Central European Journal of Physics
DOI: 10.2478/s11534-007-0014-9
Research article
CEJP 5(3) 2007 285–292

Gravitational potential in fractional space

Sami I. Muslih[1]*, Dumitru Baleanu[2]†, Eqab M. Rabei[3,4]‡

[1] *Department of Physics, Al-Azhar University, 1277 Gaza, Palestine*

[2] *Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences, Çankaya University, 06530, Ankara, Turkey*

[3] *Department of Science, Jerash Private University, Jerash, Jordan*

[4] *Physics Department, Mutah University, 61710 Karak, Jordan*

Received 10 November 2006; accepted 28 February 2007

Abstract: In this paper the gravitational potential with β-th order fractional mass distribution was obtained in α dimensionally fractional space. We show that the fractional gravitational universal constant G_α is given by $G_\alpha = \frac{2\Gamma(\frac{\alpha}{2})}{\pi^{\alpha/2-1}(\alpha-2)} G$, where G is the usual gravitational universal constant and the dimensionality of the space is $\alpha > 2$.

Keywords: Fractional space, gravitational potential, Gauss' law
PACS (2006): 04.60.Kz

1 Introduction

Fractional dimensional space represents an effective physical description of confinement in low-dimensional systems [1, 2].

In recent years, authors have redefined the integer space to the case of fractional space [3–8]. It is believed that the dimension of space plays an important role in quantum field

* E-mail: smuslih@ictp.trieste.it
† E-mail: dumitru@cankaya.edu.tr, baleanu@venus.nipne.ro
‡ E-mail: eqabrabei@yahoo.com

theory, in the Ising limit of quantum field theory, in random walks and in Casimir effect [4]. It is worth mentioning that the experimental measurement of the dimensionality α of our real world is given by $\alpha = (3 \pm 10^{-6})$ [7, 8]. The fractional value of α agrees with the experimental physical observations in the sense that in general relativity, gravitational fields are understood to be geometric perturbations (curvatures) in our space-time [9], rather than entities residing within a flat space-time. Besides, in [6] it was noted that the current discrepancy between theoretical and experimental values of the anomalous magnetic moment of the electron could be resolved if the dimensionality of space α is $\alpha = 3 - (5.3 \pm 2.5) \times 10^{-7}$.

Among several approaches used to investigate fractional dimensions, fractional calculus [10–17], which is a branch of mathematics that deal with generalization of well-known operations of differentiations and integrations to arbitrary non integer order-which can be non-integer, real or even an imaginary number, was applied recently to gravity [18]. Also fractional calculus was employed in other physical phenomena like problems in electromagnetism [19–21]. Engheta [19–21] discussed multipoles in electromagnetic theory and applied fractional calculus to show the evolution of a dipole,for instance, from a monopole by fractional derivation. The charge distribution of a dipole is the first spatial derivative of the charge distribution of a monopole, a quadrapole is the first spatial derivative of a dipole, and so on.

For these reasons a new derivation of the scalar potential in α dimensional fractional space is important from gravitational point of view.

In this paper, we use the concept of fractional calculus to obtain the solution of the gravitational problem in α dimensional space.

The paper is organized as follows:

In Section 2 the gravitational potential in α dimensional fractional space is presented. Gauss's law in α dimensional fractional space is derived in Section 3. The compact form of the fractional gravitational potential is analyzed in Section 4. Our conclusions are presented in Section 5.

2 Gravitational potential in α dimensional fractional space

Given a mass distribution ρ, the gravitational potential $\varphi(r)$, can be determined by solving the Poisson's equation

$$\nabla^2 \varphi = 4\pi G \rho, \tag{1}$$

where G is the usual gravitational universal constant and the dimensionality of the space is $\alpha > 2$. Here ∇^2 is the Laplacian in α dimensional fractional space defined as follows [8]

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{(\alpha - 1)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin^{\alpha-2}\theta}\frac{\partial}{\partial\theta}\sin^{\alpha-2}\theta\frac{\partial}{\partial\theta}. \tag{2}$$

To obtain the solution of equation (1), let us look at the solution of Laplace's equation,

$$\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{(\alpha-1)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin^{\alpha-2}\theta}\frac{\partial}{\partial\theta}\sin^{\alpha-2}\theta\frac{\partial}{\partial\theta}\right)\varphi = 0. \tag{3}$$

The equation (3) is separable and considering

$$\varphi(r,\theta) = R(r)\Theta(\theta), \tag{4}$$

one gets the angular and radial differential equations as follows:

$$\left[\frac{d^2}{d\theta^2} + (\alpha-2)\cot\theta\frac{d}{d\theta} + l(l+\alpha-2)\right]\Theta(\theta) = 0, \tag{5}$$

$$\left[\frac{d^2}{dr^2} + \frac{(\alpha-1)}{r}\frac{d}{dr} - \frac{l(l+\alpha-2)}{r^2}\right]R(r) = 0. \tag{6}$$

The solutions of the angular equation (5) are Gegenbauer polynomials in $\cos\theta$ [8], namely

$$\Theta(\theta) = C_l^{(\frac{\alpha}{2}-1)}(\cos\theta), \quad = 0,1,2,..., \tag{7}$$

which fulfill the following orthonormality relations:

$$\int_0^\pi C_l^{(\frac{\alpha}{2}-1)}(\cos\theta)C_{l'}^{(\frac{\alpha}{2}-1)}(\cos\theta)\sin^{\alpha-2}\theta\ d\theta = N(l)\delta_{l,l'}. \tag{8}$$

Here, $N(l)$ has the following form

$$N(l) = \frac{2^{3-\alpha}\pi\Gamma(l+\alpha-2)}{l!\left(l+\frac{\alpha}{2}-1\right)\left[\Gamma\left(\frac{\alpha}{2}-1\right)\right]^2}. \tag{9}$$

The forms of the first few Gegenbauer polynomials are given by

$$C_0^{(\frac{\alpha}{2}-1)}(x) = 1, \tag{10}$$

$$C_1^{(\frac{\alpha}{2}-1)}(x) = (\alpha-2)x, \tag{11}$$

$$C_2^{(\frac{\alpha}{2}-1)}(x) = \left(\frac{\alpha}{2}-1\right)\left(\alpha x^2-1\right). \tag{12}$$

From (6), the radial solutions are found to be

$$R(r) = \begin{cases} r^l \\ \frac{1}{r^{l+\alpha-2}} \end{cases}. \tag{13}$$

Therefore, the solutions $\varphi(r,\theta)$ in α dimensional fractional space have the forms

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+\alpha-2}}\right)C_l^{(\frac{\alpha}{2}-1)}(\cos\theta), \tag{14}$$

where a_l and b_l are constant coefficients, which can be determined from the boundary conditions on $\varphi(r,\theta)$. One should notice that the radial solutions (13) and the angular solutions (7) are valid for $\alpha > 2$.

The generating function for the Gegenbauer polynomials is defined as

$$\frac{1}{(1-2xt+t^2)^{\frac{\alpha}{2}-1}} = \sum_{l=0}^{\infty} C_l^{(\frac{\alpha}{2}-1)}(x)t^l, \quad |x| \leq 1,\ |t| < 1,\ \alpha > 2. \tag{15}$$

Defining the distance $|\vec{r}-\vec{r}'| = (r^2 + r'^2 - 2rr'\cos\theta)^{\frac{1}{2}}$, then we obtain

$$\sum_{l=0}^{\infty} \frac{r'^l}{r^{l+\alpha-2}} C_l^{(\frac{\alpha}{2}-1)}(\cos\theta) = \frac{1}{|\vec{r}-\vec{r}'|^{(\alpha-2)}}, \quad r > r', \tag{16}$$

$$\sum_{l=0}^{\infty} \frac{r^l}{r'^{l+\alpha-2}} C_l^{(\frac{\alpha}{2}-1)}(\cos\theta) = \frac{1}{|\vec{r}-\vec{r}'|^{(\alpha-2)}}, \quad r < r'. \tag{17}$$

In this case the gravitational scalar potential in α dimensional fractional space becomes

$$\varphi^{\alpha}(r,\theta) = -\frac{G_{\alpha} m}{|\vec{r}-\vec{r}'|^{(\alpha-2)}}, \tag{18}$$

where m represents the point mass and G_a is a universal constant in α fractional space. An interesting observation here is that according to (18), if α differs from three, the Coulomb potential of a point source falls off as $r^{(2-\alpha)}$ and the dynamical symmetry is broken. This leads to additional contributions of the Lamb shift and perihelion shift of planetary motion [22].

3 Gauss' law in α dimensional fractional space

The next step is to derive Gauss's law in α dimensional fractional space. Let us consider a closed α dimensional sphere of radius R, with its center at the origin of the coordinate system, and let us calculate the total flux of the gravitational field $\vec{g} = -\vec{\nabla}\varphi^{\alpha}(r,\theta)$, on the surface of this closed sphere. Namely, the total flux on the sphere is given by

$$\begin{aligned}\oint_{\alpha D} \vec{g}\cdot d\vec{A} &= \oint_{\alpha D} g_{radial}\, dA = \frac{2\pi^{\frac{(\alpha-1)}{2}}}{\Gamma\left(\frac{\alpha-1}{2}\right)} \int_0^{\pi} \left[-\frac{\partial\varphi^{\alpha}}{\partial r} r^{\alpha-1}\right] d\theta \sin^{\alpha-2}\theta \\ &= -mG_{\alpha}\alpha\,(\alpha-2)\,\pi^{\alpha/2}\Gamma\left(\frac{\alpha}{2}+1\right) = -4\pi G m.\end{aligned} \tag{19}$$

From (19) we identify the constant G_{α} in α dimensional space as follows

$$G_{\alpha} = \frac{2\Gamma(\frac{\alpha}{2})}{\pi^{\alpha/2-1}(\alpha-2)} G. \tag{20}$$

It may be observed that the fractional value of α agrees with the experimental physical observations so that in general relativity, gravitational fields are understood to be geometric perturbations (curvatures) in our space-time [9], rather than entities residing within a flat space-time. In this case one should take into consideration, the new effective value of universal constant G_{α}.

4 Fractional calculus approach to gravitational problem

One of the commonly used definitions of fractional integrals is known as the Riemann-Liouville integrals [10–12, 23]. The n-th order (or n-fold) integration of the given function $f(x)$ can be written as

$$\begin{aligned} {}_a\mathbf{D}_x^{-n} f(x) &= \int_a^x dx_{n-1} \int_a^{x_{n-1}} dx_{n-2} \ldots \int_a^{x_1} f(x_0) dx_0 \\ &= \frac{1}{(n-1)!} \int_a^x (x-u)^{n-1} f(u) du, \end{aligned} \tag{21}$$

where, ${}_a\mathbf{D}_x^{-n} f(x)$ denotes the n-th order integration with lower limit a. If n is replaced by a non integer number β, the Riemann-Liouville fractional integration is written as follows:

$${}_a\mathbf{D}_x^{\beta} f(x) = \frac{1}{\Gamma(-\beta)} \int_a^x (x-u)^{-\beta-1} f(u) du, \tag{22}$$

where, Γ is the Gamma function. For fractional derivatives with $\beta > 0$, a positive integer $m > \beta$ is chosen such that $\beta - m$ is negative, then the $(\beta - m)$-th order Riemann-Liouville fractional integration is performed whose m-th order derivative is the fractional derivatives of order β.

$${}_a\mathbf{D}_x^{\beta} f(x) \equiv \frac{d^m}{dx^m} {}_a\mathbf{D}_x^{\beta-m} f(x). \tag{23}$$

In Sec. 2 we investigated the solution of the Poisson's equation in α dimensional space. The gravitational potential was obtained as

$$\varphi^{\alpha}(r,\theta) = -\frac{G_\alpha m}{|\vec{r} - \vec{r'}|^{(\alpha-2)}} = \frac{G_\alpha m}{(x^2+y^2+z^2)^{(\alpha/2-1)}}, \tag{24}$$

where m represents the point mass and G_α is the universal constant and it is defined in equation (20).

We now apply a fractional βth-order partial differential operator [10–12, 18, 23] to both sides of equation (1), with the lower limit of $a = -\infty$, we get

$${}_{-\infty}\mathbf{D}_z^{\beta}[\nabla^2 \varphi^{\alpha}] = 4\pi G[{}_{-\infty}\mathbf{D}_z^{\beta} \rho], \tag{25}$$

Following the condition of commutativity of two operators ∇^2 and ${}_{-\infty}\mathbf{D}_z^{\beta}$, we can write

$$\nabla^2[{}_{-\infty}\mathbf{D}_z^{\beta} \varphi^{\alpha}] = 4\pi G[{}_{-\infty}\mathbf{D}_z^{\beta} \rho], \tag{26}$$

with $-1 \le \beta \le 0$, the mass distribution density of β th order is obtained as

$$\rho_\beta = ml^{\beta}{}_{-\infty}\mathbf{D}_z^{\beta}[\delta(x)\delta(y)\delta(z)] = ml^{\beta}\delta(x)\delta(y) \begin{cases} 0 & z < 0 \\ \frac{z^{-\beta-1}}{\Gamma(-\beta)} & z > 0 \end{cases}. \tag{27}$$

The solution of the Poisson equation (25) of β-th order mass distribution density gives the β-th order scalar potential as

$$\begin{aligned}\Phi_\beta^\alpha &= l^\beta{}_{-\infty}\mathbf{D}_z^\beta \varphi^\alpha, \\ &= -G_\alpha m l^\beta{}_{-\infty}\mathbf{D}_z^\beta \frac{1}{(x^2+y^2+z^2)^{(\alpha/2-1)}},\end{aligned} \tag{28}$$

employing the Riemann–Liouville definition of diferintegration in equation (22), for $-1 \leq \beta \leq 0$, the potential Φ_β is calculated as

$$\Phi_\beta^\alpha = -\frac{G_\alpha}{m} l^\beta \Gamma(-\beta) \int_{-\infty}^{z} \frac{1}{(x^2+y^2+t^2)^{(\alpha/2-1)}(z-t)^{1+\beta}} dt. \tag{29}$$

Using the change of variable $v = z - t$, the integral is rewritten as

$$\Phi_\beta^\alpha = -\frac{G_\alpha m l^\beta}{\Gamma(-\beta)} \int_0^\infty \frac{1}{(x^2+y^2+(z-v)^2)^{(\alpha/2-1)} v^{1+\beta}} dv. \tag{30}$$

Using the relations $z = r\cos\theta$ and $r = \sqrt{x^2+y^2+z^2}$, one can show that the integral can be written as

$$\Phi_\beta^\alpha = -\frac{G_\alpha m l^\beta}{\Gamma(-\beta)} \int_0^\infty \frac{1}{(r^2 - 2rv\cos\theta + v^2)^{(\alpha/2-1)} v^{1+\beta}} dv. \tag{31}$$

Now, let $u = \frac{v}{r}$, one obtains

$$\Phi_\beta^\alpha = -\frac{G_\alpha m l^\beta}{\Gamma(-\beta)} \frac{1}{r^{\alpha+\beta-2}} F_\alpha^\beta(\cos\theta), \tag{32}$$

where

$$F_\alpha^\beta(\cos\theta) = \int_0^\infty \frac{1}{(1 - 2u\cos\theta + u^2)^{(\alpha/2-1)} u^{1+\beta}} du. \tag{33}$$

One should notice that the dependence on the radial distance comes only through the $\frac{1}{r^{\alpha+\beta-2}}$ factor, while the function $F_\alpha^\beta(\cos\theta)$ determines the angular dependence. An important point to be specified here is that, for three dimensional space ($\alpha = 3$), the potential (32), reduced to the potential as obtained in reference [18]

$$\Phi_\beta^3 = \frac{-mG_\alpha l^\beta}{\Gamma(-\beta)} \frac{1}{r^{1+\beta}} F_3^\beta(\cos\theta), \tag{34}$$

where $F_3^\beta(\cos\theta)$ is the Legendre function of the first kind and the (noninteger) degree β.

5 Conclusions

We have introduced the form of fractional scalar gravitational potential by using the solutions of Laplace's equation in α dimensional fractional space.

Using the generating function of the Gengenbauer's polynomials the compact form of the fractional gravitational potential was obtained and we observed that Gauss's law is satisfied and leads us to redefine the constant G_α in any fractional or integer space.

Using these results, we employ the fractional calculus to obtain the fractional scalar gravitational potential of β-th order for the fractional space of order $\alpha > 2$.

We observed that for $\alpha \to 3$, the obtained results reduces to those obtained in reference [18].

Acknowledgment

This work is partially supported by the Scientific and Technical Research Council of Turkey.

References

[1] X. He: "Anisotropy and isotropy", *Solid State Commun.*, Vol. 75, (1990), pp. 111–114.

[2] X. He: "Excitons in anisotropic solids: the model of frational-dimensional space", *Phys. Rev. B*, Vol. 43, (1991), pp. 2063–2069.

[3] C. Palmer and P.N. Stavrinou: "Equations of motions in a non-integer dimensional space", *J. Phys. A: Math. Gen.*, Vol. 37, (2004), pp. 6987–7003.

[4] C.M. Bender, S. Boethtcher and L. Lipatov: "Almost zero-dimensional quantum field theories", *Phys. Rev. D*, Vol. 46, (1992), pp. 5557–5573.

[5] M.A. Lohe and A. Thilagam: "Quantum mechanical models in fractional dimensions", *J. Phys. A: Math. Gen.*, Vol. 37, (2004), pp. 6181–6199.

[6] A. Zeilinger, and K. Svozil: "Measuring the dimension of space-time", *Phys. Rev. Lett.*, Vol. 54, (1985), pp. 2553–2555; K. Svozil: "Quantum field theory on fractal space-time", *J. Phys. A: Math. Gen.*, Vol. 20, (1987), pp. 3861–3875.

[7] K.G. Willson: "Quantum field theory, models in less than 4 dimensions", *Phys. Rev. D*, Vol. 7, (1973), pp. 2911–2926.

[8] F.H. Stillinger: "Axiomatic basis for spaces with non-integer dimension", *J. Math. Phys.*, Vol. 18, (1977), pp. 1224–1234.

[9] C.W. Misner, K.S. Thorne and J.A. Wheeler: *Gravitation*, Freeman, San Francisco, 1973.

[10] K.S. Miller and B. Ross: *An Introduction to the Fractional Integrals and Derivatives-Theory and Applications*, Gordon and Breach, Longhorne, PA, 1993.

[11] J.A. Tenreiro Machado, I.S. Jesus, A. Galhano and J.B. Cunha: "Fractional order electromagnetics", *Signal Processing*, Vol. 86, (2006), pp. 2637–2644.

[12] I. Podlubny: *Fractional Differential Equations*, Academic Press, New York, 1999.

[13] R. Gorenflo, A. Vivoli and F. Mainardi: "Discrete and continuous random walk models for space-time fractional diffusion, "Nonlinear Dynamics", Vol. 38, (2004), pp. 101–116.

[14] F. Mainardi: "Fractional relaxation-oscillation and fractional diffusion-wave phenomena", *Chaos, Solitons and Fractals*, Vol. 7, (1996), pp. 1461–1477.

[15] O.P. Agrawal: "Formulation of Euler–Lagrange equations for fractional variational problems", *J. Math. Anal. Appl.*, Vol. 272, (2002), pp. 368–379.

[16] Eqab M. Rabei and T. Alhalholy: "Potentials of arbitrary forces with fractional derivatives", *Int. J. Theor. Phys. A*, Vol. 19, (2004), pp. 3083–3092.

[17] S. Muslih and D. Baleanu: "Hamiltonian formulation of systems with linear velocities within Riemann-Liouville fractional derivatives", *J. Math. Anal. Appl.*, Vol. 304, (2005), pp. 599–606.

[18] A.A. Rousan, E. Malkawi, E.M. Rabei and H. Widyan: "Application of fractional calculus to gravity", *Frac. Calc. Appl. Anal.*, Vol. 5, (2002), pp. 155–168.

[19] N. Engheta: "On fractional calculus and fractional multipoles in electromagnetism", *IEEE Transactions on Antennas and Propagation*, Vol. 44, (1996), pp. 554–566.

[20] N. Engheta: "On the role of fractional calculus in electromanetic theory", *IEEE Antennas and Propagation Magazine*, Vol. 39, (1997), pp. 35–46; "Fractional paradigm in electromagnetic theory", In: D.H. Werner and R. Mitra (Eds.): *Frontiers in Electromagnetics*, IEEE Press, New York, 2000.

[21] N. Engheta: "Fractional paradigm in electromagnetic theory", In: D.H. Werner and R. Mitra (Eds.): *Frontiers in Elctromagnetics*, IEEE Press, 2000, Chapter 12, pp. 523–552.

[22] A. Shafer and B. Miller: "Bounds of for the fractal dimension of space", *J. Phys. A: Math. Gen.*, Vol. 19, (1986), pp. 3891–3902.

[23] D. Baleanu and S. Muslih: "Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives", *Physica Scripta*, Vol. 72, (2005), pp. 119–121.