+1)}(q)} \left(\frac{(\Omega^{s+1}(q)-u)^{(\frac{\eta+\varrho n}{k}-1)p+1}}{(\frac{\eta+\varrho n}{k}-1)p+1}\right)^{\frac{q}{p}} du\right]^{\frac{1}{q}} \\ &\le \frac{(s+1)^{-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\sum_{n=0}^\infty \frac{|(\gamma)_{n,k}\lambda^n|}{|\Gamma_k(\varrho n+\eta)|n!} \\ &\quad \times \left(\int_{\Omega^{(s+1)}(a)}^{\Omega^{(s+1)}(q)} \big|w(\Omega^{-1}(u^{\frac{1}{s+1}}))\Phi(\Omega^{-1}(u^{\frac{1}{s+1}}))\big|^p du\right)^{\frac{1}{p}} \\ &\quad \times \frac{(\Omega^{s+1}(q)-\Omega^{s+1}(a))^{\frac{\eta+\varrho n}{k}}}{\frac{\eta+\varrho n}{k}},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Substituting $t = \Omega^{-1}(u^{\frac{1}{s+1}})$, we have

$$\|{}_{\Omega,k}^{\ s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi\|_{X^p_w} \le \frac{(s+1)^{-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\sum_{n=0}^\infty \frac{|(\gamma)_{n,k}\lambda^n|}{|\Gamma_k(\varrho n+\eta)|n!}$$

$$\times\left((s+1)\int_a^q \left|w(t)\Phi(t)\right|^p\Omega^s(t)\Omega'(t)dt\right)^{\frac{1}{p}}$$

$$\times\frac{(\Omega^{s+1}(q)-\Omega^{s+1}(a))^{\frac{\eta+\varrho n}{k}}}{\frac{\eta+\varrho n}{k}}$$

$$=(s+1)^{-\frac{\eta}{k}}w^{-1}(\vartheta)\sum_{n=0}^{\infty}\frac{|(\gamma)_{n,k}\lambda^n|}{|\Gamma_k(\varrho n+\eta)|n!}$$

$$\times\frac{\|\Phi\|_{X_w^p}(\Omega^{s+1}(q)-\Omega^{s+1}(a))^{\frac{\eta+\varrho n}{k}}}{\eta+\varrho n}$$

$$=(s+1)^{-\frac{\eta}{k}}w^{-1}(\vartheta)\big(\Omega^{s+1}(q)-\Omega^{s+1}(a)\big)^{\frac{\eta}{k}}$$

$$\times\mathcal{E}^{\gamma}_{k,\varrho,\eta+k}\big(\lambda(\Omega^{s+1}(q)-\Omega^{s+1}(a))^{\frac{\varrho}{k}}\big)\|\Phi\|_{X_w^p}.$$

For $p=\infty$, we have

$$\|_{\Omega,{}^s_k}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi\|$$

$$=(s+1)^{-\frac{\eta}{k}}w^{-1}(\vartheta)\big(\Omega^{s+1}(q)-\Omega^{s+1}(a)\big)^{\frac{\eta}{k}}$$

$$\times E^{\gamma}_{k,\varrho,\eta+k}\big(\lambda(\Omega^{s+1}(q)-\Omega^{s+1}(r))^{\frac{\varrho}{k}}\big)\|\Phi\|_{X_w^{\infty}}.$$

Hence the result is proved. □

We need to prove the following lemma to achieve the required goal in our next result.

Lemma 18. *Let* $s\in R/\{-1\}$, $k\in\mathbb{R}^+$, $\eta,\nu,\varrho,\lambda,\gamma,\in\mathbb{C}$, $\Re(\varrho)>0$, $\Re(\eta)>0$, $\Re(\eta_1)>0$, *then*

$$\int_a^{\vartheta}\big(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t)\big)^{\frac{\eta}{k}-1}$$

$$\times(\Omega^{s+1}(t)-\Omega^{s+1}(a))^{\frac{\eta_1}{k}-1}$$

$$\times\mathcal{E}^{\gamma}_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t)))^{\frac{\varrho}{k}}$$

$$\times\mathcal{E}^{\sigma}_{k,\varrho,\eta_1}(\lambda(\Omega^{s+1}(t)-\Omega^{s+1}(a)))^{\frac{\varrho}{k}}$$

$$\times\Omega^s(t)\Omega'(t)dt$$

$$=\frac{k\big(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a)\big)^{\frac{\eta+\eta_1}{k}-1}}{s+1}$$

$$\times\mathcal{E}^{\gamma+\sigma}_{k:\varrho;\eta+\eta_1}((\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a)))^{\frac{\varrho}{k}}. \quad (10)$$

Proof. Consider the left side of (10)

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma)_{n,k}\lambda^{n+m}}{\Gamma_k(\varrho n+\eta)n!}\frac{(\sigma)_{m,k}}{\Gamma_k(\varrho m+\eta_1)m!}$$

$$\times\int_a^{\vartheta}(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\eta+\varrho n}{k}-1}$$

$$\times(\Omega^{s+1}(t)-\Omega^{s+1}(a))^{\frac{\eta_1+\varrho m}{k}-1}$$

$$\times\Omega(t)^s\Omega'(t)dt$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma)_{n,k}\lambda^{n+m}}{\Gamma_k(\varrho n+\eta)n!}$$

$$\times\frac{(\sigma)_{m,k}}{\Gamma_k(\varrho m+\eta_1)m!}$$

$$\times\int_a^{\vartheta}\left(\frac{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t)}{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a)}\right)^{\frac{\eta+\varrho n}{k}-1}$$

$$\times(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\varrho n}{k}-1}$$

$$\times(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta_1+\varrho m}{k}-1}$$

$$\times\left(\frac{\Omega^{s+1}(t)-\Omega^{s+1}(a)}{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a)}\right)^{\frac{\eta_1+\varrho m}{k}-1}$$

$$\times\Omega(t)^s\Omega'(t)dt.$$

Substituting $u=\frac{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t)}{\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a)}$, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{(s+1)}\frac{(\gamma)_{n,k}\lambda^{n+m}}{\Gamma_k(\varrho n+\eta)n!}\frac{(\sigma)_{m,k}}{\Gamma_k(\varrho m+\eta_1)m!}$$

$$\times(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1+\varrho(m+n)}{k}-1}$$

$$\times\int_0^1 u^{\frac{\eta+\varrho n}{k}-1}(1-u)^{\frac{\eta_1+\varrho m}{k}-1}du$$

$$=\frac{k(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1}{k}-1}}{(s+1)}$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma)_{n,k}\lambda^{n+m}}{\Gamma_k(\varrho n+\eta)n!}\frac{(\gamma)_{m,k}}{\Gamma_k(\varrho m+\eta_1)m!}$$

$$\times(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\varrho(m+n)}{k}}$$

$$\times B_k(\eta+\varrho n,\eta_1+\varrho m)$$

$$=\frac{k(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1}{k}-1}}{s+1}$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\gamma)_{n,k}\lambda^{n+m}}{\Gamma_k(\varrho n+\eta)n!}\frac{(\sigma)_{m,k}}{\Gamma_k(\varrho m+\eta_1)m!}$$

$$\times\frac{\Gamma_k(\varrho n+\eta)\Gamma_k(\varrho m+\eta_1)}{\Gamma_k(\varrho(m+n)+\eta+\eta_1)}$$

$$\times(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\varrho(m+n)}{k}}$$

$$=\frac{k(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta+\eta_1}{k}-1}}{s+1}$$

$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(\gamma)_{n,k}(\sigma)_{m,k} w^{m+n}}{\Gamma_k(\varrho(m+n)+\eta+\eta_1)} \times (\Omega^{s+1}(\vartheta) - \Omega^{s+1}(a)^{\frac{\varrho}{k}})^{(m+n)}$$
$$= \frac{k(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(a))^{\frac{\eta+\eta_1}{k}-1}}{s+1} \times \mathcal{E}^{\gamma+\sigma}_{k:\varrho;\eta+\eta_1}(\lambda(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(a))^{\frac{\varrho}{k}}.$$

This completes the proof of the result. □

Example 19. Corresponding to the choice of the function

$$f(t) = w(\vartheta)w^{-1}(t)(s+1)^{\frac{\eta}{k}-1} \times (\Omega^{s+1}(t) - \Omega^{s+1}(a))^{\frac{\eta_1}{k}-1} \times \mathcal{E}^{\sigma}_{k:\varrho;\eta_1}\left(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(a)\right)^{\frac{\varrho}{k}},$$

we obtained the value of ${}_{\Omega,k}^{\ s}\mathfrak{J}^{w,\lambda,\gamma}_{a^+;\varrho,\eta} f(t)$ presented on right side of (10). Now for the choice of the parameters $s = 0, \gamma = 0, k = 1, \eta_1 = 1, a = 0$ and $w(t) = 1$, we get the graphs in Figs. 4–6 with different choices of the function $\Omega(t)$.

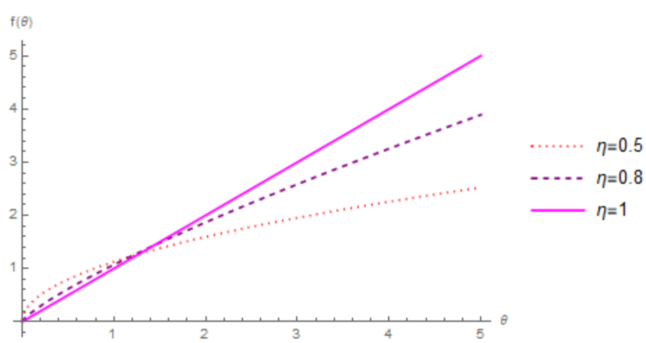

Fig. 4 For $\Omega(\vartheta) = \vartheta$, we get this figure, where $0^+ \le \vartheta \le 5$.

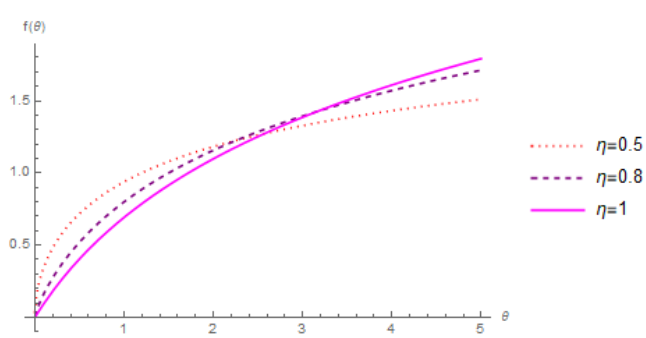

Fig. 5 For $\Omega(\vartheta) = \ln(\vartheta+1)$, we get this figure, where $0^+ \le \vartheta \le 5$.

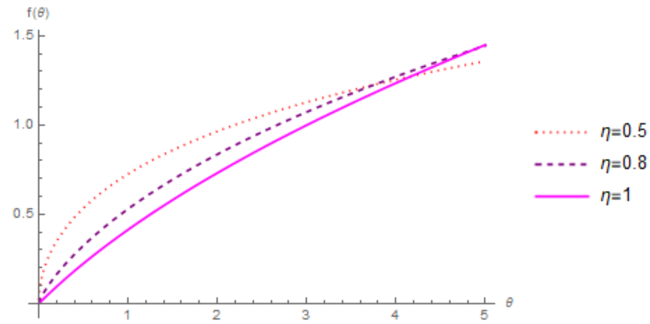

Fig. 6 For $\Omega(\vartheta) = \sqrt{\vartheta+1}$, we get this figure, where $0^+ \le \vartheta \le 5$.

Proposition 20. *Let $k \in \mathbb{R}^+$, $\eta, \eta_1, \varrho, \lambda, \gamma, \sigma \in \mathbb{C}$, $\Re(\varrho) > 0$, $\Re(\eta) > 0$, $\Re(\nu) > 0$, then for any $\Phi \in X^p_w(r,q)$, $1 \le p < \infty$, we have*

$$({}_{\Omega,k}^{\ s}\mathfrak{J}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\ {}_{\Omega,k}^{\ s}\mathfrak{J}^{w,\lambda,\sigma}_{a^+;\varrho,\eta_1}\Phi)(\nu) = ({}_{\Omega,k}^{\ s}\mathfrak{J}^{w,\lambda,\gamma+\sigma}_{a^+;\varrho,\eta+\eta_1}\Phi)(\nu).$$

Proof. By using Definition 13, interchanging the order of integration by the Dirichlet formula and Lemma 2.2 in Ref. 7, we get

$$({}_{\Omega,k}^{\ s}\mathfrak{J}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}{}_{\Omega,k}^{\ s}\mathfrak{J}^{w,\lambda,\gamma}_{a^+;\varrho,\eta_1}\Phi)(t)$$
$$= \frac{(s+1)^{1-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\int_a^{\vartheta}(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t))^{\frac{\eta}{k}-1} \times \mathcal{E}^{\gamma}_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t))^{\frac{\varrho}{k}}) \times w(t)\Omega^s(t)\Omega'(t)\left({}_{\Omega,k}^{\ s}\mathfrak{J}^{w,\lambda,\sigma}_{a^+;\varrho,\eta_1}\Phi\right)(t)dt$$
$$= \frac{(s+1)^{1-\frac{\eta}{k}}}{k} w^{-1}(\vartheta)\int_a^{\vartheta}(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t))^{\frac{\eta}{k}-1} \times \mathcal{E}^{\gamma}_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t))^{\frac{\varrho}{k}})\Omega^s(t)\Omega'(t) \times \left(\frac{(s+1)^{1-\frac{\eta_1}{k}}}{k}\int_a^t(\Omega^{s+1}(t) - \Omega^{s+1}(t_1))^{\frac{\eta_1}{k}-1} \times \mathcal{E}^{\sigma}_{k,\varrho,\eta_1}(\lambda(\Omega^{s+1}(t) - \Omega^{s+1}(t_1))^{\frac{\varrho}{k}}) \times w(t_1)\Omega^s(t_1)\Omega'(t_1)\Phi(t_1)dt_1\right)dt$$
$$= \frac{(s+1)^{2-\frac{\eta+\eta_1}{k}}}{k^2} w^{-1}(\vartheta) \times \int_a^{\vartheta} w(t_1)\Omega^s(t_1)\Omega'(t_1)\Phi(t_1)$$

$$\times \int_{t_1}^{\vartheta} (\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t))^{\frac{\eta}{k}-1}$$
$$\times (\Omega^{s+1}(t) - \Omega^{s+1}(t_1))^{\frac{\eta_1}{k}-1}$$
$$\times \mathcal{E}^{\gamma}_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t))^{\frac{\varrho}{k}})$$
$$\times \mathcal{E}^{\sigma}_{k,\varrho,\eta_1}(\lambda(\Omega^{s+1}(t) - \Omega^{s+1}(t_1))^{\frac{\varrho}{k}})$$
$$\times \Omega^s(t)\Omega'(t)dtdt_1$$
$$= \frac{(s+1)^{1-\frac{\eta+\eta_1}{k}}}{k} w^{-1}(\vartheta)$$
$$\times \int_a^{\vartheta} (\Omega^{s+1}(\vartheta) - \Omega^{s+1}(t_1))^{\frac{\eta+\eta_1}{k}-1}$$
$$\times \mathcal{E}^{\gamma+\sigma}_{k,\varrho,\eta+\eta_1}(\lambda(\Omega^{s+1}(\vartheta) - \Omega(t_1))^{\frac{\varrho}{k}})$$
$$\times w(t_1)\Omega^s(t_1)\Omega'(t_1)\Phi(t_1)dt_1$$
$$= ({}_{\Omega,k}^{\ s}\mathfrak{J}^{w,\lambda,\gamma+\sigma}_{a^+;\varrho,\eta+\eta_1}\Phi)(\vartheta).$$

The proof is done. □

3. GENERALIZED WEIGHTED LAPLACE TRANSFORM

The classical and weighted Laplace transforms are difficult to apply to the newly proposed generalized weighted fractional integral. For this purpose, we need to modify the weighted Laplace transform presented by Jarad *et al.* in Ref. 36.

Definition 21. Let ϕ and w be defined on $[a, \infty]$ and Ω be a monotonically increasing function on the interval $[a, \infty)$, then the generalized weighted Laplace transform of f is defined by

$$L^w_{\Omega^{s+1}}(\phi)(u)$$
$$= (s+1)\int_a^{\infty} e^{-u(\Omega^{s+1}(t)-\Omega^{s+1}(a))}$$
$$\times\phi(t)w(t)\Omega^s(t)\Omega'(t)dt$$

for all values of u such that the above equation is true.

Definition 22. The weighted convolution of functions Ω and ϕ is given by

$$(\Phi *^w_{\Omega^{s+1}} \Psi)(\vartheta)$$
$$= (s+1)w^{-1}(\vartheta)\int_a^{\vartheta} \Phi\big(\Omega^{-1}\big(\Omega^{s+1}(\vartheta)$$
$$+ \Omega^{s+1}(a) - \Omega^{s+1}(t)\big)^{\frac{1}{s+1}}\big)$$
$$\times w\big(\Omega^{-1}\big(\Omega^{s+1}(\vartheta) + \Omega^{s+1}(a) - \Omega^{s+1}(t)\big)^{\frac{1}{s+1}}\big)$$
$$\times w(t)\Psi(t)\Omega^s(t)\Omega'(t)dt,$$

where Ω^{-1} is inverse of Ω.

Theorem 23. *Let the generalized weighted Laplace transform of the function Φ and Ψ exist for $u > s_1$ and $u > s_2$, respectively. Then*

$$L^w_{\Omega^{s+1}}(\Phi)(u)L^w_{\Omega^{s+1}}(\Psi)(u)$$
$$= L^w_{\Omega^{s+1}}(\Phi *^w_{\Omega^{s+1}} \Psi)(u), \quad u = \max\{s_1, s_2\}.$$

Proof. By using Definition 21. We have

$$L^w_{\Omega^{s+1}}(\Phi)(u)L^w_{\Omega^{s+1}}(\Psi)(u)$$
$$= (s+1)^2\int_a^{\infty} e^{-u(\Omega^{s+1}(x)-\Omega^{s+1}(a))}\Phi(x)$$
$$\times w(x)\Omega^s(x)\Omega'(x)dx\int_a^{\infty} e^{-u(\Omega^{s+1}(\tau)-\Omega^{s+1}(a))}$$
$$\times \Psi(\tau)w(\tau)\Omega^s(\tau)\Omega'(\tau)d\tau$$
$$= \int_a^{\infty}\int_a^{\infty} e^{-u(\Omega^{s+1}(x)+\Omega^{s+1}(\tau)-2\Omega^{s+1}(a))}$$
$$\times \phi(x)w(x)\Omega^s(x)\Omega'(x)\Psi(\tau)w(\tau)$$
$$\times \Omega^s(\tau)\Omega'(\tau)d\tau dt.$$

Letting $\Omega^{s+1}(x) + \Omega^{s+1}(\tau) - \Omega^{s+1}(a) = \Omega^{s+1}(t)$, we obtain

$$(s+1)^2\int_a^{\infty}\int_t^{\infty} e^{-u(\Omega^{s+1}(t)-\Omega^{s+1}(a))}$$
$$\times w\big(\Omega^{-1}\big(\Omega^{s+1}(t) + \Omega^{s+1}(a) - \Omega^{s+1}(\tau)\big)^{\frac{1}{s+1}}\big)$$
$$\times \Phi\big(\Omega^{-1}\big(\Omega^{s+1}(t) + \Omega^{s+1}(a) - \Omega^{s+1}(\tau)\big)^{\frac{1}{s+1}}\big)$$
$$\times \Omega^s(t)\Omega'(t)\Psi(\tau)w(\tau)\Omega^s(\tau)\Omega'(\tau)dtd\tau.$$

Now interchanging the order of integration we have

$$(s+1)^2\int_a^{\infty} e^{-u(\Omega^{s+1}(t)-\Omega^{s+1}(a))}\int_a^t w\big(\Omega^{-1}(\Omega^{s+1}(t)$$
$$+ \Omega^{s+1}(a) - \Omega^{s+1}(\tau))^{\frac{1}{s+1}}\big)$$
$$\times \Phi\big(\Omega^{-1}(\Omega^{s+1}(t) + \Omega^{s+1}(a) - \Omega^{s+1}(\tau))^{\frac{1}{s+1}}\big)$$
$$\times \Psi(\tau)w(\tau)\Omega^s(\tau)\Omega'(\tau)d\tau\Omega^s(t)\Omega'(t)dt$$

$$= (s+1)\int_a^\infty e^{-u(\Omega^{s+1}(t)-\Omega^{s+1}(a))}$$
$$\times w(t)(s+1)w^{-1}(t)\int_a^t w\big(\Omega^{-1}(\Omega^{s+1}(t)$$
$$+\Omega^{s+1}(a)-\Omega^{s+1}(\tau))^{\frac{1}{s+1}}\big)$$
$$\times \Phi\big(\Omega^{-1}(\Omega^{s+1}(t)+\Omega^{s+1}(a)-\Omega^{s+1}(\tau))^{\frac{1}{s+1}}\big)$$
$$\times \Psi(\tau)w(\tau)\Omega^s(\tau)\Omega'(\tau)d\tau\Omega^s(t)\Omega'(t)dt$$
$$= (s+1)\int_a^\infty e^{-u(\Omega^{s+1}(t)-\Omega^{s+1}(a))}$$
$$\times (\Phi *^w_{\Omega^{s+1}} \Psi)(t)w(t)\Omega^s(t)\Omega'(t)dt$$
$$= L^w_{\Omega^{s+1}}(\Phi *^w_{\Omega^{s+1}} \Psi)(u).$$

Hence the result is proved. □

Theorem 24. *Let Φ be a piecewise continuous function on each interval $[a,\vartheta]$ and of w-weighted $\Omega^{(s+1)}$-exponential order. Then*

$$L^w_{\Omega^{s+1}}\{{}_{\Omega,k}^{\;\;s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi(\vartheta)\}(u)$$
$$= (s+1)^{-\frac{\eta}{k}}(ks)^{-\frac{\eta}{k}}\big(1-k\lambda(ks)^{-\frac{\varrho}{k}}\big)^{-\frac{\gamma}{k}}$$
$$\times L^w_{\Omega^{s+1}}\{\Phi(\vartheta)\}(s),$$

with $|k\lambda(ks)^{-\frac{\varrho}{k}}|<1.$

Proof. By using Definition 13, we obtain

$$L^w_{\Omega^{s+1}}\{{}_{\Omega,k}^{\;\;s}\mathfrak{I}^{w,\lambda,\gamma}_{a^+;\varrho,\eta}\Phi(\vartheta)\}(s)$$
$$= L^w_{\Omega^{s+1}}\Bigg(\frac{(s+1)^{1-\frac{\eta}{k}}}{k}w^{-1}(\vartheta)$$
$$\times\int_a^\vartheta(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\eta}{k}-1}$$
$$\times\mathcal{E}^\gamma_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(t))^{\frac{\varrho}{k}})$$
$$\times\Omega^s(t)w(t)\Omega'(t)\Phi(t)dt\Bigg)(s)$$
$$= L^w_{\Omega^{s+1}}\Bigg(\frac{(s+1)^{-\frac{\eta}{k}}}{k}w^{-1}(\vartheta)$$
$$\times(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta}{k}-1}$$
$$\times\mathcal{E}^\gamma_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\varrho}{k}})*\Phi(\vartheta)\Bigg)(s)$$
$$= \frac{(s+1)^{-\frac{\eta}{k}}}{k}\big[L^w_{\Omega^{s+1}}\{w^{-1}(\vartheta)(\Omega^{s+1}(\vartheta)$$
$$-\Omega^{s+1}(a))^{\frac{\eta}{k}-1}\mathcal{E}^\gamma_{k,\varrho,\eta}(\lambda(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\varrho}{k}})\}$$
$$\times L^w_\Omega\{\Phi(\vartheta)\}\big](s)$$
$$= \frac{(s+1)^{-\frac{\eta}{k}}}{k}\sum_{n=0}^\infty\frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta)n!}L^w_{\Omega^{s+1}}$$
$$\times\{w^{-1}(\vartheta)(\Omega^{s+1}(\vartheta)-\Omega^{s+1}(a))^{\frac{\eta}{k}+\frac{\varrho n}{k}-1}\}$$
$$\times L^w_\Omega\{\Phi(\vartheta)\}(s)$$
$$= \frac{(s+1)^{-\frac{\eta}{k}}}{k}\sum_{n=0}^\infty\frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta)n!}$$
$$\times\frac{\Gamma\big(\frac{\eta}{k}+\frac{\varrho n}{k}\big)}{s^{\frac{\eta}{k}+\frac{\varrho n}{k}}}L^w_{\Omega^{s+1}}\{\Phi(\vartheta)\}(s)$$
$$= \frac{(s+1)^{-\frac{\eta}{k}}}{k}\sum_{n=0}^\infty\frac{(\gamma)_{n,k}\lambda^n}{\Gamma_k(\varrho n+\eta)n!}$$
$$\times\frac{k^{1-\frac{\eta}{k}-\frac{\varrho n}{k}}\Gamma_k(\varrho n+\eta)}{s^{\frac{\eta}{k}+\frac{\varrho n}{k}}}L^w_{\Omega^{s+1}}\{\Phi(\vartheta)\}(s)$$
$$= \frac{(s+1)^{-\frac{\eta}{k}}}{(ks)^{\frac{\eta}{k}}}\sum_{n=0}^\infty\frac{(\gamma)_{n,k}}{n!}\left(\frac{\lambda}{(ks)^{\frac{\varrho}{k}}}\right)^n$$
$$\times L^w_{\Omega^{s+1}}\{\Phi(\vartheta)\}(s)$$
$$= \frac{(s+1)^{-\frac{\eta}{k}}}{(ks)^{\frac{\eta}{k}}}\frac{1}{(1-\frac{k\lambda}{(ks)^{\frac{\varrho}{k}}})^{\frac{\gamma}{k}}}L^w_{\Omega^{s+1}}\{\Phi(\vartheta)\}(s)$$
$$= (s+1)^{-\frac{\eta}{k}}(ks)^{-\frac{\eta}{k}}$$
$$\times\big(1-k\lambda(ks)^{-\frac{\varrho}{k}}\big)^{-\frac{\gamma}{k}}L^w_{\Omega^{s+1}}\{\Phi(\vartheta)\}(s).$$

This completes the proof of the result. □

4. CONCLUSIONS

In this era of science, we came to know about different inventions in fractional calculus and its applications. In recent decades the importance of fractional calculus is continuously increasing in almost all areas of science. In this paper, we proposed a new weighted fractional integral operator and discussed its properties. Significantly, the presented integral operator reduced to notable fractional operators that are known in the literature. We explored important aspects of this novel operator and demonstrated the conditions under which the semigroup property holds. The fascinating results are obtained corresponding to certain choices of the functions. The resulting consequences are presented via graphs. The new operator is proved to be bounded by using generalized Minkowski and Hölder's inequalities that have strong applications in theory of inequalities. The weighted Laplace transform of the operator in continuous-like spaces is evaluated. The established operator can be used to construct different physical models and their solutions obtained via Laplace transform and some other techniques. This research may motivate the researchers to extend the study of fractional calculus by exploring the more general forms of fractional operators with singular and nonsingular kernels.

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